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RATES OF CONVERGENCE FOR RENEWAL SEQUENCES IN THE NULL-RECURRENT CASE

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Abstract

Motivated by work of Garsia and Lamperti we consider null-recurrent renewal sequences with a regularly varying tail and seek information about their rate of convergence to zero. The main result shows that such sequences subject to a monotonicity condition obey a limit law whatever the value of the exponent α is, $0 < \alpha < 1$. This monotonicity property is seen to hold for a large class of renewal sequences, the so-called Kaluza sequences. This class includes moment sequences, and therefore includes the sequences generated by reversible Markov chains. Several subsidiary results are proved.

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1. Introduction

Let $\{f_n\}$, $n = 1, 2, \ldots$, be a sequence of real numbers with

$$(1.1) \qquad f_n \geq 0, \quad \sum_{n=1}^{\infty} f_n = 1, \qquad \text{g.c.d.}\{n\colon f_n > 0\} = 1.$$

Define another sequence $\{u_n\}$, $n = 0, 1, 2, \ldots$, by

$$(1.2) \qquad u_0 = 1, \qquad u_n = \sum_{k=1}^{n} f_k u_{n-k}, \qquad n \geq 1.$$

It can be seen that $0 \leq u_n \leq 1$. The sequences $\{f_n\}$ and $\{u_n\}$ are related to Markov chain theory as follows: consider a recurrent aperiodic Markov chain

$\{X_n,\ n \geq 0\}$ with state space the integers and $P(X_0 = 0) = 1$. Let T be the time of first return to the origin. If we put

$$(1.3) \qquad P(T = n) = f_n, \qquad n \geq 1,$$

then (1.1) is satisfied (the second condition there is equivalent to recurrence of the process and the third to its aperiodicity). Let

$$(1.4) \qquad P(X_n = 0 \mid X_0 = 0) = u_n.$$

Then $\{u_n\}$ satisfies (1.2).

The classical renewal theorem [2] states

$$(1.5) \qquad \lim_{n\to\infty} u_n = \frac{1}{\sum_{k=1}^{\infty} k f_k}$$

where the right side is taken to be zero when the denominator diverges. In Markov chain terminology the denominator diverges when the chain is null-recurrent, and this is the case of interest in this paper.

Garsia and Lamperti [5] studied the rate of convergence to zero in (1.5) in the null-recurrent case when T is in the domain of attraction of a stable law of index α, $0 < \alpha < 1$. Their main result (Theorem 1.1) states that if

$$(1.6) \qquad \sum_{k=n+1}^{\infty} f_k = n^{-\alpha} L(n), \qquad 0 < \alpha < 1,$$

where $L(n)$ is a slowly varying function, then

$$(1.7) \qquad \liminf_{n\to\infty} n^{1-\alpha} L(n) u_n = \frac{\sin \pi\alpha}{\pi}$$

and if $\frac{1}{2} < \alpha < 1$ then (1.7) can be sharpened to

$$(1.8) \qquad \lim_{n\to\infty} n^{1-\alpha} L(n) u_n = \frac{\sin \pi\alpha}{\pi}.$$

The principal result of this note (Theorem 3.1) is the observation that if the renewal sequence $\{u_n\}$ satisfies the monotonicity property (3.2), then (1.6) is sufficient to imply (1.8) without regard to the value of α, $0 < \alpha < 1$. In particular it follows that any renewal sequence $\{u_n\}$ such that $\{u_{nk}\}$ is a Kaluza or moment sequence for some fixed $k \geq 1$ (see Section 4) satisfies (1.8) when (1.6) is true; this includes the case of reversible Markov chains (Corollary 4.1).

Section 2 presents the mostly well-known tools on rates of growth needed for the rest of the article. Finally, Proposition 3.1 gives some information on the boundary cases $\alpha = 0$ and $\alpha = 1$, including Erickson's renewal theorem (3.14) when $\alpha = 1$.

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2. Preliminary results on rates of growth

DEFINITION. A positive function L defined on the positive real axis is slowly varying (at infinity) if, for each $\lambda > 0$

$$\lim_{t\to\infty} \frac{L(\lambda t)}{L(t)} = 1. \tag{2.1}$$

U is regularly varying with exponent ρ if

$$U(x) = x^{\rho} L(x) \tag{2.2}$$

with $-\infty < \rho < \infty$ and L slowly varying. A basic reference on slow and regular variation is [11]. We require the following results.

LEMMA 2.1. *Let $0 \le \alpha < 1$, and let $L(x)$ be slowly varying. Then*

$$\sum_{k=1}^{n} \frac{1}{k^{\alpha}} L(k) \sim \frac{1}{1-\alpha} n^{1-\alpha} L(n). \tag{2.3}$$

LEMMA 2.2. *Let $L(x)$ be slowly varying with*

$$\sum_{k=1}^{n} \frac{1}{k} L(k) \uparrow \infty. \tag{2.4}$$

Then the function

$$\int_1^x \frac{1}{y} L(y)\, dy = L_1(x) \tag{2.5}$$

is slowly varying, and

$$\sum_{k=1}^{\infty} \frac{1}{k} L(k) \sim L_1(n). \tag{2.6}$$

LEMMA 2.3. *Let $\sum_{k=1}^{n} p_k \sim n^{\alpha} L(n)$, $0 < \alpha \le 1$, where $L(n)$ is slowly varying and p_n is monotone non-increasing. Then*

$$p_n \sim \alpha n^{\alpha-1} L(n). \tag{2.7}$$

LEMMA 2.4. *Let $\sum_{k=1}^{n} p_k \sim L(n)$ where $L(n)$ is slowly varying and p_n is monotone non-increasing. Then*

$$\lim_{n\to\infty} \frac{n p_n}{L(n)} = 0. \tag{2.8}$$

From (2.8) one gets

$$\lim_{n\to\infty} n^{1-\delta} p(n) = 0, \quad \text{for all } \delta > 0. \tag{2.9}$$

The above results are either all well known or easily accessible. Observe that from [11, 4°, pages 19–21], integral test comparisons on [11, Theorem 2.1] yield Lemma 2.1, and a similar argument using [11, Exercise 2.2] proves Lemma 2.2 (see also [4, Theorem 8.9.1]). Lemma 2.3 is part of [4, Theorem 13.5.4] or [11, Exercise 2.8], and the latter reference yields (2.8). Then (2.9) follows from (2.8) and

$$\lim_{x\to\infty} x^{\delta}(L(x))^{-1} = \infty \quad \text{for } \delta > 0.$$

(See, for example, [11, 1° and 3°, page 18].)

3. Principal results

Recall the definitions of the sequences $\{f_n\}$ and $\{u_n\}$ and of the random variable T given in Section 1. Let

$$r_n = \sum_{k=n+1}^{\infty} f_k = P(T > n).$$

Throughout this section it will be assumed that (1.1) holds and that $ET = \infty$ (or equivalently, $\sum r_k$ diverges).

THEOREM 3.1. *Let T be in the domain of attraction of a stable law of index α, $0 < \alpha < 1$; more precisely, suppose*

$$r_n \sim n^{-\alpha} L(n) \tag{3.1}$$

for $L(n)$ slowly varying. If

(3.2) *there exists a fixed integer $k \geq 1$ such that the sequence $\{u_{nk}\}$ is monotone non-increasing, then*

$$u_n \sim \frac{\sin \pi\alpha}{\pi} \frac{n^{\alpha-1}}{L(n)}. \tag{3.3}$$

Conversely, suppose (3.3) *is true for some α, $0 < \alpha < 1$, and $L(n)$ slowly varying. Then* (3.1) *holds.*

PROOF. The sum

$$\sum_{j=0}^{(n+1)k-1} u_j \tag{3.4}$$

may be decomposed into the k sums

$$\sum_{j=0}^{n} u_{jk+i} = U_i(n), \qquad 0 \leq i \leq k-1.$$

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The monotonicity of $\{u_{nk}\}$ implies that the sequence $\{u_n\}$ possesses the strong ratio limit property (SRLP) (see [10]) so that

(3.5) $$u_{nk+i} \sim u_{nk+j}$$

for fixed i, j, $0 \le i, j \le k-1$. Now (3.4) diverges, in fact, by [5, Lemma 2.3.1] we know

(3.6) $$\sum_{j=0}^{n} u_j \sim \frac{\sin \pi\alpha}{\pi\alpha} \frac{n^\alpha}{L(n)}.$$

Thus at least one $U_i(n)$ diverges, and (3.5) easily implies

(3.7) $$U_i(n) \sim U_j(n), \qquad 0 \le i, j \le k-1.$$

By (3.6) and properties of slowly varying functions we obtain

$$\sum_{j=0}^{k-1} U_j(n) = \sum_{j=0}^{(n+1)k-1} u_j \sim \frac{C\{(n+1)k-1\}^\alpha}{L((n+1)k-1)} \sim \frac{C(nk)^\alpha}{L(nk)},$$
$$C = (\pi\alpha)^{-1} \sin \pi\alpha.$$

From (3.7) we conclude that

$$U_{i(n)} \sim \frac{C}{k} \frac{(nk)^\alpha}{L(nk)} = Ck^{\alpha-1} \frac{n^\alpha}{L(nk)}$$

for each i. The terms of $U_0(n)$ are monotone non-increasing and so, by Lemma 2.3

$$u_{nk} \sim \alpha C k^{\alpha-1} \frac{n^{\alpha-1}}{L(nk)} = \alpha C \frac{(nk)^{\alpha-1}}{L(nk)}.$$

Using the SRLP

$$u_{nk+i} \sim \alpha C \frac{(nk+i)^{\alpha-1}}{L(nk+i)}, \qquad 0 \le i \le k-1,$$

proving (3.3).

To prove the converse assertion, it will be sufficient to show that if

(3.8) $$u_n \sim C \frac{n^{\alpha-1}}{L(n)}$$

for some constant C, then $r_n \sim C_1 n^{-\alpha} L(n)$ for some constant C_1. Below C denotes a constant, not necessarily the same one in different relations. Since the reciprocal of a slowly varying function is also slowly varying, (3.8) can be written as $u_n \sim C n^{\alpha-1} L_1(n)$. Lemma 2.1 then gives

$$\sum_{j=1}^{n} u_j \sim C n^\alpha L_1(n).$$

An Abelian theorem [4, page 423] shows that the generating function $U(s)$ of the sequence $\{u_n\}$ satisfies

$$U(s) \sim C(1-s)^{-\alpha} L_1\left(\frac{1}{1-s}\right), \qquad s \to 1^-. \tag{3.9}$$

Use a standard renewal Tauberian argument (for example, see [5, Lemma 2.3.1], and reverse the steps) to obtain

$$\sum_{j=0}^{n} r_j \sim C n^{1-\alpha} L(n).$$

Monotonicity of $\{r_n\}$ and Lemma 2.3 allow us to deduce (3.1).

Suppose we now relax the condition on T in Theorem 3.1: let us assume that T only has a regularly varying tail. This means that (3.1) now holds where L is slowly varying and α is some real number. Since we are interested in the null-recurrent case, $\sum r_k$ diverges and hence $0 \le \alpha \le 1$. So there are two extreme cases, $\alpha = 0$ and $\alpha = 1$, not covered by Theorem 3.1. Erickson obtained the result (3.14) for $\alpha = 1$ [3]. We have the following

PROPOSITION 3.1. (a) *Let* (3.1) *hold with* $\alpha = 0$. *Then*

$$\sum_{j=0}^{n} u_j \sim (L(n))^{-1}. \tag{3.10}$$

If the monotonicity condition (3.2) *also holds, then*

$$\lim_{n\to\infty} \frac{nu_n}{L(n)} = 0 \quad \text{and} \quad \lim_{n\to\infty} n^{1-\delta} u_n = 0 \quad \text{for all } \delta > 0. \tag{3.11}$$

(b) *Let* (3.1) *hold with* $\alpha = 1$ *and let*

$$\int_1^x \frac{1}{y} L(y)\, dy = L_1(x). \tag{3.12}$$

Then

$$\sum_{k=1}^{n} u_k \sim \frac{n}{L_1(n)} \tag{3.13}$$

and

$$u_n \sim \frac{1}{L_1(n)}. \tag{3.14}$$

PROOF. Under (a), Lemma 2.1 and the Tauberian argument of [5, Lemma 2.3.1] cited previously prove (3.10). An argument similar to that in the proof of Theorem 3.1 coupled with Lemma 2.4 proves (3.11).

Under (b), divergence of $\sum r_k$ implies divergence of (3.12) so that by Lemma 2.2, $L_1(x)$ is slowly varying and (2.6) is true. Again, the Tauberian argument easily gives (3.13). Note that (3.14) does not follow immediately from (3.13), for we have not assumed monotonicity here; we refer the reader to Erickson's proof [3, page 266].

Remark 1. The failure of Lemma 2.3 for the case $\alpha = 0$ means that we are not able to obtain the exact rate of convergence of $\{u_n\}$ in this case. Lemma 2.4 gives us (3.11), but this is unsatisfactory. The case of simple random walk in the plane suggests improvement on (3.11) may be possible; there one has

$$r_n \sim \frac{\pi}{\log n}, \qquad \sum_{j=0}^{n} u_j \sim \frac{\log n}{\pi} \quad \text{and} \quad u_{2n} \sim \frac{1}{\pi n}.$$

Remark 2. It is perhaps not surprising that the case $\alpha = 1$ can be added to the Garsia-Lamperti range $\frac{1}{2} < \alpha < 1$ of values of α where renewal theorems hold automatically without further conditions. Thus there is a kind of continuity at $\alpha = 1$ of the good behavior at $\alpha = 1^-$, although (3.3) and (3.14) are different. Whether such continuity also holds at $\alpha = \frac{1}{2}$ is an open question (see [5, page 230], the discussion following (3.4.9)).

4. Applications

Throughout this section the renewal sequence $\{u_n\}$ is associated with the sequence $\{f_n\}$ where (1.1) is assumed to be valid, and $\sum r_k$ diverges.

The sequence $\{u_n\}$ is called a *Kaluza sequence* if

$$u_n^2 \le u_{n-1} \cdot u_{n+1}, \qquad n \ge 1, \tag{4.1}$$

and it is called a *moment sequence* if there exists a probability measure ν on $[0, 1]$ with $u_n = \int_0^1 x^n \nu(dx)$, $n \ge 0$. Every moment sequence is a Kaluza sequence. The most interesting property of Kaluza sequences in the present discussion is that they are non-increasing. Moreover, many renewal sequences turn out to have the Kaluza or moment properties. Perhaps the most famous case is $u_n = \binom{2n}{n} 2^{-2n}$ where $\{u_n\}$ is associated with simple random walk on the line. We refer the reader to [8] (also see [7] and [9]) for further discussion of Kaluza sequences.

A class of moment sequences arises by considering reversible Markov chains. A chain is *reversible* if $\pi(i)p(i,j) = \pi(j)p(j,i)$ for all i, j, where π is the invariant measure of the chain, and $p(\cdot,\cdot)$ is its transition probability (see for example [10, page 83]). Under our assumptions, the chain is recurrent and aperiodic and has a non-trivial σ-finite invariant measure. A result of Kendall ([6], also [10, page 83]) shows that for reversible chains u_{2n} is a moment sequence. The monotonicity property of Kaluza sequences enables us to apply Theorem 3.1 or Proposition 3.1(a). We summarize this in the following corollary.

COROLLARY 4.1. *Let* $\{u_n\}$ *be a renewal sequence such that* $\{u_{nk}\}$ *is a Kaluza sequence for a fixed integer* $k \geq 1$.

(a) *If* (3.1) *is valid for some* α, $0 < \alpha < 1$, *then* (3.3) *holds.*

(b) *If* (3.1) *is valid for* $\alpha = 0$ *then* (3.11) *holds.*

In particular, if $\{u_n\}$ *is derived from a reversible Markov chain, then* $\{u_{2n}\}$ *is a moment (hence Kaluza) sequence, so that if* T *has a regularly varying tail,* (3.3), (3.11) *or* (3.14) *holds, depending upon the value of* α.

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