ON SOME DIVISIBILITY PROPERTIES OF $\binom{2n}{n}$

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L. Moser [3] recently gave a very simple proof that

$$(1) \qquad \binom{2n}{n} = \binom{2a}{a}\binom{2b}{b}$$

has no solutions. In the present note we shall first of all prove that for $a > \frac{n}{2}$, $\binom{2a}{a} \nmid \binom{2n}{n}$, which by the fact that there is a prime p satisfying $n < p \leq 2n$ immediately implies that

$$(2) \qquad \binom{2n}{n} = \prod_{i=1}^{r} \binom{2a_i}{a_i}^{\alpha_i}, \quad \alpha_i \geq 1, \quad n > a_i \geq 1$$

has no solutions. It is easy to see on the other hand that

$$(3) \qquad \prod_{i=1}^{r_1} \binom{2a_i}{a_i}^{\alpha_i} = \prod_{i=1}^{r_2} \binom{2b_i}{b_i}^{\beta_i}, \quad a_i \geq 1, \quad b_i \geq 1$$

has infinitely many non-trivial solutions. I do not know if (3) is solvable if $\alpha_i = \beta_i = 1$. I will discuss some further divisibility properties of $\binom{2n}{n}$ and mention some unsolved problems.

THEOREM. Denote by $g(m)$ the smallest integer $n > m$ for which $\binom{2m}{m} \mid \binom{2n}{n}$. For all m we have

$$(4) \qquad g(m) \geq 2m,$$

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and for $m > m_o$

$$(5) \qquad m^{1+c} < g(m) < (2m)^{\log m/\log 2}$$

for a certain absolute constant $c > 0$.

First we prove (4). Put $n = m+k$, $0 < k < m$; then

$$(6) \qquad \frac{\binom{2n}{n}}{\binom{2m}{m}} = \prod_{i=1}^{2k} (2m+i) \Big/ \left(\prod_{i=1}^{k} (m+i)\right)^2 .$$

By a simple calculation we can show that for $n \leq 11$, (6) is never an integer. Henceforth we can thus assume $n \geq 12$. It is well known that for $n \geq 12$ there always is a prime p satisfying $\frac{2}{3}n < p < n$. Thus if $m \leq \frac{2n}{3}$, (6) cannot be an integer since the denominator is divisible by p^2 and the numerator only by p. Thus we can assume

$$n \geq 12, \qquad m > \frac{2n}{3}.$$

Miss Faulkner [2] recently proved that $\prod_{i=1}^{k} (m+i)$ always has a prime factor $q > 2k$ if $m + k \geq P$, where P is the least prime $> 2k$, except if $k = 2$, $m = 7$ or $k = 3$, $m = 7$. In our case these exceptions cannot occur since $n > 11$, $m > \frac{2}{3}n > 7$. Also, since $n > 11$ and $m > \frac{2n}{3}$, $k < \frac{n}{3}$ or $2k < \frac{2n}{3}$; hence $m+k = n > P$. Thus by the theorem of Miss Faulkner there is a prime $q > 2k$ which divides $\prod_{i=1}^{k} (m+i)$.

Let $m+j$, $0 < j < k$ be the unique value for which $m+j \equiv 0 \pmod q$ and assume $q^{\alpha} \| (m+j)$ (i.e., $q^{\alpha} | (m+j)$, $q^{\alpha+1} \nmid (m+j)$). Since $q > 2k$, $2m+2j$ is the only integer m of the sequence $2m+i$, $0 < i \leq 2k$, which is a multiple of q. Hence $q^{\alpha} \| \prod_{i=1}^{2k} (2m+i)$,

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$q^{2\alpha} \mid \prod_{i=1}^{k} (m+i)^2$, or (6) cannot be an integer, which proves (4).

It can easily be shown that $g(m) > 2m$ for $m > 1$, (i.e., $g(m) = 2m$ holds only for $m = 1$).

Now we prove the first inequality of (5). It is well known and evident that if $2k + 1 < (2n)^{1/2}$, then no prime p satisfying $\frac{2n}{2k+1} < p < \frac{n}{k}$ divides $\binom{2n}{n}$. Further, it follows from the classical theorem of Hoheisel [3] that if $\varepsilon > 0$ is sufficiently small and $k < n^{\varepsilon}$, $n > n_0(\varepsilon)$, then there always is a prime satisfying

$$(7) \qquad \frac{2n}{2k+1} < p < \frac{n}{k}.$$

Now if $c = c(\varepsilon)$ is sufficiently small and $\frac{5}{2}m < n < m^{1+c}$ then there clearly is a $k < n^{\varepsilon}$ for which

$$m < \frac{2n}{2k+1} < \frac{n}{k} < 2m,$$

or

$$p \,\Big|\, \binom{2m}{m}, \qquad p \nmid \binom{2n}{n},$$

which proves $g(m) > m^{1+c}$ (if $2m < n \leq \frac{5}{2}m$ then the interval $(\frac{2}{3}n, 2m)$ contains a prime, thus $\binom{2m}{m} \nmid \binom{2n}{n}$).

It seems very likely that for every k and $m > m_0(k)$, $g(m) > m^k$, but this is perhaps not easy to prove. It seems likely that to every $\varepsilon > 0$ there is an n_0 so that for every $m > n^{\varepsilon}$ there is a prime p, $m < p < 2m$, such that $p \nmid \binom{2n}{n}$. This would of course imply $g(m) > m^k$.

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Now we prove the second inequality of (5). L. Moser [4] observed that $\binom{2m}{m} \mid \binom{2n}{n}$ if $n = \binom{2m}{m} - 1$ (i.e. $(n+1) \mid \binom{2n}{n}$), but this only gives $g(m) < c_1^m$.

We will only outline the proof of the upper bound for $g(m)$. In fact we shall show a stronger result than (5). Let $m > m_0(\varepsilon)$ and $x > m^{\log m/\log 2}$. Then the number of integers $n < x$ for which $\binom{2m}{m} \nmid \binom{2n}{n}$ is less than εx.

It is well known that if

$$n = \sum_{i=0}^{k} a_i p^i, \qquad 0 \le a_i < p,$$

is the p-ary expansion of n, then $p^r \| \binom{2n}{n}$, where

$$(8) \qquad r = \sum_{a_i \ge p/2} 1 .$$

In other words $p \nmid \binom{2n}{n}$ if and only if all the a_i are $< p/2$. Thus by a simple calculation the number of integers $n < p^{k+1}$ for which $p \nmid \binom{2n}{n}$ equals $[\frac{p}{2}]^{k+1}$. Hence if $x > (2m)^{\log m/\log 2}$ and $p < 2m$ then the number of integers $n < x$ for which $p \nmid \binom{2n}{n}$ is less than

$$(10) \qquad \frac{x}{2^{\log m/\log 2}} = \frac{x}{m} .$$

Further, a simple combinatorial argument shows that the number of integers $n < p^{k+1}$ for which $p^r \nmid \binom{2n}{n}$ equals

$$(11) \qquad [\tfrac{p}{2}]^{k+1} \sum_{i=0}^{r-1} \binom{k+1}{i} < [\tfrac{p}{2}]^{k+1} (k+1)^r .$$

Hence by (11) we obtain by a simple computation, the details of which we suppress, that the number of integers $n < x$ $(x > (2m)^{\log m/\log 2})$ for which

$$p^r \nmid \binom{2n}{n}, \qquad p < (2m)^{1/r}$$

is also less than $\frac{x}{m}$ (as in (10)). Now it is well known and easy to prove that if $p^r \mid \binom{2m}{m}$ then $p^r < 2m$ (or $p < (2m)^{1/r}$). Hence from (10) the number of integers $n < x$ for which

$$\binom{2m}{m} \nmid \binom{2n}{n}$$

is less than

$$x\frac{\pi(2m)}{m} < \varepsilon x$$

for $m > m_0(\varepsilon)$, which completes the proof of (5).

I do not know to what extent our upper bound for $g(m)$ can be improved.

I have not been able to show that there is an infinite sequence $n_1 < n_2 < \ldots$ so that for every $i < j$, $\binom{2n_i}{n_i} \nmid \binom{2n_j}{n_j}$, but it seems certain that such a sequence exists [1].

REFERENCES

1. See P. Erdős, Quelques Problèmes de la Thèorie des Nombres Problème 57, Monographie de L'Enseignement Math. No. 6.

2. Miss Faulkner's proof is not yet published.

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3. For the strongest result in this direction see A. E. Ingham, On the difference between consecutive primes, Quarterly Journal of Math. 8 (1937), 255-266.

4. L. Moser, Insolvability of $\binom{2n}{n} = \binom{2a}{a}\binom{2b}{b}$, Can. Math. Bull. 6 (1963) 167-169.

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