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# THE NACHBIN QUASI-UNIFORMITY OF A BI-STONIAN SPACE

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**Abstract**

It is known that every frame is isomorphic to the generalized Gleason algebra of an essentially unique bi-Stonian space $(X, \sigma, \tau)$ in which $\sigma$ is $T_0$. Let $(X, \sigma, \tau)$ be as above. The specialization order $\leq_\sigma$ of $(X, \sigma)$ is $\tau \times \tau$-closed. By Nachbin's Theorem there is exactly one quasi-uniformity $\mathscr{U}$ on $X$ such that $\cap\mathscr{U} = \leq_\sigma$ and $\mathscr{T}(\mathscr{U}^*) = \tau$. This quasi-uniformity is compatible with $\sigma$ and is coarser than the Pervin quasi-uniformity $\mathscr{P}$ of $(X, \sigma)$. Consequently, $\tau$ is coarser than the Skula topology of $\sigma$ and coincides with the Skula topology if and only if $\mathscr{U} = \mathscr{P}$.

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## 1. Introduction

According to [4], a subset $G$ of *a* bitopological space $(X, \sigma, \tau)$ is *strictly regular* provided that $G = \text{int}_\sigma \, \text{cl}_\tau \, G$ and a bitopological space $(X, \sigma, \tau)$ is *bi-Stonian* provided that $\tau$ is a compact Hausdorff 0-dimensional topology containing $\sigma$, $\sigma$ has a base of strictly regular sets and each strictly regular set is $\tau$-closed. The *generalized Gleason algebra* of a bi-Stonian space $(X, \sigma, \tau)$ is the frame of its strictly regular subsets, where $A \wedge B$ is defined as $A \cap B$ and $\bigvee A_\alpha$ is defined as $\text{int}_\sigma \, \text{cl}_\tau \bigcup A_\alpha$. The principal result of [4] is that every frame is isomorphic to the generalized Gleason algebra of an essentially unique bi-Stonian space in which the coarser topology is $T_0$ [4, Theorem 3.2].

In this paper we consider only those bi-Stonian spaces $(X, \sigma, \tau)$ for which $\sigma$ is a $T_0$ topology. Under this restriction, as is well known, the specialization order of $\sigma$ is a partial order, and a simple lemma establishes that this partial order is $\tau \times \tau$-closed. Consequently, there is exactly one quasi-uniformity on $X$, which we call the Nachbin





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quasi-uniformity and denote by $\mathcal{N}$, such that $\bigcap \mathcal{N}$ is the specialization order of $\sigma$ and $\mathcal{T}(\mathcal{N}^*) = \tau$ [3, Proposition 13] and [2, Theorem 1.20]. We show that $\mathcal{N}$ is compatible with $\sigma$ and that $\mathcal{N}$ is coarser than the Pervin quasi-uniformity of $(X, \sigma)$. It follows that $\tau$ is coarser than the Skula topology of $\sigma$ [6], and we show that $\tau$ is the Skula topology of $\sigma$ if, and only if, the Nachbin quasi-uniformity and the Pervin quasi-uniformity of $(X, \sigma)$ coincide. Moreover, there is a natural base $\mathcal{B}$ for $\mathcal{N}$ such that for each $U \in \mathcal{B}$ and $x \in X$, $U(x)$ is $\tau(\mathcal{N}^{-1})$-closed and hence strictly regular and $\tau$-closed.

We refer the reader to [2] for definitions and results concerning quasi-uniformities that are assumed here.

## 2. The Nachbin quasi-uniformity

Throughout this paper, we consider a given bi-Stonian space $(X, \sigma, \tau)$ for which $\sigma$ is a $T_0$ topology. The specialization order of $\sigma$, which is denoted by $\leq_\sigma$, is defined by $x \leq_\sigma y$ if, and only if, $x \in \mathrm{cl}_\sigma\{y\}$. This order is a partial order.

LEMMA 2.1. *The specialization order of $\sigma$ is $\tau \times \tau$-closed.*

PROOF. Suppose that $x \notin \mathrm{cl}_\sigma\{y\}$ and let $G$ be a strictly regular set about $x$ such that $y \notin G$. Then $(x, y) \in G \times (X - G)$, which is $\tau \times \tau$-open, and since $G \in \sigma$, it is evident that $\leq_\sigma$ and $G \times (X - G)$ are disjoint.

PROPOSITION 2.1. *The Nachbin quasi-uniformity of $(X, \sigma, \tau)$ is compatible with $\sigma$.*

PROOF. Let $\mathcal{B}$ be the base for $\sigma$ consisting of all strictly regular sets and let $B \in \mathcal{B}$. Then $B$ is $\tau$-open and $\tau$-closed and so $U_B = (B \times B) \cup ((X - B) \times X)$ is a $\tau \times \tau$-open set containing $\leq_\sigma$. It follows that $U_B \in \mathcal{N}$ [3, Proposition 13], or [2, Theorem 1.20]. Let $\mathcal{V}$ be the quasi-uniformity on $X$ for which $\{U_B : B \in \mathcal{B}\}$ is a transitive subbase. It is evident that $\mathcal{V}$ is compatible with $\sigma$ – we complete the proof by showing that $\mathcal{V}$ is the Nachbin quasi-uniformity. Since $\mathcal{V} \subseteq \mathcal{N}$, $\leq_\sigma = \bigcap \mathcal{N} \subseteq \bigcap \mathcal{V}$. Let $(x, y) \in \bigcap \mathcal{V}$. Then for each $B \in \mathcal{B}$, $(x, y) \in U_B$ and so $x \in \mathrm{cl}_\sigma\{y\}$. Hence $\cap\mathcal{V}$ is $\leq_\sigma$. To see that $\mathcal{T}(\mathcal{V}^*) \subseteq \tau$, note that since $\mathcal{V} \subseteq \mathcal{N}$, $\mathcal{T}(\mathcal{V}^*) \subseteq \mathcal{T}(\mathcal{N}^*) = \tau$. Since $\sigma$ is a $T_0$ topology, $\mathcal{T}(\mathcal{V}^*)$ is a Hausdorff topology and so $\mathcal{T}(\mathcal{V}^*) = \tau$.

DEFINITION [6]. The *Skula topology* of a topological space $(S, \mathcal{G})$ is the topology on $S$ for which $\mathcal{G} \cup \{X - G : G \in \mathcal{G}\}$ is a subbase.

COROLLARY 2.1. *The Nachbin quasi-uniformity of $(X, \sigma, \tau)$ is a transitive quasi-uniformity coarser than the Pervin quasi-uniformity of $(X, \sigma)$ and $\tau$ is coarser than the Skula topology of $\sigma$.*

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PROOF. Let $\mathscr{P}_\sigma$ denote the Pervin quasi-uniformity for $(X, \sigma)$. The collection $\{U_B : B \in \mathscr{B}\}$ given in the proof of Proposition 2.1 is a subcollection of $\mathscr{P}_\sigma$ consisting of transitive entourages. Thus $\mathscr{N}$ is a transitive quasi-uniformity coarser than $\mathscr{P}_\sigma$. By [1, Proposition 1.4] and [5, Proposition 3.2.2.3], $\mathscr{T}(P_\sigma^*)$ is the Skula topology of $\sigma$ and so $\tau = \mathscr{T}(\mathscr{N}^*)$ is coarser than the Skula topology of $\sigma$.

Suppose for the nonce that $\sigma$ is a $T_1$ topology. Then $\leq_\sigma$ is the diagonal and so $\mathscr{N}$ is the only uniformity compatible with $\tau$. By the previous proposition, $\sigma = \mathscr{T}(\mathscr{N}) = \tau$ and the frame corresponding to $(X, \sigma, \tau)$ is the Boolean algebra of $\tau$-regular open sets. This observation is an instance of a general principle: interesting topologies are always $T_0$ and never $T_1$.

PROPOSITION 2.2. *Let $\mathscr{B}$ be a transitive base for the Nachbin quasi-uniformity of $(X, \sigma, \tau)$. Then for each $U \in \mathscr{B}$ and each $x \in X$, $U(x)$ is $\mathscr{T}(\mathscr{N}^{-1})$-closed and $U^{-1}(x)$ is $\mathscr{T}(\mathscr{N})$-closed; hence both $U(x)$ and $U^{-1}(x)$ are $\tau$-closed, $\tau$-open and strictly regular.*

PROOF. Let $U \in \mathscr{B}$ and $x \in X$. Then $\{U^{-1}(y) : y \notin U(x)\} \bigcup \{U(x)\}$ is a cover of $X$ and $\cup\{U^{-1}(y) : y \notin U(x)\}$ is disjoint from $U(x)$. It follows that $U(x)$ is $\mathscr{T}(\mathscr{N}^{-1})$-closed. The proof of the corresponding result for $U^{-1}(x)$ follows in an analogous way.

Because the Nachbin quasi-uniformity of $(X, \sigma, \tau)$ is contained in the Pervin quasi-uniformity, it is natural to consider when these quasi-uniformities coincide.

PROPOSITION 2.3. *The Nachbin quasi-uniformity of $(X, \sigma, \tau)$ is $\mathscr{P}_\sigma$ if and only if $\tau$ is the Skula topology of $\sigma$.*

PROOF. Suppose that $\tau$ is the Skula topology of $\sigma$. Then every $\sigma$-open set is $\tau$-closed and hence strictly regular. Hence the base $\mathscr{B}$ for $\sigma$ given in the proof of Proposition 2.1 is $\sigma$ itself and so $\mathscr{N} = \mathscr{V} = \mathscr{P}_\sigma$.

Now suppose that $\mathscr{P}_\sigma = \mathscr{N}$. Then $\mathscr{T}(\mathscr{P}_\sigma^*) = \mathscr{T}(\mathscr{N}^*) = \tau$ and $\mathscr{T}(\mathscr{P}_\sigma^*)$ is the Skula topology of $\sigma$.

COROLLARY 2.2. *The quasi-proximities $\delta_{\mathscr{P}_\sigma}$ and $\delta_{\mathscr{N}}$ agree if and only if $\tau$ is the Skula topology of $\sigma$.*

In light of Corollary 2.2, the last result of this section is somewhat surprising.

PROPOSITION 2.4. *Let $A$ and $B$ be $\tau$-closed sets. Then $A \delta_{\mathscr{P}_\sigma} B$ if and only if $A \delta_{\mathscr{N}} B$.*

PROOF. Since $\mathscr{N} \subseteq \mathscr{P}$, if $A\delta_{\mathscr{P}_\sigma} B$, then $A\delta_{\mathscr{N}} B$. Suppose that $A\delta_{\mathscr{N}} B$. Then for each $U \in \mathscr{N}$, $U \cap A \times B \neq \emptyset$ and we must show that $A \cap \mathrm{cl}_\sigma B \neq \emptyset$. If $\leq_\sigma \cap A \times B = \emptyset$, then $X \times X - A \times B \in \mathscr{N}$ – a contradiction. Thus there exists $x \in A$ and $y \in B$ such that $x \leq_\sigma y$ and it follows that $A \cap \mathrm{cl}_\sigma B \neq \emptyset$.

EXAMPLE. We show here that in general for a bi-Stonian space $(X, \sigma, \tau)$, the Skula modification $Sk(\sigma)$ of $\sigma$ is not $\tau$. Consider the chain $L = [0, \alpha]$ where $\alpha$ is an initial ordinal. It is readily seen that its Boolean extension, $B_L$, resides in $\mathscr{P}([0, \alpha))$ and that $L$ is embedded in $B_L$ by mapping $\beta$ to $[0, \beta)$ for $0 \leq \beta \leq \alpha$. For the ground set we take $X = \prod B_L$, the Stone space of $B_L$, which has as points all ultrafilters in $B_L$, and, for $F \in L$, we denote by $\prod_F$ the collection of all points of $\prod B_L$ that contain $F$. We take the Stone topology for $\tau$, and for $\sigma$ we take the topology for which $\{\prod_{[0,\beta)} : \beta \in [0, \alpha]\} \cup \{\prod B_L\}$ is a base. Take $P = \cap_{n \in \mathbb{N}} \prod_{[n,\alpha)}$. Evidently $P$ is in $Sk(\sigma)$, since each $\prod_{[n,\alpha)}$ is $\sigma$-closed. We show that $P \neq \prod_{[w_0,\alpha)}$, after which, by checking other cases, it is readily seen that $P \notin \tau$.

The set $\{[n, \alpha) : n \in \mathbb{N}\} \cup \{[0, w_0)\}$ has the finite intersection property, so can be extended to an ultrafilter $\mathscr{F}$ in $B_L$, which clearly contains each $[n, a)$, but not $[w_0, \alpha)$. Hence $\mathscr{F} \in P$, but $\mathscr{F} \notin \prod_{[w_0,\alpha)}$, as required.

In the case where $\alpha = w_0$, the example is particularly simple, since then $B_L$ is just all finite or cofinite subsets of $[0, w_0)$.

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