Nagoya Math. J. **199** (2010), 107–122
DOI 10.1215/00277630-2010-005

DYNAMICAL CONSTRUCTION OF KÄHLER-EINSTEIN METRICS

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Abstract. In this article, we give a new construction of a Kähler-Einstein metric on a smooth projective variety with ample canonical bundle. As a consequence, for a dominant projective morphism $f: X \longrightarrow S$ with connected fibers such that a general fiber has an ample canonical bundle, and for a positive integer m, we construct a canonical singular Hermitian metric $h_{E,m}$ on $f_*\mathcal{O}_X(mK_{X/S})$ with semipositive curvature in the sense of Nakano.

§1. Introduction

Let X be a smooth projective n-fold with ample canonical bundle defined over $\mathbb{C}$. Then by the celebrated solution of Calabi's conjecture (see [A], [Y1]), there exists a unique Kähler-Einstein C^∞-form ω_E such that

$$-\operatorname{Ric}_{\omega_E} = \omega_E \tag{1.1}$$

holds, where $\operatorname{Ric}_{\omega_E}$ denotes the Ricci form of the Kähler manifold (X, ω_E).

On the other hand, for a complex manifold with very ample L^2-canonical forms, there exists a standard Kähler form called the *Bergman Kähler form.*

Let us explain more precisely. Let M be a complex manifold of dimension n such that the space of L^2-canonical forms

$$\begin{aligned} &H^0_{(2)}\big(M, \mathcal{O}_M(K_M)\big) \\ &\quad := \Big\{\eta \in H^0\big(M, \mathcal{O}_M(K_M)\big) \,\Big|\, (\sqrt{-1})^{n^2}\int_M \eta \wedge \bar{\eta} < \infty\Big\} \end{aligned} \tag{1.2}$$

gives a very ample linear system. Then M admits a Bergman kernel,

$$B(z,w) := \sum_i \sigma_i(z) \cdot \overline{\sigma_i(w)} \qquad (z, w \in M), \tag{1.3}$$

Received December 1, 2006. Revised January 29, 2009. Accepted November 26, 2009.
2000 Mathematics Subject Classification. 53C25, 32G07, 53C55, 58E11.

where $\{\sigma_i\}$ is a complete orthonormal basis of $H^0_{(2)}(M, \mathcal{O}_M(K_M))$ with respect to the inner product

$$(\eta, \eta') := (\sqrt{-1})^{n^2} \int_M \eta \wedge \bar{\eta}'. \tag{1.4}$$

We see that $B(z, w)$ is independent of the choice of the orthonormal basis $\{\sigma_i\}$ and that

$$\omega_B(z) := \sqrt{-1}\partial\bar{\partial} \log B(z, z) \quad (z \in M) \tag{1.5}$$

is a Kähler form; ω_B is called the *Bergman Kähler form on* M. The same construction applies for the case of the adjoint bundle of a (possibly singular) Hermitian line bundle (L, h) on M (see Section 3).

Both Kähler-Einstein metrics and Bergman metrics are determined uniquely by the complex structures. In this sense, these metrics are canonical. Hence, it is natural to study the relation of these metrics.

Recently, S. K. Donaldson [Do] found a new construction of Kähler-Einstein metrics or, more generally, of Kähler metrics with constant scalar curvature; actually, he found a strong connection between the existence of Kähler metrics with constant scalar curvature and the asymptotic stability of Hilbert points of projective embeddings. In particular, this implies the connection between the existence of Kähler-Einstein metrics and the asymptotic stability of Hilbert points of projective embeddings.

Let us explain a part of his results. Let X be a smooth projective variety, and let L be an ample line bundle on X. Then for every sufficiently large positive integer m, the linear system $|mL|$ gives a projective embedding

$$\Phi_m : X \longrightarrow \mathbb{P}^{N_m}, \tag{1.6}$$

given by

$$\Phi_m(x) := [\sigma_0^{(m)} : \cdots : \sigma_{N_m}^{(m)}], \tag{1.7}$$

where $\{\sigma_0^{(m)}, \ldots, \sigma_{N_m}^{(m)}\}$ is a basis of $H^0(X, \mathcal{O}_X(mL))$. Hence, Φ_m depends on the choice of the basis. Let ω_{FS} denote the Fubini-Study Kähler form on $\mathbb{P}^{N_m}$. If, for some choice of $\{\sigma_0^{(m)}, \ldots, \sigma_{N_m}^{(m)}\}$, the equality

$$\int_X \frac{\sigma_i^{(m)} \cdot \bar{\sigma}_j^{(m)}}{\sum_{i=0}^{N_m} |\sigma_i^{(m)}|^2} (\Phi_m^* \omega_{FS})^n = \delta_{ij} \tag{1.8}$$

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holds for every $0 \leqq i, j \leqq N_m$ (i.e., if $\{\sigma_0^{(m)}, \dots, \sigma_{N_m}^{(m)}\}$ is orthonormal with respect to the L^2-inner product with respect to the Hermitian metric $\left(\sum_{i=0}^{N_m} |\sigma_i^{(m)}|^2\right)^{-2}$ on mL and the volume form $(\Phi_m^* \omega_{FS})^n$), the Kähler form

$$\omega_m := \frac{1}{m} \Phi_m^* \omega_{FS} \tag{1.9}$$

is called *balanced (or critical)*. The Hilbert point of $\Phi_m(X)$ is stable if and only if there exists a choice of the basis $\{\sigma_0^{(m)}, \dots, \sigma_{N_m}^{(m)}\}$ such that Φ_m is balanced (see [Z]). Donaldson's theorem is the following.

THEOREM 1.1 ([Do, page 482, Theorem 3]). *Let X be a smooth projective variety, and let L be an ample line bundle on X. Suppose that* $\mathrm{Aut}(X, L)$ *is discrete. If X admits a Kähler form ω cohomologous to $2\pi c_1(L)$ with constant scalar curvature, then, for every sufficiently large m, $\Phi_m(X)$ is stable (i.e., (X, L) is called* asymptotically stable*). The limit of the balanced Kähler forms $\{\omega_m\}$ exists in C^∞-topology, and the limit is a Kähler form with constant scalar curvature.*

In short, Theorem 1.1 gives a construction of a Kähler form with constant scalar curvature as the limit of a sequence of balanced Kähler forms, and Theorem 1.1 is closely related to the asymptotic expansion of Bergman kernels (see [C], [Ze]).

The purpose of this article is to construct Kähler-Einstein forms with negative Ricci curvature as a limit of Bergman Kähler forms. More precisely, the purpose of this article is to relate Kähler-Einstein forms and Bergman Kähler forms in the case of projective manifolds with ample canonical bundle.

Let us explain the construction. Let X be a smooth projective n-fold with ample canonical bundle. Let m_0 be a positive integer such that

(1) $|mK_X|$ is very ample for every $m \geqq m_0$, and

(2) for every pseudoeffective singular Hermitian line bundle (L, h_L) (see Definition 2.3), $\mathcal{O}_X(m_0 K_X + L) \otimes \mathcal{I}(h_L)$ is globally generated.

The existence of such m_0 follows from Nadel's vanishing theorem [N, page 561].

Let h_{m_0} be a C^∞-Hermitian metric on $m_0 K_X$ with strictly positive curvature. Suppose that we have constructed K_m and the C^∞-Hermitian metric h_m on mK_X. Then we define

$$K_{m+1} := K\big(X, (m+1)K_X, h_m\big) \tag{1.10}$$

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and

$$h_{m+1} := 1/K_{m+1}, \tag{1.11}$$

where $K(X,(m+1)K_X,h_m)$ denotes the diagonal part of the Bergman kernel of $(m+1)K_X$ with respect to h_m constructed as follows.

Let $\{\sigma_0^{(m+1)},\dots,\sigma_{N_{m+1}}^{(m+1)}\}$ be the complete orthonormal basis of $H^0\bigl(X, \mathcal{O}_X((m+1)K_X)\bigr)$ with respect to the inner product

$$(\sigma,\tau) := (\sqrt{-1})^{n_2}\int_X h_m\cdot\sigma\wedge\bar\tau \qquad (\sigma,\tau\in H^0(X,\mathcal{O}_X((m+1)K_X))). \tag{1.12}$$

Then for $x\in X$ we define

$$\begin{aligned} K_{m+1}(x) &= K\bigl(X,(m+1)K_X,h_m\bigr)(x) \\ &:= \sum_{i=0}^{N_{m+1}} |\sigma_i^{(m+1)}|^2(x), \end{aligned} \tag{1.13}$$

where, for a global section σ of $(m+1)K_X$, $|\sigma|^2$ denotes the global section $\sigma\cdot\bar\sigma$ of $(K_X\otimes\overline{K_X})^{\otimes(m+1)}$. We note that by the choice of m_0, $|(m+1)K_X|$ is very ample. Hence, $h_{m+1} := 1/K_{m+1}$ is a C^∞-Hermitian metric on $(m+1)K_X$. Inductively, we construct the sequences $\{h_m\}_{m\geqq m_0}$ and $\{K_m\}_{m>m_0}$. This is the same construction originated by the author in [T3].

The following theorem is the main result in this article.

THEOREM 1.2. *Let X be a smooth projective n-fold with ample canonical bundle. Let m_0 and $\{h_m\}_{m>m_0}$ be the sequence of Hermitian metrics as above. Then*

$$h_\infty := \liminf_{m\to\infty} \sqrt[m]{(m!)^n\cdot h_m} \tag{1.14}$$

is a C^∞-Hermitian metric on K_X such that

$$\omega_\infty := \Theta_{h_\infty} \tag{1.15}$$

is a Kähler form on X with

$$-\operatorname{Ric}_{\omega_\infty} = \omega_\infty. \tag{1.16}$$

REMARK 1.3. The existence of the limit h_∞ has already been proved in [T3] not only for the canonically polarized varieties but also for varieties of general type (for smooth projective varieties of nongeneral type, see [T3], [T5], [T6], and [T7]).

The construction of the Kähler-Einstein form in Theorem 1.2 is more straightforward than the one in Theorem 1.1. Also Theorem 1.2 seems to imply that the sequence of Kähler forms

$$\left\{\frac{1}{m}\Theta_{h_m}\right\}_{m \geqq m_0} \tag{1.17}$$

induced by the morphisms $\Phi^{(m)} : X \longrightarrow \mathbb{P}^{N_m} (m > m_0)$ defined by

$$\Phi^{(m)}(x) = [\sigma_0^{(m)}(x) : \cdots : \sigma_{N_m}^{(m)}(x)] \quad (x \in X) \tag{1.18}$$

is asymptotically nearly balanced.

Theorem 1.2 implies the following semipositivity theorem.

THEOREM 1.4. *Let $f : X \longrightarrow S$ be a projective morphism with connected fibers between smooth varieties. Let S° denote the maximal Zariski dense subset of S such that f is smooth over $X^\circ := f^{-1}(S^\circ)$. Suppose that a general fiber of f is a smooth projective variety with ample canonical bundle. Let $\omega_{E/S}$ be the family of relative Kähler-Einstein forms on X°. Let h_E° be the C^∞-Hermitian metric on $K_{X/S} \mid X^\circ$ defined by*

$$h_E^\circ := (\omega_{E/S}^n)^{-1}, \tag{1.19}$$

where n denotes the relative dimension of $f : X \longrightarrow S$. Then we have the following:

(1) *h_E° extends to a singular Hermitian metric h_E on $K_{X/S}$;*

(2) *the curvature current Θ_{h_E} of h_E is semipositive on X;*

(3) *$F_m := f_*\mathcal{O}_X(mK_{X/S})$ is locally free on S°, and $F_m|S^\circ$ carries the C^∞-Hermitian metric $h_{E,m}$ defined by*

(1.20)

$$h_{E,m}(\sigma,\tau) := (\sqrt{-1})^{n^2}\int_{X_s} h_E^{m-1}\cdot \sigma\wedge\bar{\tau} \quad (\sigma,\tau \in H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}))).$$

Then $(F_m|S^\circ, h_{E,m})$ is semipositive in the sense of Nakano (see [D, VII-6]).

We note that Theorems 1.2 and 1.4 can be generalized to the case of Kähler-Einstein currents (or, more generally, canonical measures; see [ST], [T7]) without any essential changes by using the existence of Kähler-Einstein currents due to Sugiyama [S] (which is a generalization of [T0]) and the recent result on finite generation of canonical rings (see [BCHM]).* (For such further generalizations, see [T6], [T7], [T8], and [T9].)

Theorem 1.4 has several applications. For example, it immediately gives canonical positive line bundles on the moduli space of canonically polarized varieties with only canonical singularities. Such applications will be discussed in a subsequent paper ([T9]).

We should note that the convergence in Theorem 1.2 is much weaker than in Theorem 1.1. Also, Theorem 1.2 does not say anything about Kähler forms with constant scalar curvature at this moment.

§2. Preliminaries

In this section, we review the basic terminologies used in this paper.

2.1. Singular Hermitian metrics

In this subection, L denotes a holomorphic line bundle on a complex manifold M.

DEFINITION 2.1. A *singular Hermitian metric* h on L is given by

$$h = e^{-\varphi} \cdot h_0, \tag{2.1}$$

where h_0 is a C^∞-Hermitian metric on L and $\varphi \in L^1_{\mathrm{loc}}(M)$ is an arbitrary function on M. We call φ a *weight function of* h.

The curvature current Θ_h of the singular Hermitian line bundle (L,h) is defined by

$$\Theta_h := \Theta_{h_0} + \sqrt{-1}\partial\bar{\partial}\varphi, \tag{2.2}$$

where $\partial\bar{\partial}$ is taken in the sense of a current and where $\Theta_{h_0} = \sqrt{-1}\bar{\partial}\partial \log h_0$. We note that in our convention, the curvature current Θ_h is always a closed real current.

The L^2-sheaf $\mathcal{L}^2(L,h)$ of the singular Hermitian line bundle (L,h) is defined by

$$\mathcal{L}^2(L,h)(U) := \left\{\sigma \in \Gamma\big(U, \mathcal{O}_M(L)\big) \;\middle|\; h(\sigma,\sigma) \in L^1_{\mathrm{loc}}(U)\right\}, \tag{2.3}$$

where U runs over the open subsets of M. In this case, there exists an ideal sheaf $\mathcal{I}(h)$ such that

$$\mathcal{L}^2(L,h) = \mathcal{O}_M(L) \otimes \mathcal{I}(h) \tag{2.4}$$

*Actually, this paper is a part of [T6].

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holds. We call $\mathcal{I}(h)$ the *multiplier ideal sheaf* of (L,h). If we write h as

$$h = e^{-\varphi} \cdot h_0, \tag{2.5}$$

where h_0 is a C^∞-Hermitian metric on L and where $\varphi \in L^1_{\mathrm{loc}}(M)$ is the weight function, we see that

$$\mathcal{I}(h) = \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi}) \tag{2.6}$$

holds. For $\varphi \in L^1_{\mathrm{loc}}(M)$, we define the multiplier ideal sheaf of φ by

$$\mathcal{I}(\varphi) := \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi}). \tag{2.7}$$

EXAMPLE 2.2. Let $\sigma \in \Gamma(X, \mathcal{O}_X(L)) - \{0\}$ be a global section. Then

$$h := \frac{1}{|\sigma|^2} = \frac{h_0}{h_0(\sigma,\sigma)} \tag{2.8}$$

is a singular Hermitian metric on L, where h_0 is an arbitrary C^∞-Hermitian metric on L (the right-hand side is obviously independent of h_0). The curvature Θ_h is given by

$$\Theta_h = 2\pi(\sigma), \tag{2.9}$$

where (σ) denotes the current of integration over the divisor of σ.

DEFINITION 2.3. Here L is said to be *pseudoeffective* if there exists a singular Hermitian metric h on L such that the curvature current Θ_h is a closed positive current. Also, a singular Hermitian line bundle (L,h) is said to be *pseudoeffective* if the curvature current Θ_h is a closed positive current.

2.2. Analytic Zariski decompositions

Let L be a pseudoeffective line bundle on a compact complex manifold X. To analyze the ring

$$R(X,L) = \bigoplus_{m=0}^{\infty} H^0\big(X, \mathcal{O}_X(mL)\big), \tag{2.10}$$

it is sometimes useful to introduce the notion of analytic Zariski decompositions.

DEFINITION 2.4. Let M be a compact complex manifold, and let L be a holomorphic line bundle on M. A singular Hermitian metric h on L is said to be an *analytic Zariski decomposition* if the following hold:

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(1) Θ_h is a closed positive current; and
(2) for every $m \geq 0$, the natural inclusion

$$H^0\big(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)\big) \rightarrow H^0\big(M, \mathcal{O}_M(mL)\big) \tag{2.11}$$

is an isomorphism.

Remark 2.5. If an analytic Zariski decomposition (AZD) exists on a line bundle L on a smooth projective variety M, L is pseudoeffective by condition (1) above.

It is known that for every pseudoeffective line bundle on a compact complex manifold, there exists an AZD on L (see [T1], [T2], [DPS]).

The advantage of the AZD is that we can handle pseudoeffective line bundle L on a compact complex manifold X as a singular Hermitian line bundle with semipositive curvature current as long as we consider the ring $R(X, L)$.

§3. Proof of Theorem 1.2

Let X be a smooth projective n-fold with ample canonical bundle. Let m_0 be a positive integer such that

(1) $|mK_X|$ is very ample for every $m \geqq m_0$; and
(2) for every pseudoeffective singular Hermitian line bundle (L, h_L), $\mathcal{O}_X(m_0K_X + L) \otimes \mathcal{I}(h_L)$ is globally generated.

Let h_{m_0} be a C^∞-Hermitian metric on m_0K_X with strictly positive curvature. Let $\{h_m\}_{m \geqq m_0}$ and $\{K_m\}_{m > m_0}$ be the sequences of Hermitian metrics and Bergman kernels constructed as in Section 1; that is, $\{h_m\}_{m \geqq m_0}$ and $\{K_m\}_{m > m_0}$ are defined inductively by

$$K_{m+1} = K(X, K_X + mK_X, h_m) \tag{3.1}$$

and

$$h_{m+1} = 1/K_{m+1}. \tag{3.2}$$

3.1. Upper estimate of K_m

Let Δ be the unit open disk in $\mathbb{C}$ with center zero. Let

$$K_1(\Delta) := K(\Delta, K_\Delta), \tag{3.3}$$

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and let $h_{1,\Delta} := 1/K_1(\Delta)$. Inductively, we define $K_m(\Delta)(m \geqq 1)$ and $h_{m,\Delta}$ by

$$K_m(\Delta) := K(\Delta, mK_\Delta, h_{m-1,\Delta}) \tag{3.4}$$

and

$$h_{m,\Delta} = 1/K_m(\Delta). \tag{3.5}$$

Then by direct calculation, we see that

$$K_m(\Delta) = \frac{(m+1)!}{(1-|z|^2)^{2m}} \Big(\frac{\sqrt{-1}}{2\pi}\, dz \wedge d\bar{z}\Big)^{\otimes m} \tag{3.6}$$

holds.

Let $x \in X$, and let $(U, z_1, \ldots, z_n)$ be a local coordinate around x such that $z_1(x) = \cdots = z_n(x) = 0$ and U is biholomorphic to the unit open polydisk Δ^n with center the origin via $(z_1, \ldots, z_n)$. Then by induction in m, it is easy to see that there exists a positive constant C_U such that

$$K_m \leqq C_U^m \cdot \big((m+1)!\big)^n \cdot \prod_{i=1}^n \frac{1}{(1-|z_i|^2)^{2m}} \cdot \Big(\bigwedge_{i=1}^n \frac{\sqrt{-1}}{2\pi}\, dz_i \wedge d\bar{z}_i\Big)^{\otimes m} \tag{3.7}$$

holds on U. Hence, moving U, by the compactness of X, we have the following lemma.

Lemma 3.1. *Let dV be a C^∞-volume form on X. Then there exists a positive constant C_+ such that*

$$K_m \leqq C_+^m \cdot (m!)^n \cdot (dV)^m \tag{3.8}$$

holds on X.

3.2. Lower estimate of K_m

Let ω_E be the Kähler-Einstein form on X such that

$$-\operatorname{Ric}_{\omega_E} = \omega_E. \tag{3.9}$$

Let $dV_E = (n!)^{-1}\omega_E^n$ be the volume form associated with (X, ω_E).

Lemma 3.2. *Here*

$$\limsup_{m\to\infty} \sqrt[m]{(m!)^{-n} K_m} \geqq (2\pi)^{-n}\, dV_E \tag{3.10}$$

holds on X.

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Proof. Let us consider the Hermitian line bundle (K_X, dV_E) on X. Let $p \in X$ be a point. Then by the Kähler-Einstein condition, there exists a holomorphic normal coordinate $(U, z_1, \ldots, z_n)$ such that

$$dV_E^{-1} = \left\{ \prod_{i=1}^{n} (1 - |z_i|^2) + O(\|z\|^3) \right\} \cdot |dz_1 \wedge \cdots \wedge dz_n|^{-2} \tag{3.11}$$

holds. Suppose that

$$C_{m-1} \cdot dV_E^{m-1} \leqq K_{m-1} \tag{3.12}$$

holds on X for some positive constant C_{m-1}. We note that

(3.13)
$$K_m(x) = \sup\left\{ |\sigma|^2(x); \sigma \in H^0\big(X, \mathcal{O}_X(mK_X)\big), (\sqrt{-1})^{n^2} \int_X h_{m-1} \cdot \sigma \wedge \bar{\sigma} = 1 \right\}$$

holds for every $x \in X$, by the extremal property of the Bergman kernel. (This is well known; see, e.g., [Kr, page 46, Proposition 1.3.16].) We note that for the open unit disk $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$,

$$\int_\Delta (1 - |t|^2)^m \, dt \wedge d\bar{t} = \frac{2\pi}{m+1} \tag{3.14}$$

holds. Then by Hörmander's L^2-estimate of $\bar{\partial}$-operator, we see that there exists a positive constant λ_m such that

$$\big(\lambda_m \cdot (2\pi)^{-n} \cdot m^n\big) \cdot C_{m-1} \cdot dV_E^m \leqq K_m \tag{3.15}$$

with

$$\lambda_m \geqq 1 - \frac{C}{\sqrt{m}}, \tag{3.16}$$

where C is a positive constant independent of m.

In fact, this can be verified as follows. Let $x \in X$ be a point on X, and let $(U, z_1, \ldots, z_n)$ be the normal coordinate as above. We may assume that U is biholomorphic to the polydisk $\Delta^n(r)$ of radius r with center O in $\mathbb{C}^n$ for some r via $(z_1, \ldots, z_n)$.

Taking r sufficiently small, we may assume that there exists a C^∞-function ρ on X such that

(1) ρ is identically 1 on $\Delta^n(r/3)$,
(2) $0 \leqq \rho \leqq 1$,

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(3) $\operatorname{Supp}\rho \subset\subset U$, and

(4) $|d\rho| < 3/r$, where $|\,|$ denotes the pointwise norm with respect to ω_E.

We note that by (3.11), the mass of $\rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m}$ concentrates around the origin as m tends to infinity. Hence, by (3.14) we see that the L^2-norm

$$\|\rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m}\| \tag{3.17}$$

of $\rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m}$ with respect to $(dV_E)^{-\otimes m}$ and ω_E is asymptotically

$$\|\rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m}\|^2 \sim \left(\frac{2\pi}{m}\right)^n \tag{3.18}$$

as m tends to infinity, where $\sim$ means that the ratio of the both sides converges to 1. We set

$$\phi := n\rho \log \sum_{i=1}^{n} |z_i|^2. \tag{3.19}$$

We may and do assume that m is sufficiently large so that

$$m \cdot \omega_E + \sqrt{-1}\partial\bar{\partial}\phi > 0 \tag{3.20}$$

holds on X.

By (3.18), the L^2-norm

$$\left\|\bar{\partial}\big(\rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m}\big)\right\|_{\phi} \tag{3.21}$$

of $\bar{\partial}(\rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m})$ with respect to $e^{-\phi} \cdot (dV_E)^{-\otimes m}$ and ω_E satisfies the inequality

$$\left\|\bar{\partial}\big(\rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m}\big)\right\|_{\phi}^2 \leqq M_0 \cdot \left(\frac{3}{r}\right)^{2n+2} \left(\frac{2\pi}{m}\right)^n \tag{3.22}$$

for every m, where M_0 is a positive constant independent of m.

By Hörmander's L^2-estimate applied to the adjoint line bundle of the Hermitian line bundle $((m-1)K_X, e^{-\phi} \cdot dV_E^{-(m-1)})$, we see that for every sufficiently large m, there exists a C^∞-solution of the equation

$$\bar{\partial}u = \bar{\partial}\big(\rho \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes m}\big) \tag{3.23}$$

such that

$$u(x) = 0 \tag{3.24}$$

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and

$$\|u\|_\phi^2 \leqq \frac{M_1}{m}\big\|\bar\partial\big(\rho\cdot(dz_1\wedge\cdots\wedge dz_n)^{\otimes m}\big)\big\|_\phi^2 \tag{3.25}$$

hold, where $\|\ \|_\phi$ denotes the L^2-norm with respect to $e^{-\phi}\cdot dV_E^{-(m-1)}$ and ω_E, respectively, and where M_1 is a positive constant independent of m. Then $\rho\cdot(dz_1\wedge\cdots\wedge dz_n)^{\otimes m}-u$ is a holomorphic section of mK_X such that

$$\big(\rho\cdot(dz_1\wedge\cdots\wedge dz_n)^{\otimes m}-u\big)(x)=(dz_1\wedge\cdots\wedge dz_n)^m \tag{3.26}$$

and

$$\|\rho\cdot(dz_1\wedge\cdots\wedge dz_n)^{\otimes m}-u\|^2 \leqq \left(1+M_0\cdot\left(\frac{3}{r}\right)^{2n+2}\sqrt{\frac{M_1}{m}}\right)\left(\frac{2\pi}{m}\right)^n. \tag{3.27}$$

Hence, by induction on m, using (3.13) and (3.15), we see that there exist positive constants C and C' such that, for every $m>m_0$,

$$K_m \geqq C'\left(\prod_{k=m_0}^{m}\left(1-\frac{C}{\sqrt{k}}\right)\right)\cdot(m!)^n\cdot(2\pi)^{-mn}\cdot dV_E^m \tag{3.28}$$

holds on X. This implies that

$$\limsup_{m\to\infty}\sqrt[m]{(m!)^{-n}K_m}\geqq(2\pi)^{-n}\,dV_E \tag{3.29}$$

holds on X. ∎

3.3. Integral estimate of K_m

Lemma 3.3. *The inequality*

$$\int_X \sqrt[m]{K_m}\leqq\left(\prod_{k=m_0}^{m}(N_k+1)\right)^{1/m}\cdot\left(\int_X\sqrt[m_0]{K_{m_0}}\right)^{m_0/m} \tag{3.30}$$

holds, where $N_k := \dim|kK_X| = \dim H^0(X,\mathcal{O}_X(kK_X))-1$.

Proof. By Hölder's inequality, we have

$$\begin{aligned}\int_X\sqrt[m]{K_m} &= \int_X\frac{K_m^{1/m}}{K_{m-1}^{1/(m-1)}}\cdot K_{m-1}^{1/(m-1)}\\ &\leqq\left(\int_X\frac{K_m}{K_{m-1}^{m/(m-1)}}\cdot K_{m-1}^{1/(m-1)}\right)^{1/m}\cdot\left(\int_X K_{m-1}^{1/(m-1)}\right)^{(m-1)/m}\end{aligned}$$

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$$= \left(\int_X \frac{K_m}{K_{m-1}}\right)^{1/m} \cdot \left(\int_X K_{m-1}^{1/(m-1)}\right)^{(m-1)/m}$$
$$= (N_m + 1)^{1/m} \cdot \left(\int_X K_{m-1}^{1/(m-1)}\right)^{(m-1)/m}.$$

Then continuing this process, by using

$$\int_X K_{m-1}^{1/(m-1)} \leqq (N_{m-1}+1)^{1/(m-1)} \cdot \left(\int_X K_{m-2}^{1/(m-2)}\right)^{(m-2)/(m-1)}, \tag{3.31}$$

we have that

$$\begin{aligned} &\int_X (K_m)^{1/m} \\ &\quad \leqq \left\{(N_m+1)\cdot(N_{m-1}+1)\right\}^{1/m} \cdot \left(\int_X (K_{m-2})^{1/(m-2)}\right)^{(m-2)/m} \end{aligned} \tag{3.32}$$

holds. Continuing this process, we obtain the lemma. □

Using Lemma 3.3, we obtain the following.

LEMMA 3.4. *The inequality*

$$\limsup_{m\to\infty} \frac{1}{(m!)^{n/m}} \int_X (K_m)^{1/m} \leqq \frac{K_X^n}{n!} \tag{3.33}$$

holds.

Proof. By the Kodaira vanishing theorem,

$$H^q\big(X, \mathcal{O}_X(mK_X)\big) = 0 \tag{3.34}$$

holds for every $m \geqq 2$ and $q \geqq 1$. Then by Hirzebruch's Riemann-Roch theorem, we have that

$$N_m + 1 = \frac{K_X^n}{n!} m^n + O(m^{n-1}) \tag{3.35}$$

holds. Then by Lemma 3.3, we have

$$\limsup_{m\to\infty} \frac{1}{(m!)^{n/m}} \int_X (K_m)^{1/m} \leqq \frac{K_X^n}{n!} \tag{3.36}$$

holds. □

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3.4. Completion of the proof of Theorem 1.2

By Lemma 3.1 and Lebesgue's bounded convergence theorem, we see that

$$\limsup_{m\to\infty} \frac{1}{(m!)^{n/m}} \int_X (K_m)^{1/m} = \int_X \left(\limsup_{m\to\infty} \frac{1}{(m!)^{n/m}} (K_m)^{1/m} \right) \tag{3.37}$$

holds. Hence, by Lemma 3.4, we have that

$$\int_X \left(\limsup_{m\to\infty} \frac{1}{(m!)^{n/m}} (K_m)^{1/m} \right) \leqq \frac{(2\pi)^n K_X^n}{n!} \tag{3.38}$$

holds. Since

$$\int_X dV_E = \frac{1}{n!} \int_X \omega_E^n = \frac{(2\pi)^n K_X^n}{n!} \tag{3.39}$$

holds by the Kähler-Einstein condition, combining Lemma 3.2 and (3.38), we have the equality

$$\limsup_{m\to\infty} \frac{1}{(m!)^{n/m}} \sqrt[m]{K_m} = (2\pi)^{-n} \, dV_E. \tag{3.40}$$

This completes the proof of Theorem 1.2. □

§4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4 by using Theorem 1.2. Roughly speaking, Theorem 1.2 implies that what we can say about Bergman kernels also holds for Kähler-Einstein volume forms.

Proof of Theorem 1.4. Let A be a sufficiently ample line bundle on X, and let h_0 be a C^∞-Hermitian metric with strictly positive curvature. Then for every $s \in S^\circ$, we define the dynamical system of the Bergman kernels $\{K_{m,s}\}$ on the fiber $X_s := f^{-1}(s)$ as in Section 1. Then by induction on m, we see that the Hermitian metric

$$h_m \mid X_s = 1/K_{m,s} \tag{4.1}$$

on $A + mK_{X/S} \mid X^\circ$ has semipositive curvature by [B] or [T6, Theorem 1.4]. It also extends to a singular Hermitian metric on $A + mK_{X/S}$ with semipositive curvature by [T6, Theorem 1.4] or [BP]. Then by Theorem 1.2, we see that h_E is a singular Hermitian metric on $K_{X/S}$ with semipositive curvature current by [T6, Theorem 1.4] or [BP]. And h_E is smooth over $f^{-1}(S^\circ)$ by the well-known standard implicit function theorem argument. Then again, by [B], we see that $h_{E,m}$ defined as (1.20) has semipositive curvature in the sense of Nakano for every $m \geqq 1$. This completes the proof of Theorem 1.4. □

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