ON THE MT^*- AND λ-CONJUGATES OF $\mathfrak{L}^\lambda$ SPACES

H. W. ELLIS

1. Introduction. Marston Morse and William Transue **(9, 10)**, motivated by their theory of bilinear functions, introduced and studied vector function spaces called MT-spaces for which each element of the dual is represented by an integral with respect to a suitable (C) measure. In this paper the definition of real MT-spaces is generalized to give spaces, called MT^*-spaces, for which part but not all of the dual is of integral type and this part is called the *MT^*-conjugate* of the space. In the theory of $\mathfrak{L}^\lambda$ spaces **(6)** a conjugate space is also defined. It will be called the λ-conjugate below. An $\mathfrak{L}^\lambda$ space is an MT^*-space if and only if it contains $\mathfrak{K}$, the space of all continuous functions with compact support. The purpose of this paper is to compare the MT^*- and λ-conjugates of $\mathfrak{L}^\lambda$ spaces that are MT^*-spaces .

When E is countable at infinity the MT^*-conjugates and λ-conjugates coincide. Conditions ensuring that the MT^*-conjugate contains the λ-conjugate are given in Theorem 3.1, that the λ-conjugate includes the MT^*-conjugate in Theorems 4.1 and 4.2. Examples are given of $\mathfrak{L}^\lambda$ spaces, including the space $\bar{\mathfrak{L}}^1$ **(4**, p. 13**)**, for each of which the λ-conjugate strictly contains the MT^*-conjugate. Theorem 4.2 shows that, for a class of spaces E more general than the spaces E that are countable at infinity, the λ-conjugate always contains the MT^*-conjugate. Whether or not this is true for all E is not known. This paper makes essential use of many of the results of references **(3, 4)** and **(8)**. The author wishes to express his thanks to Professor Morse for making available pre-publication copies of **(8)** and **(10)**.

2. MT^*- and $\mathfrak{L}^\lambda$-spaces and their conjugates. Let E be an arbitrary locally compact space and let $\mathbf{R}^E$ denote the space of real valued functions on E.

Definition 2.1. We call A a (real) MT^*-space if: (i) A is a vector subspace of $\mathbf{R}^E$, (ii) A contains $\mathfrak{K}$, (iii) A contains $|x|$ if it contains x and (iv) there is a non-trivial, monotone semi-norm $\mathfrak{N}^A$ defined on A.

We note that a real MT^*-space satisfies condition (i) in the definition of real MT-spaces apart from the requirement that $\mathfrak{K}$ be dense in A **(9**, p. 168**)**. If A' denotes the topological dual of A then, as in **(9**, Corollary 10.1**)**, for every $\Phi \in A'$, the restriction of Φ to $\mathfrak{K}$ defines a Radon measure ϕ on E such that for every $f \in \mathfrak{K}$,

$$\Phi(f) = \int f d\phi. \tag{2.1}$$

Received August 12, 1957. Written while the author was an Alfred P. Sloan Research Fellow.

The set of elements Φ in A' for which (2.1) holds for every $f \in A$ will be called the MT^*-conjugate of A and denoted by A^*. The mapping $\Phi \rightarrow \phi$ of A' into the vector space of all Radon measures on E is an isomorphism defining the MT^*-measure conjugate $\mathfrak{A}^*$ of A (cf. **9**, p .169). If $\phi \in \mathfrak{A}^*$, every $f \in A$ is integrable (ϕ) and to $\Phi(f)$, $\Phi \in A^*$. The definition

$$||\phi||_{\mathfrak{A}^*} = \sup_{x \in A, \mathfrak{N}^A(x) > 0} \frac{|\int x d\phi|}{\mathfrak{N}^A(x)},$$

for all $\phi \in \mathfrak{A}^*$, gives $||\phi||_{\mathfrak{A}^*} = ||\Phi||_{A'}$.

A real MT-space is an MT^*-space for which $A' = A^*$. Conversely, if A is an MT^*-space for which $A' = A^*$, it is a real MT-space. Condition (ii) for MT-spaces is then satisfied by hypothesis. If $\mathfrak{K}$ is assumed to be non-dense in A then (**1**, Lemme, p. 57) implies the existence of $\Phi \in A'$, not the zero element, vanishing in $\mathfrak{K}$. The corresponding ϕ is then the zero measure so that for some $f \in A$, $0 = \int f d\phi \neq \Phi(f)$ contradicting the fact that $A' = A^*$. Examples 3.1 and 3.2 of (**10**) show that there are MT^*-spaces in which $\mathfrak{K}$ is dense that are not real MT-spaces.

Definition 2.2 (**11**, p. 53). *Let $\mathfrak{M}$, $\mathfrak{M}'$ be families of subsets of E closed under the formation of countable unions and complements where $\mathfrak{M}'$ is a proper subfamily of $\mathfrak{M}$ and has the additional property that $M \in \mathfrak{M}'$, $A \in E$, $A \subset \mathfrak{M}$ imply that $A \in \mathfrak{M}'$. For an $\mathfrak{M}$-measurable real valued function $f(P)$ on E, let $||f||_\infty$, the* ess sup *of $|f(P)|$, be the infimum of the set of numbers α such that $E_\alpha = (P:|f(P)| < \alpha)$ is in $\mathfrak{M}'$, if this set is non-void. Otherwise let $||f||_\infty$. Let $\mathfrak{L}_\infty = \mathfrak{L}_\infty(E, \mathfrak{M}, \mathfrak{M}')$ denote the subspace of $\mathbf{R}^E$ for which $||f||_\infty$ is defined and finite.*

The space $\mathfrak{L}_\infty$ is an MT^*-space with semi-norm $||\cdot||_\infty$ if every continuous function with compact support is $\mathfrak{M}$-measurable and in particular if $\mathfrak{M}$ contains all the relatively compact Borel sets on E. The elements of A' to which correspond finitely additive measures that are not countably additive are not in A^*. If $E = (0, 1)$, if $\mathfrak{M}$ denotes the Borel sets on E and $\mathfrak{M}'$ the Borel sets of the first category on E, $\mathfrak{L}_\infty$ is an MT^*-space for which A^* reduces to the zero element of A' (**2**, Corollary 1, p. 186).

If A is an MT^*-space, $\mathfrak{N}^A$ the semi-norm on A, $\mathfrak{N}^A$ will be called reflexive if

$$\mathfrak{N}^A(x) = \sup\left(\left|\int x d\phi\right| ; ||\phi||_{\mathfrak{A}^*} \leqslant 1, \phi \in \mathfrak{A}^*\right). \tag{2.2}$$

The left side of (2.2) is never less than the right side. When $\mathfrak{N}^A$ is reflexive and the supremum in (2.2) can be replaced by $\sup\int|x|d|\phi|$, for all $\phi \in \mathfrak{A}^*$ with $||\phi||_{\mathfrak{A}^*} \leqslant 1$, $\mathfrak{N}^A$ has a natural extension to all of $\bar{\mathbf{R}}^E$. If A is an MT-space $\mathfrak{N}^A$ is always reflexive and has such an extension (**9**, § 11). For the example above with $A^* = 0$, $\mathfrak{N}^A$ is not reflexive.

Length functions and the corresponding function spaces were introduced in terms of non topological spaces E and general measures in (**6**). In this paper we suppose given a positive Radon measure μ on an arbitrary locally compact space E. If $\lambda(f)$ is defined and $0 \leqslant \lambda(f) \leqslant \infty$ for every μ-measurable function $f(P)$ with $0 \leqslant f(P) \leqslant \infty$ almost everywhere, λ is called a length function if:

(L 1) $\lambda(f) = 0$ whenever f is μ-negligible,
(L 2) $\lambda(f) \leqslant \lambda(g)$ whenever $f(P) \leqslant g(P)$ for all $P \in E$,
(L 3) $\lambda(f+g) \leqslant \lambda(f) + \lambda(g)$,
(L 4) $\lambda(kf) = k\lambda(f)$ for all $k > 0$,
(L 5) $f_1(P) \leqslant f_2(P) \leqslant \dots$ for all P implies that $\lambda(\sup f_n) = \sup \lambda(f_n)$.

A length function λ will be called continuous at infinity if, for every f,

(L 6) $\lambda(f) = \sup_K \lambda(f\chi_K)$,

for all compact sets $K \in E$, where χ_K denotes the characteristic function of K. If $|f(P)|$ is μ-measurable, $\lambda(f)$ will mean $\lambda(|f|)$. A function f (set B) will be called λ-negligible if $\lambda(f) = 0$ $(\lambda(\chi_B) = 0)$. $\mathfrak{L}^\lambda = \mathfrak{L}^\lambda(E, \mu)$ will denote the vector subspace of $\mathbf{R}^E$ consisting of all μ-measurable $f \in \mathbf{R}^E$ with $\lambda(f) < \infty$. $\mathfrak{L}^\lambda$ will be an MT^*-space if and only if it contains $\mathfrak{K}$. $L^\lambda = L^\lambda(E, \mu)$ will denote the normed space associated with $\mathfrak{L}^\lambda$. Every space L^λ is a Banach space (**6**, Theorem 3.1).

For every length function λ a λ-conjugate length function λ^* is defined by

$$\lambda^*(g) = \sup \int f(P)g(P)d\mu \leqslant \infty, \tag{2.3}$$

the supremum being taken for all $f \in \mathfrak{L}^\lambda$ with $\lambda(f) \leqslant 1$. The space $\mathfrak{L}^{\lambda^*}$ will be called the λ-conjugate of $\mathfrak{L}^\lambda$ and L^λ. A length function is reflexive if $\lambda(f) = \lambda^{**}(f)$ for every non-negative μ-measurable f. Necessary and sufficient conditions for the reflexivity of λ are given in (**7**, (4.1)) (for a general measure space). It can be shown that when λ is reflexive and $\mathfrak{L}^\lambda$ is an MT^*-space for which the MT^*-conjugate contains the λ-conjugate then λ is also reflexive as a semi-norm on $\mathfrak{L}^\lambda$, $|\phi| \in \mathfrak{A}^*$ if $\phi \in \mathfrak{K}^*$ and

$$\lambda(f) = \sup \left(\int |f|d|\phi|;\ ||\phi||_{\mathfrak{A}^*} \leqslant 1,\ \phi \in \mathfrak{A}^* \right)$$

permitting a natural extension of λ to all of $\bar{\mathbf{R}}_+{}^E$.

Let μ denote a positive Radon measure on E. For $1 \leqslant p < \infty$, $\mathfrak{N}_p(f) = (\int f^p d\mu)^{1/p}$ is defined and non-negative for every μ-measurable f that is defined and non-negative almost everywhere. $\mathfrak{N}_p$ then defines a length function. (L 1) follows from (**3**, Théorème 1, p. 119), (L 2) from (**3**, Proposition 10, p. 109) and (L 5) from (**3**, Théorème 3, p. 110) all applied to f^p, and (L 3) and (L 4) from (**3**, Proposition 2, p. 127). The μ-negligible and $\mathfrak{N}_p$-negligible sets coincide. If E is countable at infinity λ will be continuous at infinity. If E is arbitrary the length function $\mathfrak{N}_p$ will not be continuous at infinity if E contains a locally negligible set that is not μ-negligible. By (**3**, Théorème 5, p. 194) the spaces $\mathfrak{L}^p$ and $\mathfrak{L}^\lambda$ with $\lambda = \mathfrak{N}_p$ coincide.

For every μ-measurable $f(P)$, with $0 \leqslant f(P) \leqslant \infty$ almost everywhere, write

$$\overline{\mathfrak{N}}_p(f) = \sup_K \mathfrak{N}_p(f\chi_K),$$

where K runs through the compact subsets of E. It is easily verified that $\overline{\mathfrak{N}}_p$ is a length function that is continuous at infinity and that the $\overline{\mathfrak{N}}_p$-negligible sets are the locally negligible sets of E. We write $\overline{\mathfrak{L}}^p$ and $\bar{L}^p$ for the spaces $\mathfrak{L}^\lambda$, L^λ with $\lambda = \overline{\mathfrak{N}}_p$. From **(4**, Proposition 7, p. 13**)** it follows that $\overline{\mathfrak{L}}^1$ is the space of essentially integrable functions for μ.

Lemma 2.1. *If $f \in \overline{\mathfrak{L}}^p$, the set $E(f) = (P:f(P) \neq 0)$ is the union of a countable sequence of compact sets and a locally negligible set E_0.*

Theorem 2.1. *If every locally null (μ) subset of E is μ-negligible, in particular if E is countable at infinity, $\mathfrak{L}^p$ and $\overline{\mathfrak{L}}^p$, $1 \leqslant p < \infty$ coincide. If E contains a locally null subset that is not μ-negligible, $\overline{\mathfrak{L}}^p$ strictly contains $\mathfrak{L}^p$, $\bar{L}^p$ and L^p, $1 \leqslant p < \infty$, are equivalent.*

Lemma 2.1 is the analogue of **(3**, Lemme 2, p. 194**)**. It is implied by **(4**, Corollaire, p .13**)** for $p = 1$. Theorem 2.1 for $p = 1$ is a consequence of results in **(4**, § 2**)**. In both cases the extension to all p, $1 < p < \infty$ is not difficult.

When $p = \infty$ and $\mathfrak{M}$ denotes the μ-measurable subsets of E, two length functions are obtained from Definition 2.2 by taking: (1) $\lambda = \mathfrak{N}_\infty = ||\,.\,||_\infty$ with $\mathfrak{M}'$ the μ-negligible subsets of E; and (2) $\lambda = \overline{\mathfrak{N}}_\infty = ||\,.\,||_\infty$ with $\mathfrak{M}'$ the locally negligible subsets of E. $\overline{\mathfrak{N}}_\infty$ is continuous at infinity, $\mathfrak{N}_\infty$ is not if E contains a locally negligible set that is not μ-negligible. The $\mathfrak{L}^\lambda$ spaces $\mathfrak{L}_\infty(E, \mathfrak{M}, \mathfrak{M}')$ corresponding to (1) and (2) respectively will be denoted below by $\mathfrak{L}^\infty$ and $\overline{\mathfrak{L}}^\infty$. These spaces are MT^*-spaces. Since $\overline{\mathfrak{N}}_\infty(f) \leqslant \mathfrak{N}_\infty(f)$, $\mathfrak{L}^\infty$ is always contained in $\overline{\mathfrak{L}}^\infty$. In contrast to the case with $p < \infty$, L^∞ and $\bar{L}^\infty$ need not coincide. For example, if E contains a locally negligible set D with $\mu^*(D) = \infty$, $f_a(P) = a\chi_D$ is in $\mathfrak{L}^\infty$ for every finite a and the equivalence classes $\hat{f}_a$ in L^∞ are different for different positive values of a. Each f_a is in $\overline{\mathfrak{L}}^\infty$ but all belong to the equivalence class of $g(P) \equiv 0$. We note that the dual of $\mathfrak{L}^1$ is $\overline{\mathfrak{L}}^\infty$.

3. The λ-conjugate of $A = \mathfrak{L}^\lambda$. In the sequel E will denote an arbitrary locally compact space, μ a positive Radon measure on E, K, K_i compact subsets of E, and λ will be an arbitrary length function for which $\mathfrak{L}^\lambda$ contains $\mathfrak{K}$. $\lambda(B)$ will be an abbreviation for $\lambda(\chi_B)$. A function g is locally integrable if it is μ-measurable and if $g\chi_K$ is integrable (μ) for every K. (Since E is locally compact this is equivalent to the definition in **(8)**).

When $\mathfrak{L}^\lambda$ is an MT^*-space it contains the characteristic function of every relatively compact μ-measurable subset of E. $\mathfrak{L}^\lambda$ contains $\mathfrak{K}$ and since, given K, there exists a continuous function $f \in \mathfrak{K}$ coinciding with χ_K in K, if $B \subset K$ is μ-measurable $\lambda(B) \leqslant \lambda(K) \leqslant \lambda(f) < \infty$ and $\chi_B \in L^\lambda$. It then follows from

(2.3) that every $g \in \mathfrak{L}^{\lambda *}$ is locally integrable. Thus g defines a Radon measure $g \,.\, \mu$ (**8**, § 1) with values

$$\int f \, d(g \,.\, \mu) = \int fg \, d\mu = g(f), g \in (L^\lambda)', \tag{3.1}$$

for all $f \in K$. If for every $g \in \mathfrak{L}^{\lambda *}$, (3.1) extends to all $f \in \mathfrak{L}^\lambda$, the λ-conjugate of $\mathfrak{L}^\lambda$ is contained in the MT^*-conjugate. Since the right side of (3.1) is finite for all $f \in \mathfrak{L}^\lambda$ if $g \in \mathfrak{L}^{\lambda *}$, (**8**, Theorem 1.1) shows that it is sufficient to show that the left side is also always finite. There is no loss of generality in assuming that g is positive (that is non-negative) so that $g \,.\, \mu$ is a positive Radon measure on E.

Lemma 3.1. *If $f \in \bar{\mathbf{R}}_+^E$ is measurable (μ) and $g \geqslant 0$ is locally integrable and if $X = \cup_1^\infty K_i$, then*

$$\int^* fg\chi_X d\mu = \int^* f\chi_X d(g \,.\, \mu) \leqslant \infty, \tag{3.2}$$

and both are finite if $f \in \mathfrak{L}^\lambda$ and $g \in \mathfrak{L}^{\lambda *}$.

Proof. The equality (3.2) is a corollary of (**8**, Lemma 5.1). When $f \in \mathfrak{L}^\lambda$ and $g \in \mathfrak{L}^{\lambda *}$, $\int^* fg\chi_X d\mu \leqslant \int fg \, d\mu < \infty$.

Lemma 3.2. *If $f \in \mathfrak{L}^\lambda$ is a positive lower semi-continuous function and $g \in \mathfrak{L}^{\lambda *}$ then* (3.1) *holds.*

Proof. There is no loss of generality in supposing that g is non-negative. Then, using (3.1),

$$\int f \, d(g \,.\, \mu) = \sup_{\substack{h \in \mathfrak{K} \\ 0 \leqslant h \leqslant f}} \int h \, d(g \,.\, \mu) = \sup_{\substack{h \in \mathfrak{K} \\ 0 \leqslant h \leqslant f}} \int hg \, d\mu \leqslant \int fg \, d\mu < \infty.$$

Note. Lemmas 3.1 and 3.2 are also a consequence of (**4**, (Propositions 2 and 3, p. 9, and Theorem 1, p. 43)). See also (**8**, Note, p. 478).

Lemma 3.3. *If $g \in \overline{\mathbf{R}}_+^E$ is locally integrable every μ-negligible set is $g \,.\, \mu$-negligible.*

Proof. Suppose that B is μ-null. If $g \in \overline{\mathbf{R}}_+^E$, $\int g\chi_B \, d\mu = 0$ by (**3**, Théorème 1, p. 119). Given $\epsilon > 0$, there exists an open set $U \supset B$ with $\mu(U) < \frac{1}{2}\epsilon$ and a l.s.c. function $h \geqslant \chi_B g$ with $\mu(h) < \frac{1}{2}\epsilon$. Then env. sup. (χ_U, h) is l.s.c., $\geqslant \chi_U g$ and

$$\int^* \chi_U g \, d\mu \leqslant \int^* h \, d\mu + \int^* \chi_U d\mu < \epsilon.$$

If g is locally integrable, the measure $g \,.\, \mu$ is defined $\geqslant 0$ and, by (**3**, Corollaire 4, p .158), and Lemma 3.1,

$$(g \,.\, \mu)^*(B) \leqslant (g \,.\, \mu)^*(U) = \sup_{K \subset U} g \,.\, \mu(K) = \sup_{K \subset U} \int \chi_K d(g \,.\, \mu)$$
$$= \sup_{K \subset U} \int \chi_K g \, d\mu \leqslant \int \chi_U g \, d\mu < \epsilon.$$

Since ϵ is arbitrary B is $g \,.\, \mu$-null.

THEOREM 3.1. *Let $g \in \mathfrak{L}^{\lambda *}$. Then* (i) *$\hat{g}$ is in the λ-conjugate $\mathfrak{L}^{\lambda *}$ and is in the MT^*-conjugate if and only if $\int f\, d(g \,.\, \mu) = 0$ for every $f \in \mathfrak{L}^{\lambda}$ for which fg vanishes in E and* (ii) *if the MT^*-conjugate does not contain the λ-conjugate, E contains a locally negligible set B with $\lambda(B) < \infty$, with $\lambda(U) = \infty$ for every open set $U \supset B$ and $\mathfrak{L}^{\lambda *}$ contains g with $g\chi_B = 0$ and with $(g \,.\, \mu)^*(B) = \infty$.*

Proof. (i) Suppose that $g \in \mathfrak{L}^{\lambda *}$ is positive. If $f \in \mathfrak{L}^{\lambda}$, $\int fg\, d\mu < \infty$ and, for fixed f, the set where $fg \neq 0$ is the union of X, the union of a sequence of compact sets, and a μ-negligible set E'. If $E_0 = (P{:}fg(P) = 0), E = E_0 \cup E' \cup X$ and E_0 is measurable (μ). Suppose that f is non-negative. Using Lemma 3.1,

$$\int fg \, d\mu = \int fg\chi_X d\mu = \int f\chi_X d(g \,.\, \mu) \leqslant \int^* f \, d(g \,.\, \mu)$$

and, using Lemma 3.3, we have

$$\int^* f\, d(g \,.\, \mu) \leqslant \int^* f \,\chi_{E'} d(g \,.\, \mu) + \int^* f \,\chi_{E_0} d(g \,.\, \mu) + \int^* f \,\chi_X d(g \,.\, \mu)$$
$$= \int^* f \,\chi_{E_0} d(g \,.\, \mu) + \int fg \, d\mu.$$

Equality then holds if

$$\int^* f \,\chi_{E_0} d(g \,.\, \mu) = 0.$$

Conversely, if $g \in \mathfrak{L}^{\lambda *}$ and if $\hat{g}$ is in the MT^*-conjugate, then, whenever $f \in \mathfrak{L}^{\lambda}$ and fg vanishes in E, $\int f\, d(g \,.\, \mu) = \hat{g}(f) = \int fg\, d\mu = 0$. When one or both of f, g is not positive the extension is trivial.

(ii) Suppose that $\mathfrak{L}^{\lambda *}$ contains g with $\hat{g}$ not in the MT^*-conjugate A^*. Since g^+ and g^-, the positive and negative parts of g are also in $\mathfrak{L}^{\lambda *}$ and $\hat{g}^+$ and $\hat{g}^-$ in A^* would imply that $\hat{g}$ was in A^*, there is no loss of generality in assuming that g is positive. There then exists a positive $f \in \mathfrak{L}^{\lambda}$ with $fg(P) = 0$ in E but with $\int^* f\, d(g \,.\, \mu) > 0$ and, since $\int fg\, d\mu = 0$ (**8**, Lemma 2.1) implies that $\int^* f\, d(g \,.\, \mu) = \infty$,

Let $E_n = (P{:}f(P) > 1/n)$. The μ-measurability of f implies that E_n is μ-measurable. Since $\chi_{E_n} \leqslant n f(P)$, $\lambda(E_n) \leqslant n\lambda(f)$ for each n. Furthermore

$$\int \chi_{E_n} g \, d\mu \leqslant n \int fg \, d\mu = 0, \qquad n = 1, 2, \ldots,$$
$$\sup_{K \subset E} \int \chi_{E_n \cap K} d(g \,.\, \mu) = \sup_{K \subset E} \int \chi_{E_n \cap K} g \, d\mu = 0,$$

so that each E_n is locally negligible.

Let $f_n(P) = \min.\,(n, f(P))$. Then f_n increases to f and by **(3**, Théorème 3, p. 110**)**,

$$\sup_n \int^* f_n \chi_{E_n} d(g\,.\,\mu) = \int^* f\,d(g\,.\,\mu) = \infty.$$

Thus, for all sufficiently large n,

$$\int^* f_n \chi_{E_n} d(g\,.\,\mu) > 0,$$

$$\int^* \chi_{E_n} d(g\,.\,\mu) \geqslant n^{-1} \int^* f_n \chi_{E_n} d(g\,.\,\mu) > 0,$$

and **(8**, Lemma 2.1**)** implies that

$$\int^* \chi_{E_n} d(g\,.\,\mu) = \infty. \tag{3.3}$$

Finally, for all n for which (3.3) holds, $\lambda(U) = \infty$ for every open set U containing E_n since otherwise Lemma 3.2 implies that

$$\int^* \chi_{E_n} d(g\,.\,\mu) \leqslant \int \chi_U d(g\,.\,\mu) = \int \chi_U g\,d\mu \leqslant \lambda(U)\lambda^*(g) < \infty,$$

giving a contradiction.

As a corollary we list some of the conditions that imply that the MT^*-conjugate A^* of $A = \mathfrak{L}^\lambda$ contains the λ-conjugate $\mathfrak{L}^{\lambda*}$:

(1) E is countable at infinity.

(2) For every $g \in \mathfrak{L}^{\lambda*}$, $g\,.\,\mu$ is bounded.

(3) For every $g \in \mathfrak{L}^{\lambda*}$ and every μ-measurable B, $(g\,.\,\mu)^*(B) = \infty$ implies that $\lambda(B) = \infty$.

(4) For every locally integrable g, $(g\,.\,\mu)^*(B) = \infty$, $\lambda(B) < \infty$, imply that $\lambda^*(g) = \infty$.

(5) If B is μ-measurable and if $\lambda(U) = \infty$ for every open set U containing B then $\lambda(B) = \infty$.

(6) $\lambda(E) < \infty$ or $\lambda(K)$ is bounded for all K in E.

If $A = \mathfrak{L}^p$, $1 \leqslant p < \infty$, and B is μ-measurable then $\mathfrak{N}_p(B) = \mu^{*1/p}(B)$ which implies (5). To prove (6) suppose that $\lambda(E) = M < \infty$ or that $\lambda(K) \leqslant M < \infty$ for every $K \subset E$. The existence of B with $(g\,.\,\mu)^*(B) = \infty$ implies that $(g\,.\,\mu)^*(E) = \infty$ and **(3**, Corollaire 4, p. 158**)** implies that E contains compact subsets with arbitrarily large $(g\,.\,\mu)$-measure. Then

$$\lambda^*(g) \geqslant \sup_K \int \chi_K g\,d\mu/\lambda(K) \geqslant \sup_K g\,.\,\mu(K)/M = \infty,$$

so that (4) applies. The spaces $\mathfrak{L}^\infty$, $\overline{\mathfrak{L}}^\infty$ satisfy (6).

THEOREM 3.2. *There exist MT^*-spaces $\mathfrak{L}^\lambda$, with λ reflexive and continuous at infinity, for which the λ-conjugate strictly contains the MT^*-conjugate. In particular the spaces $\overline{\mathfrak{L}}^p$, $1 \leqslant p < \infty$, are of this type for suitable μ, E.*

Proof. Let E, D, $T_n(y)$ and μ be defined as in the example of a locally compact space E that is not countable at infinity given in (**3**, Exercice 4, p. 116) and let $g = \chi_{E-D}$. Then $\overline{\mathfrak{N}}_\infty(g) = 1$ and $\hat{g} \in \bar{L}^\infty = \mathfrak{L}^{\lambda *}$, $\lambda = \overline{\mathfrak{N}}_1$. As in (**8**, Exercice 4) $\chi_D \in \bar{\mathfrak{L}}^1$ but is not integrable $(g \,.\, \mu)$ so that $\hat{g}$ does not belong to the MT^*-conjugate of $\bar{\mathfrak{L}}^1$

To give an example for $1 < p < \infty$ let E, D be as before but for δ fixed, $0 < \delta < 1$, P_{ni} the point $(1 - n, i/n^2)$, define $\beta(P_{ni}) = n^{-2-\delta}$ and define $\beta(P) = 0$ for the points $(0, y)$ in E. Let μ' denote the measure determined by the masses $\beta(P)$. Define $g(P) = n^{\delta-1}$ for $P = P_{ni}$ $(i = 0, 1, \ldots, n^2;$ $n = 1, 2, \ldots;)$, $g(P) = 0$ elsewhere in E. Actual computation shows that every compact subset of E has finite μ'-measure and that, for $\lambda = \overline{\mathfrak{N}}^p$, $\lambda^*(g) = \overline{\mathfrak{N}}^q(g) < \infty$ so that $\hat{g} \in \mathfrak{L}^{\lambda *}$. Suppose that B is a subset of $(0, y)$ dense (in the usual topology on $\mathbf{R}$) on some interval (a, b), $-1 < a < b < 1$ and let $U_{B,n} = \cup\, T_i(y)$ for all $i \geqslant n$, $y \in B$. Computation shows that $(g \,.\, \mu')^*(U_{B,n}) = \infty$ for every n. As in the Bourbaki example every open set containing D contains some set $U_{B,n}$ and therefore $(g \,.\, \mu')^*(D) = \infty$ although D is locally negligible $(g \,.\, \mu')$. Since the dual of $\bar{\mathfrak{L}}^p$ coincides with the λ-conjugate, $1 < p < \infty$, the λ-conjugate strictly contains the MT^*-conjugate. The length functions $\overline{\mathfrak{N}}_p$ are continuous at infinity and reflexive.

Remark. The right side of (2.1) is $\mathfrak{N}_1(f, \phi)$. Replacing $\mathfrak{N}_1$ by $\overline{\mathfrak{N}}_1$ in the definition of the MT^*-conjugate gives a conjugate which always contains the λ-conjugate when $A = \mathfrak{L}^\lambda$.

4. The MT^*-conjugate of $A = \mathfrak{L}^\lambda$. In this section Φ denotes an arbitrary element of the MT^*-conjugate of $A = \mathfrak{L}^\lambda$, ϕ the Radon measure corresponding to Φ determined by the restriction of Φ to K.

LEMMA 4.1. *If* $\Phi \in A^*$ *and* $A = \mathfrak{L}^\lambda(\mu)$ *there exists a locally* μ*-integrable function* g *for which the measure* $g \,.\, \mu$ *coincides with* ϕ.

Proof. There is no loss of generality in assuming $\Phi \geqslant 0$. By hypothesis every $f \in A$ is integrable (ϕ) with

$$\Phi(f) = \int f \, d\phi. \tag{4.1}$$

For each compact set K every μ-measurable subset e is in A and $\Phi(\chi_e) = \int \chi_e d\phi = \phi(e)$. If $\mu(e) = 0$,

$$|\phi(e)| = |\Phi(e)| \leqslant |\Phi| \mathfrak{N}^A(\chi_e) = |\Phi| \lambda(e).$$

Since $\lambda(e) = 0$ whenever $\mu(e) = 0$, every set that is locally μ-negligible is locally ϕ-negligible. The Lebesgue-Nikodym theorem (**4**, Théorème 2, p. 47) then implies that ϕ is a measure of base μ, that is, that there exists a locally integrable point function g with $\phi = g.\ \mu$ (**4**, p. 42).

THEOREM 4.1. *Let* $\Phi \in A^*$ *and let* g *in* $\mathbf{R}^E$ *be locally integrable with* $g \,.\, \mu \equiv \phi$. *Then* (i) $g \in \mathfrak{L}^{\lambda *}$ *if and only if* $\mu(g\chi_B) = 0$ *for every* ϕ*-negligible set* B *that is* *the union of a sequence* $\{B_i\}$ *with each function* χ_{B_i} *in* $\mathfrak{L}^\lambda$, *and* (ii). *If* $g \notin \mathfrak{L}^{\lambda *}$, *E contains a set D, negligible* $(g \,.\, \mu)$ *and locally negligible* (μ) *with* $\lambda(D) < \infty$ *and* $\mu^*(D) = \infty$.

Proof. (i) There is no loss of generality in considering only positive f and Φ in the proof of (i) and (ii). If $f \in \mathfrak{L}^\lambda$, $\Phi \in A^*$, then $f \in \mathfrak{L}^1(\phi)$ and $E = E_0 \cup E' \cup X$, where $E_0 = (P{:}f(P) = 0)$, E' is ϕ-negligible and X is the union of a countable sequence of compact sets. Since E, E_0 and X are measurable (μ) so is E'. Since $\int f\chi_X d(g\,.\,\mu) \leqslant \int f\, d(g\,.\,\mu) = \int f\, d\phi < \infty$, (3.2) holds finitely. Since

$$fg\chi_{E_0}$$

vanishes,

$$\int fg\chi_{E_0} d\mu = 0.$$

Thus

$$\begin{aligned}(4.4)\quad \int f\, d\phi = \int f\, d(g\,.\,\mu) &= \int fg\chi_X d\mu \leqslant \int^* fg\, d\mu \\ &\leqslant \int^* fg\chi_{E_0} d\mu + \int^* fg\chi_{E'} d\mu + \int fg\chi_X d\mu = \int^* fg\chi_{E'} d\mu + \int f\, d\phi.\end{aligned}$$

If $E_i' = (P \in E'{:}f(P) > 1/i)$, $E' = \cup_1^\infty E_i'$, each E_i' is μ-measurable and $\lambda(E_i') \leqslant i\lambda(f) < \infty$ so that each $\chi_{E_i'} \in \mathfrak{L}^\lambda$. If the hypothesis of (i) is satisfied, $\int \chi_{E'} g\, d\mu = 0$ which implies that $\int \chi_{E'} fg\, d\mu = 0$. Then $\int fg\, d\mu = \int f\, d\phi = \Phi(f)$ for every $f \in \mathfrak{L}^\lambda$ and $g \in \mathfrak{L}^{\lambda *}$. Conversely, if $g \in \mathfrak{L}^{\lambda *}$, $\int fg\, d\mu < \infty$ for every $f \in \mathfrak{L}^\lambda$ and, by **(8**, Theorem 1.1**)**,

$$(4.5)\qquad \int fg\, d\mu = \int f\, d(g\,.\,\mu) = \Phi(f)$$

for all $f \in \mathfrak{L}^\lambda$. If B_i is ϕ-negligible with $\chi_{B_i} \in \mathfrak{L}^\lambda$ (4.4) and (4.5) imply that

$$\int g\chi_{B_i}\, d\mu = 0$$

whence $\int g\chi_B\, d\mu = 0$ if $B = \cup_1^\infty B_i$.

(ii) If $g \notin \mathfrak{L}^{\lambda *}$ (i) implies that there exists a ϕ-negligible set $B = \cup_1^\infty B_i$, where $\lambda(B_i) < \infty$, $i = 1, 2, \ldots$, and such that $\mu^*(g\chi_B) > 0$. Then **(8**, Lemma 2.1**)** implies that $\mu^*(g\chi_B) = \infty$. Writing $B(n) = \cup_1^n B_i$, since $B(n) \uparrow B$, $\mu^*(g\chi_{B)n)})$ is positive and therefore infinite for all sufficiently large n, say $n > n_0$. We show that for a fixed $n > n_0$, $D = (P \in B(n) : g(P) > 0)$ satisfies all the conditions (ii). We note that $\lambda(D) \leqslant \lambda(B(n)) < \infty$ and that $\mu^*(g\chi_D) = \mu^*(g\chi_{B(n)}) = \infty$. Consider $g_m(P) = \min\,(m, g(P))$. It is locally integrable and defines a Radon measure $g_m\,.\,\mu$ with $0 < g_m\,.\,\mu \leqslant g\,.\,\mu$ and $g_m\,.\,\mu(D) \leqslant g\,.\,\mu(D) \leqslant g\,.\,\mu(B) = 0$. Now

$$\infty = \int^* g\chi_D d\mu = \sup_m \int^* g_m\chi_D d\mu,$$

and (**8**, Lemma 2.1) implies that $\mu^*(g_m\chi_D) = 0$ or ∞ for each m and therefore is infinite for all sufficiently large m so that

$$\int^* \chi_D d\mu \geqslant m^{-1}\int^* g_m \chi_D d\mu = \infty.$$

Finally, if $D_i = (P \in D : g(P) > 1/i)$, $D = \cup_1^\infty D_i$ and, for an arbitrary compact set K,

$$\int \chi_{D\cap K} d\mu = \sup_i \int \chi_{D_i\cap K} d\mu \leqslant \sup_i i \int g\chi_{D_i\cap K} d\mu$$
$$\leqslant \sup_i i \int \chi_D \, d(g\,.\,\mu) = 0.$$

Thus D is locally μ-negligible.

A variety of conditions, sufficient to ensure that the λ-conjugate contains the MT^*-conjugate, follow from Theorem 4.1. We mention only: (i) E is countable at infinity, (ii) μ is bounded and (iii) $\mu^*(B) = \infty$ implies that $\lambda(B) = \infty$. Condition (iii) shows that if $A = \mathfrak{L}^p$, $1 \leqslant p < \infty$, the λ-conjugate always contains the MT^*-conjugate so that these conjugates then coincide. Actually, each of (i)-(iii) implies that every locally integrable $g \in \mathbf{R}^E$ is in $\mathfrak{L}^{\lambda *}$ whereas $\mathfrak{L}^{\lambda *}$ will contain A^* if to each $\Phi \in A^*$ corresponds one $g \in \mathfrak{L}^{\lambda *}$ with $g\,.\,\mu \equiv \phi$. If g is locally integrable with $g\,.\,\mu \equiv \phi$, every g' that is locally equivalent to g is also locally integrable and $g'\,.\,\mu \equiv \phi$. Consider (**8**, Example 5.2) where $g = \chi_H$, $f = \chi_E$ and the length functions $\lambda = \overline{\mathfrak{N}}_p$, $1 \leqslant p < \infty$. Then $g \notin \mathfrak{L}^{\lambda *}$ but g is locally equivalent to the zero element of $\mathfrak{L}^{\lambda *}$. More generally, let E, D, and μ be defined as in Theorem 3.2 and let λ denote an arbitrary length function. Then D is locally negligible with $\mu^*(D) = \infty$ but every locally negligible subset of $E - D$ is μ-negligible. Replacing g by $g\chi_D$ gives a g' locally equivalent to g with $\int^* g'\chi_B \, d\mu = \int^* g\chi_{B\cap D} d\mu = 0$ for every locally negligible set B with $\lambda(B) < \infty$. Since

$$(g'\,.\,\mu)^*(B) \leqslant \int^* g'\chi_B \, d\mu$$

by (4.4) this contradicts Theorem 4.1 (ii) if g' is not in $\mathfrak{L}^{\lambda *}$.

Edwards (**5**, p. 143) defines the μ-measure of a μ-measurable set B to be $\mu(B) = \sup_K \int \chi_{B\cap K} d\mu$, where K runs through the compact subsets of E.

THEOREM 4.2. *If E contains E^* with $\mu^*(A) < \infty$ for every locally negligible set contained in $E - E^*$ and if E^* is the union of a countable collection of sets of finite μ-measure, then for every MT^*-space $\mathfrak{L}^\lambda$ on E, the λ-conjugate contains the MT^*-conjugate.*

Proof. First suppose that $\mu(E^*) < \infty$. By the argument of (**5**, Theorem 7 (4)), $E^* = Q_1 \cup Q_2$ where Q_1 is the union of a countable collection of compact sets and Q_2 is locally null. If $A \subset E^*$ with $\mu(A) = 0$, $\mu^*(A) = \infty$, $\mu(Q_1 \cap A) = 0$. Set $g' = g\chi_X$, where $X = CE^* \cup Q_1$. If $K \subset CE$ or $K \subset Q_1$, $g(P) = g'(P)$ for $P \in K$. Suppose that $\mu(K \cap CE^*) > 0$ and $\mu(K \cap E^*) > 0$.

Then $K = N \cup_1^{\infty} K_n$ where N is μ-negligible and each K_n is compact and contained in one of E^*, CE^* (**3**, pp. 181-2). Since $\mu(K_n \cap Q_2) = 0, n = 1, 2, \ldots$, $g = g'$ almost everywhere in each K and g' is locally equivalent to g. For each such $A \subset E^*$,

$$\mu^*(g'\chi_A) = \mu^*(g\chi_{Q_1 \cup A}) = 0$$

contradicting Theorem 4.2 (ii) if $g' \notin \mathfrak{L}^{\lambda *}$. If $E^* = \cup_1^{\infty} E_n$ with $\mu(E_n) < \infty$, $n = 1, 2, \ldots$, each $E_n = Q_{1n} \cup Q_{2n}$ as above and the preceding argument holds when applied to $\cup_n Q_{1n}$, $\cup_n Q_{2n}$ in place of Q_1 and Q_2 respectively.

References

1. S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932).

2. G. Birkhoff, *Lattice Theory*, Am. Math. Soc. Colloquium Publications, vol. XXV (New York, 1948).

3. N. Bourbaki, *Eléments de Mathématique*, Livre VI, Integration, chaps. I-IV (Paris, 1952).

4. ———, *Eléments de Mathématique*, Livre VI, Integration, chap. V (Paris, 1956).

5. R. E. Edwards, *A theory of Radon measures on locally compact spaces*, Acta. Math., *89* (1953), 133–164.

6. H. W. Ellis and I. Halperin, *Function spaces determined by a levelling length function*, Can. J. Math., *5* (1953), 576–592.

7. I. Halperin and W. A. J. Luxemburg, *Reflexivity of the length function*, Proc. Amer. Math. Soc., *8* (1957), 496–9.

8. M. Morse and W. Transue, *Products of a C-measure and a locally integrable mapping*, Can. J. Math., *9* (1957), 475–486.

9. ———, *Semi-normed vector spaces with duals of integral type*, J. d'Analyse Math., *4*(1954-5), 149–186.

10. ———, *Vector subspaces A of C^E with duals of integral type*, J. Math. pures et appl., to appear.

11. K. Yosida and E. Hewitt, *Finitely additive measures*, Trans. Amer. Math. Soc., *72* (1952), 46–66.

Queen's University
Kingston