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ON SQUARE PSEUDO-LUCAS NUMBERS

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J. H. E. Cohn (1) has shown that

$$F_1 = F_2 = 1 \quad \text{and} \quad F_{12} = 144$$

are the only square Fibonacci numbers in the set of Fibonacci numbers defined by

$$F_1 = F_2 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 3.$$

If n is a positive integer, we shall call the numbers defined by

(1) $$u_1 = 1, \qquad u_2 = 6, \qquad u_{n+2} = u_{n+1} + u_n$$

pseudo-Lucas numbers.

In this paper we describe a method to show that the only square pseudo-Lucas numbers are,

$$u_1 = 1 \quad \text{and} \quad u_{10} = 225.$$

If we remove the restriction $n > 0$, we obtain exactly one more square,

$$u_{-2} = 9.$$

It can be easily shown that the general solution of the difference equation (1) is given by

(2) $$u_n = \tfrac{1}{5}(11L_n - 3L_{n-1}),$$

where n is an integer.

Then we easily obtain the following relations:

(3) $$L_r = L_{r-1} + L_{r-2}, \qquad L_1 = 1, \qquad L_2 = 3,$$

(4) $$F_r = F_{r-1} + F_{r-2}, \qquad F_1 = 1, \qquad F_2 = 1,$$

(5) $$L_r^2 - 5F_r^2 = (-1)^r 4,$$

(6) $$L_{2r} = L_r^2 + (-1)^{r+1} 2,$$

(7) $$2L_{m+n} = 5F_m F_n + L_m L_n,$$

(8) $$2F_{m+n} = L_n F_m + L_m F_n,$$

(9) $$F_{2r} = L_r F_r,$$

(10) $$u_n = \tfrac{1}{2}(5L_n - 3F_n).$$

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The following conguences hold:

(11) $$u_{n+2r} \equiv (-1)^{r+1} u_n (\text{mod } L_r 2^{-s}),$$

(12) $$u_{n+2r} \equiv (-1)^{r} u_n (\text{mod } F_r 2^{-s}),$$

where $s = 0$ or 1. Let

$$\phi_t = L_2 t,$$

where t is a positive integer. Then we get

(13) $$\phi_{t+1} = \phi_t^2 - 2$$

We also need the following results concerning ϕ_t:

(14) ϕ_t is an odd integer,

(15) $\phi_t \equiv 3 (\text{mod } 4)$,

(16) $\phi_t \equiv 2 (\text{mod } 5)$, $t \geqslant 2$.

(17) $\phi_t \equiv 2 (\text{mod } 3)$, $t \geqslant 3$.

We have the following tables of values:

n	-12	-5	-2	-1	0	1	2	3	4	6	7	10	11	20
u_n	1021	-35	9	-4	5	1	6	7	13	33	53	225	364	27670

t	4	5	8
F_t	3	5	3.7

t	5	10	20
L_t	11	3.41	7.2161

Let

(18) $$x^2 = u_n$$

The proof is now accomplished in eighteen stages:

(a) (18) is impossible if $n \equiv 0$ (mod 8). For, using (12) we find that

$$u_n \equiv u_0 (\text{mod } F_4).$$
$$\equiv 5 (\text{mod } 3).$$

Since

$$\left(\frac{5}{3}\right) = -1,$$

(18) is impossible.

(b) (18) is impossible if $n \equiv 2 (\text{mod } 16)$. For, using (12) we find that

$$u_n \equiv u_2 (\text{mod } F_8)$$
$$\equiv 6 (\text{mod } 7), \qquad \text{since } 7/F_8$$

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Since

$$\left(\frac{6}{7}\right) = -1,$$

(18) is impossible.

(c) (18) is impossible if $n \equiv 3 \pmod{10}$. For, using (12) in this case

$$\begin{aligned} u_n &\equiv \pm u_3 \pmod{F_5} \\ &\equiv \pm 7 \pmod{5} \end{aligned}$$

Since

$$\left(\frac{-7}{5}\right) = \left(\frac{7}{5}\right) = -1,$$

(18) is impossible.

(d) (18) is impossible if $n \equiv 4 \pmod{16}$. For, using (12) we find that

$$\begin{aligned} u_n &\equiv u_4 \pmod{F_8} \\ &\equiv 13 \pmod{7}, \qquad \text{since } 7/F_8 \end{aligned}$$

Since

$$\left(\frac{13}{7}\right) = -1,$$

(18) is impossible.

(e) (18) is impossible if $n \equiv 6 \pmod{16}$. For, using (12) in this case

$$\begin{aligned} u_n &\equiv u_6 \pmod{F_8} \\ &\equiv 33 \pmod{7}, \qquad \text{since } 7/F_8 \end{aligned}$$

Since

$$\left(\frac{33}{7}\right) = -1,$$

(18) is impossible.

(f) (18) is impossible if $n \equiv 7 \pmod{8}$. For, using (12) in this case

$$\begin{aligned} u_n &\equiv u_7 \pmod{F_4} \\ &\equiv 53 \pmod{3} \end{aligned}$$

Since

$$\left(\frac{53}{3}\right) = -1,$$

(18) is impossible.

(g) (18) is impossible if $n \equiv 11 \pmod{40}$. For, using (11) we find that

$$\begin{aligned} u_n &\equiv \pm u_{11} \pmod{L_{20}} \\ &\equiv \pm 364 \pmod{2161}, \qquad \text{since } 2161/L_{20} \end{aligned}$$

Since

$$\left(\frac{-364}{2161}\right) = \left(\frac{364}{2161}\right) = -1,$$

(18) is impossible.

(h) (18) is impossible if $n \equiv 2 \pmod{10}$. For, using (11) in this case

$$\begin{aligned} u_n &\equiv u_2 \pmod{L_5} \\ &\equiv 6 \pmod{11} \end{aligned}$$

Since

$$\left(\frac{6}{11}\right) = -1,$$

(18) is impossible.

(i) (18) is impossible if $n \equiv -1 \pmod{10}$. For, using (11) we find that

$$\begin{aligned} u_n &\equiv u_{-1} \pmod{L_5} \\ &\equiv -4 \pmod{11} \end{aligned}$$

Since

$$\left(\frac{-4}{11}\right) = -1,$$

(18) is impossible.

(j) (18) is impossible if $n \equiv -5 \pmod{20}$. For, using (11) in this case

$$\begin{aligned} u_n &\equiv \pm u_{-5} \pmod{L_{10}} \\ &\equiv \mp 35 \pmod{41}, \qquad \text{since } 41/L_{10} \end{aligned}$$

Since

$$\left(\frac{-35}{41}\right) = \left(\frac{35}{41}\right) = -1,$$

(18) is impossible.

(k) (18) is impossible if $n \equiv 7 \pmod{10}$. For, using (12) in this case

$$\begin{aligned} u_n &\equiv \pm u_7 \pmod{F_5} \\ &\equiv \pm 53 \pmod{5} \end{aligned}$$

Since

$$\left(\frac{-53}{5}\right)=\left(\frac{53}{5}\right)=-1,$$

(18) is impossible.

(l) (18) is impossible if $n\equiv-12 \pmod{40}$. For, using (11) we find that

$$\begin{aligned} u_n &\equiv \pm u_{-12} \pmod{L_{20}} \\ &\equiv \pm 1021 \pmod{2161}, \qquad \text{since } 2161/L_{20} \end{aligned}$$

Since

$$\left(\frac{-1021}{2161}\right)=\left(\frac{1021}{2161}\right)=-1,$$

(18) is impossible.

(m) (18) is impossible if $n\equiv 4 \pmod{10}$. For, using (12) in this case

$$\begin{aligned} u_n &\equiv \pm u_4 \pmod{F_5} \\ &\equiv \pm 13 \pmod 5 \end{aligned}$$

Since

$$\left(\frac{-13}{5}\right)=\left(\frac{13}{5}\right)=-1,$$

(18) is impossible.

(n) (18) is impossible if $n\equiv 6 \pmod{10}$. For, using (12) we get

$$\begin{aligned} u_n &\equiv \pm u_6 \pmod{F_5} \\ &\equiv \pm 33 \pmod 5 \end{aligned}$$

Since

$$\left(\frac{-33}{5}\right)=\left(\frac{33}{5}\right)=-1,$$

(18) is impossible.

(o) (18) is impossible if $n\equiv 20 \pmod{40}$. For, using (11) we find that

$$\begin{aligned} u_n &\equiv \pm u_{20} \pmod{L_{20}} \\ &\equiv \pm 27670 \pmod{2161}, \qquad \text{since } 2161/L_{20} \end{aligned}$$

Since

$$\left(\frac{-27670}{2161}\right)=\left(\frac{27670}{2161}\right)=-1,$$

(18) is impossible.

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(p) (18) is impossible if $n \equiv 1 \pmod 4$, $n \neq 1$, that is if $n = 1 + 2^t r$, where r is odd and t is a positive integer $\geqslant 2$. For, using (11) in this case

$$u_n \equiv -u_1 (\text{mod } L_2^{t-1})$$
$$\equiv -1 (\text{mod } \phi_{t-1})$$

Now using (15), we have

$$\phi_{t-1} = 4k + 3,$$

where k is a non-negative integer.
Since

$$\left(\frac{-1}{\phi_{t-1}}\right) = \left(\frac{-1}{4k+3}\right) = -1,$$

(18) is impossible.

(q) (18) is impossible if $n \equiv 10 \pmod{16}$, $n \neq 10$, that is if $n = 10 + 2^t r$, where r is odd and t is a positive integer $\geqslant 4$. For, using (11) we find that

$$u_n \equiv -u_{10} (\text{mod } L_2^{t-1})$$
$$\equiv -225 (\text{mod } \phi_{t-1})$$

Now using (16) and (17) we get

$$(\phi_{t-1}, 5) = 1 \quad \text{and} \quad (\phi_{t-1}, 3) = 1$$

respectively.

By virtue of (15), we get $\phi_{t-1} = 4k + 3$, where k is a positive integer $\geqslant 11$.
Next, since

$$\left(\frac{-225}{\phi_{t-1}}\right) = \left(\frac{-225}{4k+3}\right) = -1,$$

(18) is impossible.

(r) (18) is impossible if $n \equiv -2 \pmod{16}$, $n \neq -2$, that is if $n = -2 + 2^t r$, where r is odd and t is a positive integer $\geqslant 4$. For, using (11) we find that

$$u_n \equiv -u_{-2} (\text{mod } L_2^{t-1})$$
$$\equiv -9 (\text{mod } \phi_{t-1})$$

Now using (17) we get

$$(\phi_{t-1}, 3) = 1$$

By virtue of (15) we get

$$\phi_{t-1} = 4k + 3,$$

where k is a positive integer $\geqslant 11$.

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Next, since

$$\left(\frac{-9}{\phi_{t-1}}\right)=\left(\frac{-9}{4k+3}\right)=-1,$$

(18) is impossible.

We have now three further cases $n=-2, 1, 10$ to consider

When $n=-2$, $u_n=9$ is a perfect square.

When $n=1$, $u_n=1$ is a perfect square.

When $n=10$, $u_n=225$ is a perfect square.

References

1. J. H. E. Cohn "*On Square Fibonacci numbers*". J. London Math. Soc. **39** (1964), 537–540.

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