Canad. J. Math. Vol. **51** (6), 1999 pp. 1135–1148

Endoscopic L-Functions and a Combinatorial Identity

James Arthur

Abstract. The trace formula contains terms on the spectral side that are constructed from unramified automorphic L-functions. We shall establish an identify that relates these terms with corresponding terms attached to endoscopic groups of G. In the process, we shall show that the L-functions of G that come from automorphic representations of endoscopic groups have meromorphic continuation.

1 Introduction

In this paper we shall prove a combinatorial identity for certain functions attached to reductive algebraic groups over number fields. The functions are built out of logarithmic derivatives of L-functions, and occur as terms on the spectral side of the trace formula. The identity is suggested by the problem of stabilizing the trace formula.

We shall say nothing about the general problem, since we will be dealing with only a small part of it here. For a given group G (which for the introduction we assume is semisimple and simply connected), together with a Levi subgroup M, we shall define a function $r_M^G(c_\lambda)$ of a complex variable λ. The symbol c represents a family $\{c_v : v \notin V\}$ of semisimple conjugacy classes from the local L-groups ${}^L M_v$. The function $r_M^G(c_\lambda)$ is constructed in a familiar way from the quotients

$$r_{Q|P}(c_\lambda) = L(0, c_\lambda, \rho_{Q|P}) L(1, c_\lambda, \rho_{Q|P})^{-1}, \quad Q, P \in \mathcal{P}(M),$$

of unramified L-functions

$$L(s, c, \rho_{Q|P}) = \prod_a L(s, c, \rho_a)$$

that are part of the theory of Eisenstein series. We assume initially that c is automorphic, in the sense that it comes from an automorphic representation of M. As is well known, the quotients $r_{Q|P}(c_\lambda)$ will then have meromorphic continuation for λ in a complex vector space $\mathfrak{a}^*_{M,\mathbb{C}}$. This property is required for the definition of $r_M^G(c_\lambda)$, and implies that $r_M^G(c_\lambda)$ is also a meromorphic function of λ.

In the present context, the stabilization problem is to relate $r_M^G(c_\lambda)$ with "stable" functions $s_{M'}^{G'}(c'_\lambda)$ attached to endoscopic groups of G. Here M' is a fixed endoscopic group for M, $c' = \{c'_v : v \notin V\}$ is an automorphic family of conjugacy classes attached to M', and c is now assumed to be the image of c' under a fixed embedding ${}^L M' \hookrightarrow {}^L M$. The functoriality principle asserts that c is an automorphic family for M. This would immediately imply the meromorphic continuation of the functions $r_{Q|P}(c_\lambda)$, on which the definition of $r_M^G(c_\lambda)$

Received by the editors June 29, 1998.
Supported in part by NSERC Operating Grant A3483.
AMS subject classification: 22E45, 22E46.

depends. However, even such a relatively accessible case of functoriality is far from known. We shall instead prove directly that the basic endoscopic L-functions $L(s, c, \rho_a)$ have meromorphic continuation in $s \in \mathbb{C}$ (Proposition 1). We will then be able to define $r_M^G(c_\lambda)$ without knowing that c is automorphic. Once we have defined $r_M^G(c_\lambda)$, we shall construct functions $s_{M'}^{G'}(c'_\lambda)$ by an inductive procedure that is typical of the general stabilization of terms in the trace formula.

The meromorphic continuation of endoscopic L-functions is related to results of Shahidi in [13] and [14], which were a part of his proof of some important properties of local L-functions and ε-factors [14, Theorems 3.5, 7.9 and 8.1]. We shall apply similar inductive arguments, but we will avoid case by case considerations by using the global form of a set $\mathcal{E}_{M'}(G)$ introduced in [6] and [7]. The elements in $\mathcal{E}_{M'}(G)$ are endoscopic groups G' for G which contain M' as a Levi subgroup. They determine a simple decomposition of the endoscopic L-functions (Lemma 4), of which the required meromorphic continuation will be a straightforward consequence. The set $\mathcal{E}_{M'}(G)$, and Lemma 4 in particular, will also be the basis of our stabilization of $r_M^G(c_\lambda)$. The construction, which includes our combinatorial identity, will be the content of Theorem 5.

I have profited from many enlightening conversations with Shahidi on L-functions and intertwining operators. In particular, I would like to thank him for pointing out an error in my original manuscript.

2 Endoscopic L-functions

Let G be a connected, reductive algebraic group over a number field F. For reasons of induction it is also convenient to introduce a central torus Z in G over F, together with a character ζ on $Z(F) \setminus Z(\mathbb{A})$. We assume that Z is induced, in the sense that it is a finite product of tori of the form $\mathrm{Res}_{E/F}\big(\mathrm{GL}(1)\big)$. We also fix a finite set of valuations V of F such that the local components G_v, Z_v and ζ_v are all unramified at any v not in V.

We consider families

$$c = \{c_v : v \notin V\},$$

where each c_v is a semisimple conjugacy class in the local L-group ${}^LG_v$ of G_v, whose image in the local Weil group W_{F_v} is a Frobenius element. Let $\mathcal{C}(G^V, \zeta^V)$ be the set of families c that satisfy the following two conditions. First of all, each c_v must be compatible with ζ_v. In other words, the image of c_v under the projection ${}^LG_v \to {}^LZ_v$ gives the unramified Langlands parameter of ζ_v. Secondly, we require that for any $\hat{G}$-invariant polynomial A on ${}^LG$, c satisfy an estimate

$$|A(c_v)| \leq q_v^{r_A}, \quad v \notin V,$$

for some $r_A > 0$. As usual, q_v stands for the order of the residue class field of F_v. Suppose that c belongs to $\mathcal{C}(G^V, \zeta^V)$, and that ρ is a finite dimensional representation of ${}^LG$. Then the Euler product

$$L(s, c, \rho) = \prod_{v \notin V} \det\big(1 - \rho(c_v) q_v^{-s}\big)^{-1}, \quad s \in \mathbb{C}, \tag{1}$$

converges to an analytic function of s in some right half plane.

A fundamental conjecture of Langlands [10] asserts that if c comes from an automorphic representation, $L(s, c, \rho)$ can be continued to a meromorphic function of s in the entire complex plane. To be more precise, we recall that the local components of c determine unramified irreducible representations $\pi_v(c) = \pi(c_v)$ of $G(F_v)$, and hence an unramified representation $\pi^V(c) = \bigotimes_{v \notin V} \pi_v(c)$ of the subgroup $G(\mathbb{A}^V)$ of points in $G(\mathbb{A})$ that are 1 at each v in V. Let $\mathcal{C}^V_{\mathrm{aut}}(G, \zeta)$ be the set of c for which there exists an irreducible representation π_V of $G(F_V) = \prod_{v \in V} G(F_v)$ such that the representation $\pi_V \otimes \pi^V(c)$ of $G(\mathbb{A})$ is automorphic [11]. The Langlands conjecture asserts that if c belongs to $\mathcal{C}^V_{\mathrm{aut}}(G, \zeta)$, $L(s, c, \rho)$ has meromorphic continuation. The conjecture actually asserts more, namely that one can add Euler factors at the places $v \in V$ so that the completed L-function satisfies a suitable functional equation. Our concern in this paper, however, is with the unramified L-functions.

Let M be a Levi subgroup of G, and let $\hat{M} \subset \hat{G}$ be a dual Levi subgroup of $\hat{G}$ [7, Section 1]. Then there is a bijection $P \to \hat{P}$, from the set $\mathcal{P}(M)$ of parabolic subgroups of G over F with Levi component M, to the set $\mathcal{P}(\hat{M})$ of $\Gamma = \mathrm{Gal}(\overline{F}/F)$-stable parabolic subgroups of $\hat{G}$ with Levi component $\hat{M}$. If P and Q lie in $\mathcal{P}(M)$, let $\rho_{Q|P}$ be the adjoint representation of ${}^LM$ on the Lie algebra of the intersection of the unipotent radicals of $\widehat{\overline{P}}$ and $\hat{Q}$. Suppose that $c \in \mathcal{C}^V_{\mathrm{aut}}(M, \zeta)$. It then follows from results of Shahidi that $L(s, c, \rho_{Q|P})$ has meromorphic continuation in s. We shall show that this property holds for families $c \in \mathcal{C}(M^V, \zeta^V)$ that come from automorphic representations of endoscopic groups for M.

Let M' stand for an elliptic endoscopic datum $(M', \mathcal{M}', s'_M, \xi'_M)$ for M over F [12, (1.2)], that is unramified outside of V. Recall that $\mathcal{M}'$ is a split extension of W_F by $\hat{M}'$ that need not be L-isomorphic to ${}^LM'$. To take care of this problem, one has to fix a central extension $\tilde{M}'$ of M' by an induced torus $\tilde{C}'$ over F, and an L-embedding $\tilde{\xi}'\colon \mathcal{M}' \to {}^L\tilde{M}'$. (See [12, (4.4)] and [5, Section 2].) If $W_F \to \mathcal{M}'$ is any section, the composition

$$W_F \longrightarrow \mathcal{M}' \xrightarrow{\tilde{\xi}'} {}^L\tilde{M}' \longrightarrow {}^L\tilde{C}'$$

is then a global Langlands parameter that is dual to a character $\tilde{\eta}'$ on $\tilde{C}'(F) \backslash \tilde{C}'(\mathbb{A})$. We can assume that $\tilde{M}'$ and $\tilde{\eta}'$ are also unramified outside of V. The preimage $\tilde{Z}'$ of Z in $\tilde{M}'$ is also a central induced torus. It is easy to see that the construction above (applied to $\overline{M} = M/Z$ in place of M) provides a canonical extension of $\tilde{\eta}'$ to a character on $\tilde{Z}'(F) \backslash \tilde{Z}'(\mathbb{A})$. The product of $\tilde{\eta}'$ with the pullback of ζ to $\tilde{Z}'(\mathbb{A})$ is then another character $\tilde{\zeta}'$ in $\tilde{Z}'(F) \backslash \tilde{Z}'(\mathbb{A})$, that is unramified outside of V.

Suppose that c' belongs to $\mathcal{C}\big((\tilde{M}')^V, (\tilde{\zeta}')^V\big)$. If $v \notin V$, the projection of c'_v onto ${}^L\tilde{C}'_v$ is the conjugacy class that corresponds to the Langlands parameter of the unramified representation $\tilde{\eta}'_v$. Therefore c'_v equals $\tilde{\xi}'_v(\mathfrak{c}'_v)$, for a semisimple conjugacy class $\mathfrak{c}'_v$ in $\mathcal{M}'_v$. Let $c_v = \xi'_{M,v}(\mathfrak{c}'_v)$ be the image of $\mathfrak{c}'_v$ in ${}^LM_v$. The family $c = \{c_v : v \notin V\}$ then belongs to $\mathcal{C}(M^V, \zeta^V)$. We thus obtain a map $c' \to c$ from $\mathcal{C}\big((\tilde{M}')^V, (\tilde{\zeta}')^V\big)$ to $\mathcal{C}(M^V, \zeta^V)$. Langlands' functoriality principle [10] implies that the subset $\mathcal{C}^V_{\mathrm{aut}}(\tilde{M}', \tilde{\zeta}')$ of $\mathcal{C}\big((\tilde{M}')^V, (\tilde{\zeta}')^V\big)$ is mapped into the subset $\mathcal{C}^V_{\mathrm{aut}}(M, \zeta)$ of $\mathcal{C}(M^V, \zeta^V)$. However, this is far from known. We shall nevertheless prove that if c is the image of a family c' in $\mathcal{C}^V_{\mathrm{aut}}(\tilde{M}', \tilde{\zeta}')$, the L-functions $L(s, c, \rho_{Q|P})$ all have meromorphic continuation.

We shall first construct a subset $\mathcal{C}^V_+(M, \zeta)$ of $\mathcal{C}(M^V, \zeta^V)$ that contains the images of the sets $\mathcal{C}^V_{\mathrm{aut}}(\tilde{M}', \tilde{\zeta}')$. If $\psi_\alpha\colon M \to M_\alpha$ is an inner twist over F that is unramified outside of V, let (Z_α, ζ_α) be the image of (Z, ζ), and let $\psi^*_\alpha\colon {}^LM_\alpha \to {}^LM$ be an L-isomorphism that

https://doi.org/10.4153/CJM-1999-050-x Published online by Cambridge University Press

is dual to ψ_α. Then ψ_α^* maps $\mathcal{C}^V_{\mathrm{aut}}(M_\alpha,\zeta_\alpha)$ onto a subset of $\mathcal{C}(M^V,\zeta^V)$. The set $\mathcal{C}^V_+(M,\zeta)$ will contain these images, as well as the endoscopic contributions. We assume inductively that for any elliptic endoscopic datum M' for M as above, that is proper in the sense that it is not equal to a quasisplit inner form of M, the set $\mathcal{C}^V_+(\tilde{M}',\tilde{\zeta}')$ has been defined. We define $\mathcal{C}^V_+(M,\zeta)$ to be the union, over all such M', of the images in $\mathcal{C}(M^V,\zeta^V)$ of the sets $\mathcal{C}^V_+(\tilde{M}',\tilde{\zeta}')$, together with the union, over all M_α as above, of the images of the sets $\mathcal{C}^V_{\mathrm{aut}}(M_\alpha,\zeta_\alpha)$. Note that $\mathcal{C}^V_+(M,\zeta)$ contains the image of $\mathcal{C}^V_{\mathrm{aut}}(\tilde{M}',\tilde{\zeta}')$ for each M'. The functoriality principle implies that $\mathcal{C}^V_+(M,\zeta)$ actually equals $\mathcal{C}^V_{\mathrm{aut}}(M,\zeta)$, at least when M is quasisplit, but we are of course not free to assume this.

Following standard notation, we write $\bigl(Z(\hat{M})^\Gamma\bigr)^0$ for the identity component of the group of $\Gamma = \mathrm{Gal}(\overline{F}/F)$-invariant elements in the center of $\hat{M}$. Then $\bigl(Z(\hat{M})^\Gamma\bigr)^0$ is a complex torus in $\hat{G}$, which plays the role of the split component of the center of a rational group. Suppose that a is a nontrivial (continuous) character on $\bigl(Z(\hat{M})^\Gamma\bigr)^0$. We write ρ_a for the representation of ${}^LM$ on the root space $\hat{\mathfrak{g}}_a$ of a on the Lie algebra of $\hat{G}$. For any $c \in \mathcal{C}(M^V,\zeta^V)$,

$$L_G(s,c,a) = L(s,c,\rho_a), \quad s \in \mathbb{C},$$

is an analytic function of s in some right half plane, which of course equals 1 unless a is actually a root of $\Bigl(\hat{G},\bigl(Z(\hat{M})^\Gamma\bigr)^0\Bigr)$. If $\Sigma(\hat{P})$ denotes the set of roots attached to a parabolic subgroup $P \in \mathcal{P}(M)$, we have the product formula

$$L(s,c,\rho_{Q|P}) = \prod_{a \in \Sigma(\overline{\hat{P}}) \cap \Sigma(\hat{Q})} L_G(s,c,a), \quad P,Q \in \mathcal{P}(M).$$

Using the theory of Eisenstein series, Shahidi has shown that if c belongs to the subset $\mathcal{C}^V_{\mathrm{aut}}(M,\zeta)$, the functions $L_G(s,c,a)$ have meromorphic continuation [13, Section 4], [14, Proposition 4.1]. The functions $L(s,c,\rho_{Q|P})$ therefore also have meromorphic continuation in this case. We shall establish the same properties for any c in the set $\mathcal{C}^V_+(M,\zeta)$.

Proposition 1 *For any character a on $\bigl(Z(\hat{M})^\Gamma\bigr)^0$, and any element $c \in \mathcal{C}^V_+(M,\zeta)$, the function*

$$s \longrightarrow L_G(s,c,a)$$

has meromorphic continuation to the complex plane.

3 Proof of Meromorphic Continuation

Suppose that M' represents an elliptic endoscopic datum $(M',\mathcal{M}',s'_M,\xi'_M)$ for M, in which $\mathcal{M}'$ is actually an L-subgroup of ${}^LM$, and ξ'_M is the identity embedding of $\mathcal{M}'$ into ${}^LM$. The key to Proposition 1, and to our later combinatorial identity as well, will be the set $\mathcal{E}_{M'}(G)$ of endoscopic data that was introduced for local fields in [6, Section 4] and [7, Section 3]. For our global field F, we define $\mathcal{E}_{M'}(G)$ the same way. Then $\mathcal{E}_{M'}(G)$ is the set of endoscopic data $(G',\mathcal{G}',s',\xi')$ for G over F, taken modulo translation of s' by $Z(\hat{G})^\Gamma$, in which s' lies in $s'_M Z(\hat{M})^\Gamma$, $\hat{G}'$ is the connected centralizer of s' in $\hat{G}$, $\mathcal{G}'$ equals $\mathcal{M}'\hat{G}'$, and ξ' is the identity embedding of $\mathcal{G}'$ into ${}^LG$. For each G' in $\mathcal{E}_{M'}(G)$, we fix an embedding $M' \subset G'$ for which

https://doi.org/10.4153/CJM-1999-050-x Published online by Cambridge University Press

$\hat{M}' \subset \hat{G}'$ is a dual Levi subgroup. The central extension $\tilde{M}'$ of M' in Section 2 determines a central extension $\tilde{G}'$ of G' by $\tilde{C}'$ that contains $\tilde{M}'$ as a Levi subgroup. We define coefficients

$$\iota_{M'}(G,G') = |Z(\hat{M}')^\Gamma/Z(\hat{M})^\Gamma|\,|Z(\hat{G}')^\Gamma/Z(\hat{G})^\Gamma|^{-1}, \quad G' \in \mathcal{E}_{M'}(G),$$

just as in the local case.

We have simply copied the definition for local fields in [7, Section 3]. However, there is one point to be verified in the global case. We need to show that $\mathcal{E}_{M'}(G)$ is bijective with $Z(\hat{M})^\Gamma/Z(\hat{G})^\Gamma$. To do so, we must verify that any point s' in $s'_M Z(\hat{M})^\Gamma$ actually does define a datum G' in $\mathcal{E}_{M'}(G)$. The point to check is that $\mathrm{Int}(s)\circ\xi'$ equals $a\otimes\xi'$, where a is a locally trivial 1 cocycle of W_F in $Z(\hat{G})$. Since M' is a global endoscopic datum, the restriction of $\mathrm{Int}(s)\circ\xi'$ to $\mathcal{M}'$ equals $a_M\otimes\xi'$, where a_M is a locally trivial 1 cocycle of W_F in $Z(\hat{M})$. The existence of a is an immediate consequence of the following lemma, in which $\ker^1\bigl(F,Z(\hat{G})\bigr)$ denotes the subgroup of locally trivial elements in $H^1\bigl(F,Z(\hat{G})\bigr)$.

Lemma 2 *The map*

$$\ker^1\bigl(F,Z(\hat{G})\bigr) \longrightarrow \ker^1\bigl(F,Z(\hat{M})\bigr)$$

is an isomorphism.

Proof By the obvious transitivity property, we can assume that G is quasisplit and that M is a minimal Levi subgroup. Then $\hat{M}$ is a torus, and $Z(\hat{M})/Z(\hat{G})$ is a maximal torus in the adjoint group $\hat{G}/Z(\hat{G})$. The action of Γ in $Z(\hat{M})/Z(\hat{G})$ is dual to a direct sum of permutation representations. The required bijectivity of the given map then follows from the exact sequence

$$\pi_0\Bigl(\bigl(Z(\hat{M})/Z(\hat{G})\bigr)^\Gamma\Bigr) \to H^1\bigl(F,Z(\hat{G})\bigr) \to H^1\bigl(F,Z(\hat{M})\bigr) \to H^1\bigl(F,Z(\hat{M})/Z(\hat{G})\bigr),$$

and its analogues for the completions of F. (See the proof of [9, Lemma 4.3.2(a)].) ■

Corollary 3 *The set $\mathcal{E}_{M'}(G)$ is bijective with $Z(\hat{M})^\Gamma/Z(\hat{G})^\Gamma$.* ■

Suppose that a is a nontrivial character on $\bigl(Z(\hat{M})^\Gamma\bigr)^0$, as above. Observe that the kernel

$$Z_a = \{z \in \bigl(Z(\hat{M})^\Gamma\bigr)^0 : a(z) = 1\}$$

acts by translation on $Z(\hat{M})^\Gamma/Z(\hat{G})^\Gamma$, and therefore also on $\mathcal{E}_{M'}(G)$. We write $\mathcal{E}_{M'}(G)/Z_a$ for the set of orbits. We also note that $\bigl(Z(\hat{M})^\Gamma\bigr)^0$ equals $\bigl(Z(\hat{M}')^\Gamma\bigr)^0$, since M' is elliptic. But $\bigl(Z(\hat{M}')^\Gamma\bigr)^0$ is a subgroup of $\bigl(Z(\widehat{\tilde{M}'})^\Gamma\bigr)^0$, and for any $G' \in \mathcal{E}_{M'}(G)$, this injection determines an isomorphism

$$(2) \qquad \bigl(Z(\hat{M}')^\Gamma\bigr)^0/\bigl(Z(\hat{M}')^\Gamma\bigr)^0 \cap Z(\hat{G}')^\Gamma \xrightarrow{\sim} \bigl(Z(\widehat{\tilde{M}'})^\Gamma\bigr)^0/\bigl(Z(\widehat{\tilde{M}'})^\Gamma\bigr)^0 \cap Z(\widehat{\tilde{G}'})^\Gamma.$$

If a is trivial on $\bigl(Z(\hat{M}')^\Gamma\bigr)^0 \cap Z(\hat{G}')^\Gamma$, let $a' = a^{G'}$ be the unique character on $\bigl(Z(\widehat{\tilde{M}'})^\Gamma\bigr)^0$ that is trivial on $\bigl(Z(\widehat{\tilde{M}'})^\Gamma\bigr)^0 \cap Z(\widehat{\tilde{G}'})^\Gamma$. Otherwise, we take a' to be any character on

https://doi.org/10.4153/CJM-1999-050-x Published online by Cambridge University Press

$\big(Z(\widehat{\tilde{M}'})^\Gamma\big)^0$ whose restriction to $\big(Z(\hat{M}')^\Gamma\big)^0$ equals a. The character a thus determines a family of L-functions

$$L_{\tilde{G}'}(s, c', a'), \quad c' \in \mathcal{C}\big((\tilde{M}')^V, (\tilde{\zeta}')^V\big),$$

for each $G' \in \mathcal{E}_{M'}(G)$. In the case that a' is not uniquely determined by a, a' is not a root of $\Big(\widehat{\tilde{G}'}, \big(Z(\widehat{\tilde{M}'})^\Gamma\big)^0\Big)$, and $L_{\tilde{G}'}(s, c', a')$ is equal to 1. The L-function is therefore uniquely determined by a.

Lemma 4 *Suppose that c' is an element in $\mathcal{C}\big((\tilde{M}')^V, (\tilde{\zeta}')^V\big)$ with image c in $\mathcal{C}(M^V, \zeta^V)$. Then*

$$L_G(s, c, a) = \prod_{G' \in \mathcal{E}_{M'}(G)/Z_a} L_{\tilde{G}'}(s, c', a'). \tag{3}$$

Proof We can assume that a is a root of $\Big(\hat{G}, \big(Z(\hat{M})^\Gamma\big)^0\Big)$, since both sides of (3) would otherwise be equal to 1. If G' corresponds to the element s' in $s'_M Z(\hat{M})^\Gamma / Z(\hat{G})^\Gamma$, $\mathrm{Ad}(s')$ stabilizes the root space $\hat{\mathfrak{g}}_a$ in the Lie algebra of $\hat{G}$. The root space of a' in the Lie algebra of $\widehat{\tilde{G}'}$ can be identified with the root space $\hat{\mathfrak{g}}'_a$ of a in the Lie algebra of $\hat{G}'$. This is in turn just the $(+1)$-eigenspace of $\mathrm{Ad}(s')$ in $\hat{\mathfrak{g}}_a$. If s' is replaced by a Z_a-translate, this eigenspace remains the same. In particular, the factor $L_{\tilde{G}'}(s, c', a')$ on the right hand side of (3) does depend only on the Z_a-orbit of G', since this factor is determined by the adjoint representation of $\mathcal{M}'$ on $\hat{\mathfrak{g}}'_a$.

Let

$$\hat{\mathfrak{g}}_a = \bigoplus_i \hat{\mathfrak{g}}_{a,i}$$

be the decomposition of $\hat{\mathfrak{g}}_a$ into distinct eigenspaces under the action of $\mathrm{Ad}(s'_M)$. We can identify $s'_M Z(\hat{M})^\Gamma / Z(\hat{G})^\Gamma$ with $s'_M \big(Z(\hat{M})^\Gamma\big)^0 / \big(Z(\hat{M})^\Gamma\big)^0 \cap Z(\hat{G})^\Gamma$, by [7, Lemma 1.2]. If $s' = s'_M t$ is any point in this set, we have

$$\mathrm{Ad}(s') X_i = \lambda_i a(t) X_i, \quad X_i \in \hat{\mathfrak{g}}_{a,i},$$

where λ_i is the eigenvalue of $\mathrm{Ad}(s'_M)$ in $\hat{\mathfrak{g}}_{a,i}$. Since a defines a nontrivial character on the complex torus $\big(Z(\hat{M})^\Gamma\big)^0$, there is a point s'_i in $s'_M \big(Z(\hat{M})^\Gamma\big)^0$ that acts as the identity on $\hat{\mathfrak{g}}_{a,i}$. If $G' \in \mathcal{E}_{M'}(G)$ is the endoscopic datum corresponding to the image of s'_i in $s'_M Z(\hat{M})^\Gamma / Z(\hat{G})^\Gamma$, the root space $\hat{\mathfrak{g}}'_a$ equals $\hat{\mathfrak{g}}_{a,i}$. It is clear that s'_i is uniquely determined up to translation by Z_a, and that the distinct eigenspaces $\hat{\mathfrak{g}}_{a,i}$ determine distinct Z_a-orbits in $\mathcal{E}_{M'}(G)$. Moreover, if $G' \in \mathcal{E}_{M'}(G)$ does not correspond to the Z_a-orbit of some s'_i, $\hat{\mathfrak{g}}'_a$ equals $\{0\}$. It follows that

$$\hat{\mathfrak{g}}_a = \bigoplus_i \hat{\mathfrak{g}}_{a,i} = \bigoplus_{G' \in \mathcal{E}_{M'}(G)/Z_a} \hat{\mathfrak{g}}'_a. \tag{4}$$

This corresponds to a decomposition of the restriction of ρ_a to $\mathcal{M}'$ into a direct sum

$$\bigoplus_{G' \in \mathcal{E}_{M'}(G)} \rho'_a$$

of subrepresentations. The decomposition of $L_G(s,c,a)$ into a finite product (3) follows easily from this decomposition, and the multiplicative property of L-functions. ■

We can now prove Proposition 1. Suppose first that c is the image in $\mathcal{C}^V_+(M,\zeta)$ of an element $c_\alpha \in \mathcal{C}^V_{\mathrm{aut}}(M_\alpha,\zeta_\alpha)$, for an inner twist $\psi_\alpha \colon M \to M_\alpha$. We can always extend ψ_α to an inner twist from G to a group G_α over F. Then ρ_α corresponds to a representation ρ_{a_α} of ${}^L M_\alpha$ such that

$$L_G(s,c,a) = L_{G_\alpha}(s,c_\alpha,a_\alpha).$$

We may therefore assume that $M_\alpha = M$ and $\psi_\alpha = 1$, or in other words, that c belongs to the subset $\mathcal{C}^V_{\mathrm{aut}}(M,\zeta)$ of $\mathcal{C}^V_+(M,\zeta)$. The meromorphic continuation of $L_G(s,c,a)$ in this case is due to Shahidi. We shall give a case free proof based on the set $\mathcal{E}_{M^*}(G)$, where M^* is a quasisplit inner form of M.

We shall assume that the root space $\hat{\mathfrak{g}}_a$ is non-empty, since $L_G(s,c,a)$ would otherwise be equal to 1. Then a is a positive integral multiple of a unique reduced root a_1 of $\Big(\hat{G}, \big(Z(\hat{M})^\Gamma\big)^0\Big)$. Consider the set

$$\{a_k = ka_1 : 1 \leq k \leq m\}$$

of all roots that are positive integral multiples of a_1. We shall prove Proposition 1, in the special case that c belongs to $\mathcal{C}^V_{\mathrm{aut}}(M,\zeta)$, by induction on the length $m = m(G,a)$.

Choose a root a_k, with $1 \leq k \leq m$. The kernel Z_{a_k} contains Z_{a_1}, and a_1 maps the quotient Z_{a_k}/Z_{a_1} isomorphically onto the group of k-th roots of 1 in $\mathbb{C}^*$. Let s'_k be an element in Z_{a_k} that maps to a primitive k-th root of 1. As an element in $Z(\hat{M})^\Gamma/Z(\hat{G})^\Gamma$, s'_k determines an endoscopic datum G'_k in $\mathcal{E}_{M^*}(G)$. Consider Lemma 4, particularly the decomposition (4) established during its proof, in the special case that $M' = M^*$ and $s'_M = 1$. We observe immediately that the Lie algebra $\hat{\mathfrak{g}}'_k$ of $\hat{G}'_k$ has the property that an intersection

$$\hat{\mathfrak{g}}'_k \cap \hat{\mathfrak{g}}_{a_\ell}, \quad 1 \leq \ell \leq m,$$

is equal to either $\hat{\mathfrak{g}}_{a_\ell}$ or $\{0\}$, according to whether ℓ is an integral multiple of k or not. In particular, a_k is a reduced root of $\Big(\hat{G}'_k, \big(Z(\hat{M})^\Gamma\big)^0\Big)$, and the length $m(\tilde{G}'_k, a_k)$ equals the greatest integer in m/k. Moreover, if $a = a_k$, we have

$$L_G(s,c,a) = L_{G'_k}(s,c,a_k).$$

If $k > 1$, the meromorphic continuation of $L_G(s,c,a)$ follows from our induction hypothesis.

To deal with the remaining case that $a = a_1$, we recall that the product

$$\phi(s) = \prod_{k=1}^{m} \big(L_G(ks,c,a_k) L_G(1+ks,c,a_k)^{-1}\big)$$

is the unramified part of a constant term of any Eisenstein series attached to c. The ramified components of constant terms have meromorphic continuation, by general properties of local intertwining operators. Since Langlands' theory of Eisenstein series includes the

https://doi.org/10.4153/CJM-1999-050-x Published online by Cambridge University Press

memomorphic continuation of all constant terms, the function $\phi(s)$ has meromorphic continuation. Now we have already established the meromorphic continuation of the factors with $k > 1$ in the product for $\phi(s)$. We conclude that the remaining factor

$$L_G(s, c, a_1)L_G(1 + s, c, a_1)^{-1}$$

also has meromorphic continuation. The meromorphic continuation of the function $L_G(s, c, a) = L_G(s, c, a_1)$ follows.

We have established Proposition 1 if c belongs to the image of $\mathcal{C}^V_{\mathrm{aut}}(M_\alpha, \zeta_\alpha)$, for an inner twist M_α of M. For the remaining case of an endoscopic image, we have already made all the necessary preparations. Suppose that c is the image of an element $c' \in \mathcal{C}^V(\tilde{M}', \tilde{\zeta}')$, for an elliptic endoscopic datum $M' \neq M^*$. We can assume by induction on the dimension of the derived group of M that the proposition holds if (M, ζ) is replaced by $(\tilde{M}', \tilde{\zeta}')$. The meromorphic continuation of $L_G(s, c, a)$ then follows from Lemma 4. ■

4 Stabilization of $r^G_M(c_\lambda)$

We come now to our combinatorial identity. In preparation, we note that there is a canonical map from the real vector space

$$\mathfrak{a}^*_M = X(M)_F \otimes \mathbb{R}$$

onto the corresponding space $\mathfrak{a}^*_Z$ for Z. Let $\mathfrak{a}^*_{M,Z}$ be its kernel. The complexification $\mathfrak{a}^*_{M,Z} \otimes \mathbb{C}$ can be identified with a subspace of the Lie algebra of $Z(\hat{M})^\Gamma$, which has an action

$$c = \{c_v\} \longrightarrow c_\lambda = \{c_{v,\lambda} = c_v q_v^{-\lambda} = c_v \exp\bigl(-(\log q_v)\lambda\bigr)\}, \quad \lambda \in \mathfrak{a}^*_{M,Z} \otimes \mathbb{C},$$

on $\mathcal{C}(M^V, \zeta^V)$. It is clear that

$$L_G(s, c_\lambda, a) = L_G\bigl(s + (da)(\lambda), c, a\bigr),$$

where da is the linear form on $\mathfrak{a}^*_M \otimes \mathbb{C}$ associated to a given character a on $\bigl(Z(\hat{M})^\Gamma\bigr)^0$. The action $c \to c_\lambda$ leaves invariant the subset $\mathcal{C}^V_+(M, \zeta)$ of $\mathcal{C}(M^V, \zeta^V)$. It follows from Proposition 1 that for any fixed $s_0 \in \mathbb{C}$, the functions

$$\lambda \longrightarrow L_G(s_0, c_\lambda, a), \quad \lambda \in \mathfrak{a}^*_{M,Z} \otimes \mathbb{C}, c \in \mathcal{C}^V_+(M, \zeta),$$

are meromorphic.

Suppose that c lies in $\mathcal{C}^V(M, \zeta)$. The quotients

$$r(c_\lambda, a) = L_G(0, c_\lambda, a)L_G(1, c_\lambda, a)^{-1}$$

are then meromorphic functions of λ, as are the quotients

$$\begin{aligned} r_{Q|P}(c_\lambda) &= L(0, c_\lambda, \rho_{Q|P})L_{Q|P}(1, c_\lambda, a)^{-1} \\ &= \prod_{\alpha \in \Sigma(\hat{\overline{P}}) \cap \Sigma(\hat{Q})} r(c_\lambda, a), \quad P, Q \in \mathcal{P}(M). \end{aligned}$$

https://doi.org/10.4153/CJM-1999-050-x Published online by Cambridge University Press

Motivated by the local results of [6], we introduce a (G, M)-family of functions

$$r_Q(\Lambda, c_\lambda) = r_{Q|\overline{Q}}(c_\lambda)^{-1} r_{Q|\overline{Q}}(c_{\lambda+\frac{1}{2}\Lambda}), \quad Q \in \mathcal{P}(M),$$

of $\Lambda \in i\mathfrak{a}^*_{M,Z}$. (As a function of Λ, $r_Q(\Lambda, c_\lambda)$ takes values in the space of meromorphic functions of λ.) The limit

$$r^G_M(c_\lambda) = \lim_{\Lambda \to 0} \sum_{Q \in \mathcal{P}(M)} r_Q(\Lambda, c_\lambda) \theta_Q(\Lambda)^{-1},$$

with the function

$$\theta_Q(\Lambda) = \mathrm{vol}\big(\mathfrak{a}^G_M / \mathbb{Z}(\Delta^\vee_Q e)\big)^{-1} \prod_{\alpha \in \Delta_Q} \Lambda(\alpha^\vee e)$$

defined as for example in [1, Section 2], is then defined as a meromorphic function of $\lambda \in \mathfrak{a}^*_{M,Z} \otimes \mathbb{C}$. (See [1, Lemma 6.2].) The functions $r^G_M(c_\lambda)$ occur in the invariant global trace formula obtained from the normalized weighted characters of [6]. They are the unramified spectral terms, which take the place of the functions $r^G_M(\pi_\lambda)$ [3, (4.5)] from the original invariant trace formula. (See [8, Section 3].)

Given that they occur in the trace formula, it makes sense to stabilize the functions $r^G_M(c_\lambda)$. There are no invariant distributions here to be made into stable distributions. The question is rather that of carrying out a construction that is forced on us by the stabilization of more serious terms in the trace formula. We shall follow the prescription in [6, Section 4] and [7, Section 3] for stabilizing weighted orbital integrals.

The construction consists of a definition and an identity to be proved. In the case of weighted orbital integrals, the identity is quite deep, and was left as a conjecture. The corresponding identity here is simpler, and will be the content of the next theorem. The theorem applies to a fixed set V, and variable objects (G, M, ζ) and M' that are as above. Thus, V is a finite set of valuations of F, G is a connected reductive group over F, M is a Levi subgroup, ζ is an automorphic character on a central induced torus Z of G, and M' is an elliptic endoscopic datum for M, all of which are unramified at the places outside of V. In order that it be more closely parallel to the case of weighted orbital integrals, the definition part of the construction will be restricted to triplets (G, M, ζ) that are quasisplit (which is to say that G, M and Z are quasisplit over F), while the identity will apply to any triplet. The distinction is just cosmetic in the present situation, since none of the objects depend on a choice of inner form.

Theorem 5 *For each quasisplit triplet (G, M, ζ) and each $c \in \mathcal{C}^V_+(M, \zeta)$, there is a meromorphic function*

$$s^G_M(c_\lambda), \quad \lambda \in \mathfrak{a}^*_{M,Z} \otimes \mathbb{C},$$

with the property that for any (G, M, ζ), any M', and any element $c' \in \mathcal{C}^V_+(\tilde{M}', \tilde{\zeta}')$ with image c in $\mathcal{C}^V_+(M, \zeta)$, the identity

$$r^G_M(c_\lambda) = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') s^{\tilde{G}'}_{\tilde{M}'}(c'_\lambda) \tag{5}$$

holds.

https://doi.org/10.4153/CJM-1999-050-x Published online by Cambridge University Press

Proof If (G, M, ζ) is quasisplit and c belongs to $\mathcal{C}^V_+(M, \zeta)$, the function $s^G_M(c_\lambda)$ is uniquely determined by the required identity. We define it inductively by setting

$$s^G_M(c_\lambda) = r^G_M(c_\lambda) - \sum_{G' \in \mathcal{E}^0_M(G)} \iota_M(G, G') s^{G'}_M(c_\lambda),$$

where $\mathcal{E}^0_M(G)$ denotes the set of elements $G' \in \mathcal{E}_M(G)$ with $G' \neq G$. Since the coefficient $\iota_M(G, G')$ vanishes unless G' is elliptic, the sum can be taken over a finite set.

Now suppose that (G, M, ζ) is any triplet, and that c is the image in $\mathcal{C}^V_+(M, \zeta)$ of an element $c' \in \mathcal{C}^V_+(\tilde{M}', \tilde{\zeta}')$. Our task is to show that $r^G_M(c_\lambda)$ equals the endoscopic expression

$$r^{G,\mathcal{E}}_M(c'_\lambda) = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') s^{\tilde{G}'}_{\tilde{M}'}(c'_\lambda).$$

We shall actually prove a more general identity.

Suppose that A is a finite set of continuous characters on $\big(Z(\hat{M})^\Gamma\big)^0$. Then

$$r_Q(\Lambda, c_\lambda, A) = \prod_{a \in A \cap \Sigma(\widehat{Q})} r(c_\lambda, a)^{-1} r(c_{\lambda + \frac{1}{2}\Lambda}, a), \qquad Q \in \mathcal{P}(M),$$

is a (G, M)-family of functions of $\Lambda \in i\mathfrak{a}^*_{M,Z}$, with values in the space of mermorphic functions of λ. The limit

$$r^G_M(c_\lambda, A) = \lim_{\Lambda \to 0} \sum_{Q \in \mathcal{P}(M)} r_Q(\Lambda, c_\lambda, A) \theta_Q(\Lambda)^{-1}$$

is then a meromorphic function of λ. If $A' = \{a' : a \in A\}$, we define generalizations $s^G_M(c_\lambda, A)$ and $r^{G,\mathcal{E}}_M(c'_\lambda, A)$ of $s^G_M(c_\lambda)$ and $r^{G,\mathcal{E}}_M(c'_\lambda)$ inductively by setting

$$s^G_M(c_\lambda, A) = r^G_M(c_\lambda, A) - \sum_{G' \in \mathcal{E}^0_M(G)} \iota_M(G, G') s^{G'}_M(c_\lambda, A)$$

for (G, M, ζ) quasisplit, and

$$r^{G,\mathcal{E}}_M(c'_\lambda, A') = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G') s^{\tilde{G}'}_{\tilde{M}'}(c'_\lambda, A')$$

in general. There is actually no distinction to be made between the quasisplit and the general case. For if (G^*, M^*, ζ^*) is a quasisplit inner twist of (G, M, ζ), there is a bijection $c \to c^*$ from $\mathcal{C}^V_+(M, \zeta)$ onto $\mathcal{C}^V_+(M^*, \zeta^*)$ such that $r^G_M(c_\lambda, A) = r^{G^*}_{M^*}(c^*_\lambda, A)$. Since $r^{G,\mathcal{E}}_M(c'_\lambda, A') = r^{G^*,\mathcal{E}}_{M^*}(c'_\lambda, A')$, we may as well then assume that (G, M, ζ) is quasisplit. We shall show that $r^{G,\mathcal{E}}_M(c'_\lambda, A')$ equals $r^G_M(c_\lambda, A)$ by induction on A.

The main step is the case that A is a set $\{a\}$ of one element. In this case, the function $r^G_M(c_\lambda, A) = r^G_M(c_\lambda, a)$ vanishes, unless a is a root of $\Big(\hat{G}, \big(Z(\hat{M})^\Gamma\big)^0\Big)$ and spans the kernel $\mathfrak{a}^G_M$ of the natural map $\mathfrak{a}_M \to \mathfrak{a}_G$. The same assertion follows inductively for the functions

$s_{\tilde{M}'}^{\tilde{G}'}(c'_\lambda, A') = s_{\tilde{M}'}^{\tilde{G}'}(c'_\lambda, a')$, and hence also for $r_M^{G,\mathcal{E}}(c'_\lambda, A') = r_M^{G,\mathcal{E}}(c'_\lambda, a')$. We can therefore assume that M is a maximal Levi subgroup of G. But in this case, $r_M^G(c_\lambda, a)$ is a logarithmic derivative of the function $r(c_\lambda, a)$, (relative to the coordinate of $\frac{1}{2}\lambda$ defined by the unit vector in $\mathfrak{a}_M^G$ determined by a). On the other hand, Lemma 4 implies that $r(c_\lambda, a)$ is a product over $G' \in \mathcal{E}_{M'}(G)/Z_a$ of the functions $r(c'_\lambda, a')$ attached to $\tilde{G}'$. Since logarithmic derivatives transform products to sums, we obtain

$$r_M^G(c_\lambda, a) = \sum_{G' \in \mathcal{E}_{M'}(G)/Z_a} r_{\tilde{M}'}^{\tilde{G}'}(c'_\lambda, a').$$

To deal with the other side of the required identity, we set

$${}^*r_M^{G,\mathcal{E}}(c'_\lambda, a') = \sum_{G' \in \mathcal{E}_{M'}(G)} \iota_{M'}(G, G')\,{}^*s_{\tilde{M}'}^{\tilde{G}'}(c'_\lambda, a'),$$

where

$${}^*s_{\tilde{M}'}^{\tilde{G}'}(c'_\lambda, a') = |Z_{a'}/Z_{a'} \cap Z(\widehat{\tilde{G}'})^\Gamma|^{-1} r_{\tilde{M}'}^{\tilde{G}'}(c'_\lambda, a').$$

It will be enough to show that ${}^*r_M^{G,\mathcal{E}}(c'_\lambda, a')$ equals $r_M^G(c_\lambda, a)$. Indeed, in the case that $M' = M$, this would establish inductively that ${}^*s_M^G(c_\lambda, a) = s_M^G(c_\lambda, a)$. For arbitrary M', we would then obtain

$$r_M^{G,\mathcal{E}}(c'_\lambda, a') = {}^*r_M^{G,\mathcal{E}}(c'_\lambda, a') = r_M^G(c_\lambda, a),$$

which is the required identity (for $A = \{a\}$).

Consider the product of $\iota_{M'}(G, G')$ and ${}^*s_{\tilde{M}'}^{\tilde{G}'}(c'_\lambda, a')$ that occurs in the last sum. We can assume that a is a root of $\left(\hat{G}', \left(Z(\hat{M}')^\Gamma\right)^0\right)$, or equivalently, that a' is a root of $\left(\widehat{\tilde{G}'}, \left(Z(\widehat{\tilde{M}'})^\Gamma\right)^0\right)$, since the function ${}^*s_{\tilde{M}'}^{\tilde{G}'}(c'_\lambda, a')$ would otherwise vanish. In particular, Z_a contains $\left(Z(\hat{M}')^\Gamma\right)^0 \cap Z(\hat{G}')^\Gamma$, and $Z_{a'}$ contains $\left(Z(\widehat{\tilde{M}'})^\Gamma\right)^0 \cap Z(\widehat{\tilde{G}'})^\Gamma$. It follows from the isomorphism (2) that

$$Z_{a'}/Z_{a'} \cap Z(\widehat{\tilde{G}'})^\Gamma \cong Z_a/Z_a \cap Z(\hat{G}')^\Gamma.$$

To deal with the coefficient $\iota_{M'}(G, G')$, we recall that

$$Z(\hat{M}')^\Gamma = \left(Z(\hat{M}')^\Gamma\right)^0 Z(\hat{G}')^\Gamma = \left(Z(\hat{M})^\Gamma\right)^0 Z(\hat{G}')^\Gamma,$$

by [7, Lemma 1.2] and the ellipticity of M'. In particular, the canonical map

$$Z(\hat{G}')^\Gamma / Z(\hat{G})^\Gamma \longrightarrow Z(\hat{M}')^\Gamma / Z(\hat{M})^\Gamma$$

is surjective. It follows easily that

$$\begin{aligned}\iota_{M'}(G, G') &= |Z(\hat{M}')^\Gamma / Z(\hat{M})^\Gamma|\,|Z(\hat{G}')^\Gamma / Z(\hat{G})^\Gamma|^{-1}\\ &= |\left(Z(\hat{M})^\Gamma\right)^0 \cap Z(\hat{G}')^\Gamma / \left(Z(\hat{M})^\Gamma\right)^0 \cap Z(\hat{G})^\Gamma|^{-1}.\end{aligned}$$

Since $\big(Z(\hat{M})^\Gamma\big)^0 \cap Z(\hat{G}')^\Gamma$ is contained in Z_a, we obtain

$$\iota_{M'}(G,G') = |Z_a \cap Z(\hat{G}')^\Gamma / Z_a \cap Z(\hat{G})^\Gamma|^{-1}.$$

The product becomes

$$\begin{aligned}\iota_{M'}(G,G')^* s^{\tilde{G}'}_{\tilde{M}'}(c'_\lambda, a') &= |Z_a \cap Z(\hat{G}')^\Gamma / Z_a \cap Z(\hat{G})^\Gamma|^{-1}\, |Z_a / Z_a \cap Z(\hat{G}')^\Gamma|^{-1} r^{\tilde{G}'}_{\tilde{M}'}(c'_\lambda, a')\\ &= |Z_a / Z_a \cap Z(\hat{G}')^\Gamma|^{-1} r^{\tilde{G}'}_{\tilde{M}'}(c'_\lambda, a').\end{aligned}$$

We conclude that

$$\begin{aligned}{}^* r^{G,\mathcal{E}}_M(c'_\lambda, a') &= \sum_{G' \in \mathcal{E}_{M'}(G)} |Z_a / Z_a \cap Z(\hat{G})^\Gamma|^{-1} r^{\tilde{G}'}_{\tilde{M}'}(c'_\lambda, a')\\ &= \sum_{G' \in \mathcal{E}_{M'}(G)/Z_a} r^{\tilde{G}'}_{\tilde{M}'}(c'_\lambda, a')\\ &= r^G_M(c_\lambda, a),\end{aligned}$$

since $r^{\tilde{G}'}_{\tilde{M}'}(c'_\lambda, a')$ depends only on the orbit of Z_a in $\mathcal{E}_{M'}(G)$, and since the stabilizer of G' in Z_a is $Z_a \cap Z(\hat{G})^\Gamma$. We have established the required identity in the case that A contains one element.

Having established the case that $|A| = 1$, we now suppose that A is a disjoint union of two nonempty proper subsets A_1 and A_2. We assume inductively that

$$r^{L_i,\mathcal{E}}_M(c'_\lambda, A'_i) = r^{L_i}_M(c_\lambda, A_i), \quad L_i \in \mathcal{L}(M), i = 1,2,$$

where $\mathcal{L}(M)$ denotes the set of Levi subgroups of G that contain M. We shall use splitting formulas to reduce the case of A to those of A_1 and A_2. Indeed

$$r_Q(\Lambda, c_\lambda, A) = r_Q(\Lambda, c_\lambda, A_1) r_Q(\Lambda, c_\lambda, A_2), \quad Q \in \mathcal{P}(M),$$

is a product of (G,M)-families. It follows from the splitting formula [2, Corollary 7.4] that

$$r^G_M(c_\lambda, A) = \sum_{L_1,L_2 \in \mathcal{L}(M)} d^G_M(L_1,L_2) r^{L_1}_M(c_\lambda, A_1) r^{L_2}_M(c_\lambda, A_2),$$

for certain coefficients $d^G_M(L_1,L_2)$. There is also a splitting formula for $r^{G,\mathcal{E}}_M(c_\lambda, A)$, which can be established in exactly the same manner as Theorem 6.1 of [7]. One has to assume inductively that the appropriate splitting formula holds for each of the terms $s^{\tilde{G}'}_{\tilde{M}'}(c'_\lambda, A')$,

https://doi.org/10.4153/CJM-1999-050-x Published online by Cambridge University Press

$G' \neq G$, in the definition of $r_M^{G,\mathcal{E}}(c'_\lambda, A')$. The proof of [7, Theorem 6.1] then leads directly to the formula

$$r_M^{G,\mathcal{E}}(c'_\lambda, A') = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) r_M^{L_1,\mathcal{E}}(c'_\lambda, A'_1) r_M^{L_2,\mathcal{E}}(c'_\lambda, A'_2).$$

It follows from our induction assumption that $r_M^{G,\mathcal{E}}(c'_\lambda, A')$ equals $r_M^G(c_\lambda, A)$. This is what we wanted to show. To complete the proof of the proposition, we simply take A to be the set of roots of $\left(\hat{G}, \left(Z(\hat{M})^\Gamma\right)^0\right)$. We obtain

$$r_M^{G,\mathcal{E}}(c'_\lambda) = r_M^{G,\mathcal{E}}(c'_\lambda, A') = r_M^G(c_\lambda, A) = r_M^G(c_\lambda),$$

as required. ■

Remarks 1. As we have already noted, the construction in Theorem 5 is parallel to the stabilization of weighted orbital integrals. In particular, the required identities in each case are of similar form. Where they differ is in their degree of difficulty. That we could prove an identity here with relative ease may be regarded as evidence for the deeper identity that was stated as a conjecture in [6] and [7].

2. There is an important special case of the stabilization problem for weighted orbital integrals. It is the weighted analogue of the (conjectural) fundamental lemma. (See [8].) This is in even closer analogy with the construction here. The formal similarity between the two seems to suggest some role for the functions $r_M^G(c_\lambda)$, or rather local (unramified) forms of them, in the fundamental lemma.

3. Theorem 5 is reminiscent of another combinatorial identity. We are thinking of the construction for Weyl groups in [4, Theorem 8.1], which is actually analogous to a twisted form of Theorem 5. While Theorem 5 amounts to a stabilization of the unramified spectral terms in the trace formula, the identity in [5] would be part of the stabilization of another set of terms, the ones that occur discretely on the spectral side.

References

[1] J. Arthur, *The trace formula in invariant form.* Ann. of Math. (2) **114**(1981), 1–74.
[2] ———, *The invariant trace formula I: Local theory.* J. Amer. Math. Soc. **1**(1988), 323–383.
[3] ———, *The invariant trace formula II: Global theory.* J. Amer. Math. Soc. **1**(1988), 501–554.
[4] ———, *Unipotent automorphic representations: Global motivation.* In: Automorphic Forms, Shimura Varieties, and L-functions, Vol. **I**, Academic Press, 1–75.
[5] ———, *On local character relations.* Selecta Math. **2**(1996), 501–579.
[6] ———, *Canonical normalization of weighted characters and a transfer conjecture.* C. R. Math. Rep. Acad. Sci. Canada (2) **20**(1998), 33–52.
[7] ———, *On the transfer of distributions: weighted orbital integrals.* Duke Math. J., to appear.
[8] ———, *A stable trace formula.* In preparation.
[9] R. Kottwitz, *Stable trace formula: cuspidal tempered terms.* Duke Math. J. **51**(1984), 611–650.
[10] R. P. Langlands, *Problems in the theory of automorphic forms.* In: Lecture Notes in Modern Analysis and Applications III, Lecture Notes in Math. **170**(1970), 18–86.
[11] ———, *On the notion of an automorphic representation.* In: Automorphic Forms, Representations and L-functions, Part I, Proc. Sympos. Pure Math. **33**(1979), 203–207.
[12] R. P. Langlands and D. Shelstad, *On the definition of transfer factors.* Math. Ann. **278**(1987), 219–271.

[13] F. Shahidi, *On the Ramanujan conjecture and finiteness of poles for certain L-functions.* Ann. of Math. **127**(1988), 547–584.

[14] ______, *A proof of Langlands' conjecture on Plancherel measures; Complementary series for p-adic groups.* Ann. of Math. **132**(1990), 273–330.

Department of Mathematics
University of Toronto
Toronto, Ontario
M5S 3G3

https://doi.org/10.4153/CJM-1999-050-x Published online by Cambridge University Press