Proceedings of the Edinburgh Mathematical Society (1996) 39, 491–503 ©

PRINCIPAL CONGRUENCES ON DISTRIBUTIVE DOUBLE p-ALGEBRAS

by JÚLIA VAZ DE CARVALHO*

(Received 29th September 1994)

We use Priestley's duality to characterize, via their dual space, the distributive double p-algebras on which all congruences are principal.

1991 *Mathematics subject classification*: 06D15.

1. Introduction

An algebra L is said to have property PC if all its congruences are principal. This notion was introduced by Blyth and Varlet in [3]. There the authors characterize the distributive lattices and the Stone, the de Morgan and the Heyting algebras that have PC.

In [2], Beazer characterizes the quasi-modular p-algebras that have PC and solves the same problem for some special classes of distributive double p-algebras, using purely algebraic tools.

In this paper we use Priestley's duality to characterize, via their dual space, all the distributive double p-algebras that have PC. As a consequence we obtain not only the characterization for double Stone algebras established by the author in [9] and by Beazer in [2], but also the characterizations stated in [2] for some special classes of distributive double p-algebras.

In order to obtain such results, we start by determining the subsets of the dual space of a distributive double p-algebra that represent principal congruences and also "translate" some properties on the dual space into properties of the algebra and vice versa.

2. Preliminaries

A distributive p-algebra is an algebra $A=(A;\wedge,\vee,*,0,1)$ of type $(2,2,1,0,0)$ such that $(A;\wedge,\vee,0,1)$ is a bounded distributive lattice and $*$ is a unary operation that satisfies

*This work was done within the activities of Centro de Álgebra da Universidade de Lisboa.

$$y \wedge x = 0 \Leftrightarrow y \leqq x^*,$$

i.e., x^* is the pseudocomplement of x.

A distributive double p-algebra is an algebra $A=(A; \wedge, \vee, *, +, 0, 1)$ of type $(2,2,1,1,0,0)$ such that $A=(A; \wedge, \vee, *, 0, 1)$ is a distributive p-algebra and $+$ is a unary operation that satisfies

$$y \vee x = 1 \Leftrightarrow x^+ \leqq y,$$

i.e., x^+ is the dual psudocomplement of x.

We denote the variety of distributive double p-algebras by $\mathbf{B}_\omega^\omega$.

Let L be a distributive (double) p-algebra. We say that L is a (double) Stone algebra if $x^* \vee x^{**}=1$ ($x^* \vee x^{**}=1$ and $x^+ \wedge x^{++}=0$), for every $x \in L$.

We start by giving a brief outline of Priestley's duality as it applies to distributive double p-algebras. For more details see [**4**, **5**, **6**, **7**].

First, let P be a partially ordered set and $Q \subseteq P$. We define $\uparrow Q = \{x \in P \mid (\exists y \in Q)\, x \geqq y\}$ and $\downarrow Q = \{x \in P \mid (\exists y \in Q)\, x \leqq y\}$. When $Q=\{z\}$, we simply write $\uparrow z$ and $\downarrow z$ for $\uparrow\{z\}$ and $\downarrow\{z\}$, respectively. The subset Q is said to be an *up set* if $Q=\uparrow Q$, a *down set* if $Q=\downarrow Q$ and an *up-down set* if $Q=\uparrow Q=\downarrow Q$. We denote by $\operatorname{Min} P$ and $\operatorname{Max} P$ the sets of minimal and of maximal elements of P, respectively. We say that P has length less than or equal to $k \in N_0$, and write $l(P) \leqq k$, if every chain in P has at most $k+1$ elements. Let $X=(X; \tau, \leqq)$ be an ordered topological space, X is called a *Priestley space* if it is compact and totally order disconnected, that is, if $x, y \in X$ and $x \nleqq y$, there exists a clopen down set U such that $y \in U$ and $x \notin U$ (thus X is Hausdorff). If X is a Priestley space, then, for every $x \in X$, there exist $y \in \operatorname{Min} X$ and $z \in \operatorname{Max} X$ such that $y \leqq x \leqq z$ and, for every closed set Q, the subsets $\uparrow Q$ and $\downarrow Q$ are closed.

Now, we say that a Priestley space X is a *double p-space* if $\uparrow U$ and $\downarrow V$ are open (and therefore clopen), whenever U is a clopen down subset of X and V is a clopen up subset of X. Let X_1 and X_2 be double p-spaces. A map $f: X_1 \to X_2$ is said to be a *double p-morphism* if it is continuous, order-preserving and $f(\operatorname{Min} X_1 \cap \downarrow x) = \operatorname{Min} X_2 \cap \downarrow f(x)$ and $f(\operatorname{Max} X_1 \cap \uparrow x) = \operatorname{Max} X_2 \cap \uparrow f(x)$, for every $x \in X_1$. Next, notice that the category of distributive double p-algebras together with homomorphisms is dually equivalent to the category of double p-spaces together with double p-morphisms. Given a double p-space X, its dual algebra $\mathcal{O}(X)$ is the distributive double p-algebra whose elements are the clopen down subsets of X and whose operations are the intersection, union, $\emptyset$, X and $*$ and $+$ defined as follows: $U^* = X - \uparrow U$ and $U^+ = \downarrow(X-U)$, for every clopen down set U. Given a distributive double p-algebra L, its dual space is $(X; \tau, \leqq)$ where X is the set of prime ideals of L, the topology τ has as a sub-basis $\{X_a \mid a \in L\} \cup \{X - X_a \mid a \in L\}$ (for each $a \in L$, $X_a = \{I \in X \mid a \notin I\}$) and the order is the inclusion. If X is a double p-space, then a subset Q of X is called a *double p-subset* if $\uparrow(Q \cap \operatorname{Min} X) \subseteq Q$ and $\downarrow(Q \cap \operatorname{Max} X) \subseteq Q$. There is an isomorphism ψ between the congruence lattice of $L \in \mathbf{B}_\omega^\omega$ and the lattice of open double p-subsets of its dual space X which assigns to each congruence θ the open double p-subset $\bigcup_{(a,b)\in\theta}(X_a - X_b)$. If Q is an open double p-subset of X, then $(a,b) \in \psi^{-1}(Q)$ if and only if $X_a - Q = X_b - Q$. We ought to observe that in [4] and [6] the authors represent the congruences by closed sets whereas we do it by open sets; we just take the complements.

Let X be a double p-space. In [8, Lemma 1, Corollary 6], Priestley showed that $\operatorname{Min} X$ is closed in X and if C is a clopen subset of X, then $\uparrow(C \cap \operatorname{Min} X)$ is clopen. Similarly, we can prove that $\operatorname{Max} X$ is closed in X and $\downarrow(C \cap \operatorname{Max} X)$ is clopen, whenever C is clopen. We denote by $\operatorname{Mid} X$ the open double p-subset $X-(\operatorname{Min} X \cup \operatorname{Max} X)$. This set plays an important role since it represents the determination congruence of $\mathcal{O}(X)$, i.e., for all $U, V \in \mathcal{O}(X)$,

$$U-\operatorname{Mid} X = V-\operatorname{Mid} X \Leftrightarrow U^* = V^* \text{ and } U^+ = V^+.$$

There is an isomorphism between the lattice of filters of $L \in \mathbf{B}_\omega^\omega$ and the lattice of open up subsets of its dual space X which assigns to each filter F of L the open up set $\Gamma_F = \bigcup_{a \in F}(X - X_a)$. By [6, Theorem 2.3.5], the filter F is normal (i.e., $a^{+*} \in F$, whenever $a \in F$) if and only if Γ_F is a down set.

By [7, Proposition 3], if X is the dual space of a double Stone algebra, then, for every $x \in X$, there is a unique $m(x) \in \operatorname{Min} X$ such that $m(x) \leqq x$; similarly, we prove that there is a unique $M(x) \in \operatorname{Max} X$ such that $x \leqq M(x)$. Notice that the partial order we defined on X is the reverse of that used in [7].

If L is a finite distributive double p-algebra, we may consider its dual space to be the pair $(J(L); \leqq)$, where $J(L)$ is the set of nonzero join irreducible elements of L and $\leqq$ is the partial order induced by the lattice order. Here we drop the topology as it is the discrete one.

3. Double p-algebras

Let X be a double p-space and $C \subseteq X$. For $n \in N_0$, we define $B_n(C)$, $B'_n(C)$ and $(\uparrow\downarrow)^n(C)$ as follows:

$$B_0(C) = B'_0(C) = C, (\uparrow\downarrow)^0(C) = C,$$

$$B_{n+1}(C) = \downarrow(B'_n(C) \cap \operatorname{Max} X), B'_{n+1}(C) = \uparrow(B_n(C) \cap \operatorname{Min} X) \text{ and } (\uparrow\downarrow)^{n+1}(C) = \uparrow\downarrow((\uparrow\downarrow)^n(C)).$$

We define $(\downarrow\uparrow)^n(C)$ similarly. For every $n \in N$, it is obvious that $B_n(C)$ is a down set and $B'_n(C)$ is an up set.

Lemma 3.1. *Let X be a double p-space and $C \subseteq X$.*

(i) *If C is a down set, then $\uparrow(C \cap \operatorname{Min} X) = \uparrow C$;*

(iii) *If C is an up set, then $\downarrow(C \cap \operatorname{Max} X) = \downarrow C$;*

(iii) *For every $n \in N$, $B'_n(C) \subseteq B_{n+1}(C)$ and $B_n(C) \subseteq B'_{n+1}(C)$;*

(iv) *For every $n \geqq 3$, $B_n(C) = \downarrow\uparrow B_{n-2}(C)$;*

(v) *If C is an up set, then $\bigcup_{n \in N_0} B_n(C) = \bigcup_{n \in N_0} (\uparrow\downarrow)^n(C)$;*

(vi) *If C is clopen, then, for every $n\in N_0$, $B_n(C)$ is clopen and $\bigcup_{n\in N_0} B_n(C)$ is open;*

(vii) $Q=\bigcup_{n\in N_0} B_n(C)$ *is a double p-subset of X;*

(viii) *If X is the dual space of a double Stone algebra, then* $\bigcup_{n\in N_0} B_n(C)=\bigcup_{n=0}^{2} B_n(C)$.

Proof. (i) Let $y\in\uparrow C$. There exist $c\in C$ and $m\in \operatorname{Min} X$ such that $m\leqq c\leqq y$. Since C is a down set, the element $m\in C\cap \operatorname{Min} X$. Therefore, $y\in\uparrow(C\cap \operatorname{Min} X)$.

The proof of (ii) is similar.

(iii) Let $n\in N$. By the definitions of the Bs, B's and (i) and (ii),

$$B_{n+1}(C)=\downarrow(B'_n(C)\cap \operatorname{Max} X)=\downarrow B'_n(C)\supseteq B'_n(C),$$

$$B'_{n+1}(C)=\uparrow(B_n(C)\cap \operatorname{Min} X)=\uparrow B_n(C)\supseteq B_n(C).$$

(iv) Let $n\geqq 3$. By the definitions of the Bs, B's and (i) and (ii), we get

$$B_n(C)=\downarrow(B'_{n-1}(C)\cap \operatorname{Max} X)=\downarrow B'_{n-1}(C)=\downarrow\uparrow(B_{n-2}(C)\cap \operatorname{Min} X)=\downarrow\uparrow B_{n-2}(C).$$

(v) Let C be an up set. First we prove, inductively, that, for every $n\in N$, we have $B_{2n}(C)\subseteq B_{2n-1}(C)$. Since C is an up set, then, by the definition of the Bs, B's and (ii),

$$B_2(C)=\downarrow(B'_1(C)\cap \operatorname{Max} X)=\downarrow B'_1(C)=\downarrow\uparrow(C\cap \operatorname{Min} X)\subseteq\downarrow\uparrow C=\downarrow C=\downarrow(C\cap \operatorname{Max} X)=B_1(C).$$

Suppose

$B_{2n}(C)\subseteq B_{2n-1}(C)$. By (iv), we have $B_{2n+2}(C)=\downarrow\uparrow B_{2n}(C)\subseteq\downarrow\uparrow B_{2n-1}(C)=B_{2n+1}(C)$.

Now, applying (ii) and (iv),

$$\bigcup_{n\in N_0} B_n(C)=\bigcup_{n\in N_0} B_{2n+1}(C)=\bigcup_{n\in N_0} (\downarrow\uparrow)^n(\downarrow C)=\bigcup_{n\in N_0} (\uparrow\downarrow)^n(C).$$

(vi) By hypothesis, $B_0(C)=B'_0(C)=C$ is clopen. Now, it is clear, from the observations in Section 2, that if $B_n(C)$ and $B'_n(C)$ are clopen, so are $B_{n+1}(C)$ and $B'_{n+1}(C)$. Therefore, for every $n\in N_0$, the set B_n is clopen and the set $\bigcup_{n\in N_0} B_n(C)$ is open.

(vii) The set

$$\uparrow(Q\cap \operatorname{Min} X)=\uparrow\left(\left(\bigcup_{n\in N_0} B_n(C)\right)\cap \operatorname{Min} X\right)=\bigcup_{n\in N_0}\uparrow(B_n(C)\cap \operatorname{Min} X)=\bigcup_{n\in N} B'_n(C)$$

is contained in Q, by (iii). And

$$\begin{aligned}\downarrow(Q\cap \operatorname{Max} X)&=\downarrow\left(\left(\bigcup_{n\in N_0} B_n(C)\right)\cap \operatorname{Max} X\right)=\bigcup_{n\in N_0}\downarrow(B_n(C)\cap \operatorname{Max} X)\\&\subseteq\downarrow(B_0(C)\cap \operatorname{Max} X)\cup\bigcup_{n\in N}\downarrow B_n(C)\\&=\downarrow(B'_0(C)\cap \operatorname{Max} X)\cup\bigcup_{n\in N} B_n(C)\\&=\bigcup_{n\in N} B_n(C)\subseteq Q.\end{aligned}$$

(viii) It is sufficient to show that $B_n(C) \subseteq B_{n-2}(C)$, for $n \geqq 3$. Let $n \geqq 3$ and $y \in B_n(C)$. By the definitions of $B_n(C)$, $B'_{n-1}(C)$ and $B_{n-2}(C)$, there are elements $x \in B'_{n-1}(C) \cap \operatorname{Max} X$, $z \in B_{n-2}(C) \cap \operatorname{Min} X$ and $t \in B'_{n-3}(C) \cap \operatorname{Max} X$ such that $z \leqq t$, $z \leqq x$ and $y \leqq x$. But $t, x \in \operatorname{Max} X$, and so $t = x$ and $y \in \downarrow(B'_{n-3}(C) \cap \operatorname{Max} X) = B_{n-2}(C)$. □

Proposition 3.2. *Let $L \in \mathbf{B}_\omega^\omega$ and $a, b \in L$, $a \leqq b$.*

(i) *The congruence $\theta(a,b)$ is represented by $Q = \bigcup_{n \in N_0} B_n(X_b - X_a)$;*

(ii) *If L is a double Stone algebra, then $\theta(a,b)$ is represented by the clopen set $Q = \bigcup_{n=0}^{2} B_n(X_b - X_a)$.*

Proof. (i) First notice that the open double p-subset of X that represents $\theta(a,b)$ must contain

$$Q = \bigcup_{n \in N_0} B_n(X_b - X_a).$$

Hence, it is sufficient to show that Q is an open double p-subset of X but this follows from Lemma 3.1 (vi) and (vii).

(ii) This is an immediate consequence of (i) and Lemma 3.1 (viii), (vi). □

Proposition 3.3. *Let $L \in \mathbf{B}_\omega^\omega$ and X be its dual space.*

(i) *The subset Q of X represents a principal congruence if and only if there is a clopen convex set C such that $Q = \bigcup_{n \in N_0} B_n(C)$;*

(ii) *If L is a double Stone algebra, then the subset Q of X represents a principal congruence if and only if there is a clopen convex set C such that $Q = \bigcup_{n=0}^{2} B_n(C)$.*

Proof. (i) Suppose that Q represents a principal congruence $\theta(a,b)$. We may suppose that $a \leqq b$. By Proposition 3.2, we have $Q = \bigcup_{n \in N_0} B_n(X_b - X_a)$ and $X_b - X_a$ is clopen and convex.

Conversely, let $Q = \bigcup_{n \in N_0} B_n(C)$, for some clopen convex set C. Then Q is an open double p-subset, by Lemma 3.1 (vi) and (vii). Now, the result follows as in Lemma 3 of [1].

(ii) This follows from (i) and Lemma 3.1 (viii). □

Proposition 3.4. *If $L \in \mathbf{B}_\omega^\omega$ and L has PC, then D is an open up-down subset of its dual space X if and only if there is a clopen up set A such that $D = \bigcup_{n \in N_0} (\uparrow\downarrow)^n(A)$.*

Proof. Let D be an open up-down set, then D represents a congruence on L which must be principal. By Proposition 3.3 (i), there is a clopen convex set C such that $D = \bigcup_{n \in N_0} B_n(C)$. However, D is an open up-down set, and so it also represents a filter F of L, that is, $D = \bigcup_{a \in F} (X - X_a)$. The clopen C is contained in $\bigcup_{a \in F} (X - X_a)$. Since X is compact, there exist $t \in N_0$ and $a_1, \ldots, a_t \in F$ such that C is contained in

$\bigcup_{1 \leqq i \leqq t}(X - X_{a_i}) = X - X_{a_1 \wedge \ldots \wedge a_t}$. Let $b = a_1 \wedge \ldots \wedge a_t$ and $A = X - X_b$. Then the element $b \in F$ and D is up and down, hence $\bigcup_{n \in N_0}(\uparrow\downarrow)^n(A) \subseteq D$. By applying Lemma 3.1 (v) to the up set A, we obtain $D = \bigcup_{n \in N_0} B_n(C) \subseteq \bigcup_{n \in N_0} B_n(A) = \bigcup_{n \in N_0}(\uparrow\downarrow)^n(A)$.

Now, let $D = \bigcup_{n \in N_0}(\uparrow\downarrow)^n(A)$, where A is a clopen up set. Clearly D is up and down, hence, by Lemma 3.1 (v) and (vi), D is open. □

Proposition 3.5. *Let $L \in \mathbf{B}_\omega^\omega$ and X be its dual space. If L has PC, then every subset of* Mid X *is clopen and convex.*

Proof. First we show that if $Q \subseteq Mid\, X$ and Q is open in X, then Q is clopen and convex. Let $Q \subseteq \operatorname{Mid} X$ and Q be open. Then Q represents a congruence on L which must be principal. By Proposition 3.3 (i), there is a clopen convex set C such that $Q = \bigcup_{n \in N_0} B_n(C)$. But, for every $n \in N$, we have $B_n(C) = \emptyset$. Therefore $Q = C$ and Q is clopen and convex.

To complete the proof it remains to show that every subset of Mid X is open. Let $x \in \operatorname{Mid} X$, the set $\operatorname{Mid} X - \{x\}$ is open and so it is clopen. Thus $\{x\} = \operatorname{Mid} X - (\operatorname{Mid} X - \{x\})$ is open. □

Corollary 3.6. *Let $L \in \mathbf{B}_\omega^\omega$ and X be its dual space. If L has PC, then* Mid X *is finite.*

Proof. By Proposition 3.5, the set Mid X is closed and every subset of Mid X is clopen in X and so clopen in Mid X. Thus Mid X is a compact Hausdorff space whose subsets are clopen. Therefore, by [5, Lemma 10.9A], Mid X is finite. □

Corollary 3.7. *Let $L \in \mathbf{B}_\omega^\omega$ and X be its dual space. If L has PC, then $l(X) \leqq 3$.*

Proof. Suppose that $l(X) > 3$. Then, there are elements $a_1, a_2, a_3, a_4, a_5 \in X$ such that $a_1 < a_2 < a_3 < a_4 < a_5$. Hence, the set $\{a_2, a_4\}$ is contained in Mid X and is not convex. This contradicts Proposition 3.5. □

Lemma 3.8. *Let X be a double p-space such that $l(X) \leqq 3$ and* Mid X *is finite. Let A and T be subsets of X and* Mid X, *respectively, and $T_1 = \{x \in T \mid (\exists y \notin A \cup T)(\exists z \in A) x < y < z\}$. If A is a clopen up set and T is open, then $A \cup (T - T_1)$ is clopen and convex.*

Proof. The sets T, T_1 and $T - T_1$ are closed, since Mid X is finite and T, T_1 and $T - T_1$ are subsets of Mid X. However T is open and so $T - T_1$ is also open. Therefore, $A \cup (T - T_1)$ is clopen.

To show that $A \cup (T - T_1)$ is convex, let $u, v \in A \cup (T - T_1)$ and $x \in X$ be such that $u < x < v$. If $u \in A$, then $x \in A$, since A is an up set. Suppose that $u \in T - T_1$. Now, as $l(X) \leqq 3$ and $T - T_1 \subseteq \operatorname{Mid} X$, the element v must belong to A. If $x \notin A \cup T$, then $u \in T_1$, a contradiction. If $x \in T_1$, then there are elements $y \notin A \cup T$ and $z \in A$ such that $u < x < y < z$, contradicting the fact that $u \in T - T_1$. Thus $x \in A \cup (T - T_1)$ and so $A \cup (T - T_1)$ is convex. □

Theorem 3.9. *Let $L \in \mathbf{B}_\omega^\omega$ and X be its dual space. Then L has PC if and only if X satisfies the following conditions*

(i) $\mathrm{Mid}\, X$ *is finite*;

(ii) $l(X) \leqq 3$;

(iii) *The open up-down subsets of X are the sets of the form $\bigcup_{n \in N_0} (\uparrow\downarrow)^n(A)$, where A is a clopen up set.*

Proof. Suppose that L has PC. By Proposition 3.4 and Corollaries 3.6 and 3.7, the conditions (i), (ii) and (iii) are satisfied.

Conversely, suppose that X satisfies (i), (ii) and (iii) and let $Q \subseteq X$ be an open double p-subset of X. By [**6**, Theorems 2.3.5, 2.3.6, 2.3.7], we have $Q = Q_1 \cup (Q \cap \mathrm{Mid}\, X)$, where Q_1 is an open up-down set. Hence, there is a clopen up set A such that $Q_1 = \bigcup_{n \in N_0} (\uparrow\downarrow)^n(A)$. By Lemma 3.1 (v), we obtain $Q_1 = \bigcup_{n \in N_0} B_n(A)$. Next, notice that the set $T = Q \cap \mathrm{Mid}\, X$ is open and so, by Lemma 3.8, the set $A \cup (T - T_1)$ is clopen and convex. If $x \in T_1$, then $x \in \downarrow A = B_1(A)$, by the definition of T_1 and Lemma 3.1 (ii). Therefore

$$Q = \left(\bigcup_{n \in N_0} B_n(A) \right) \cup T = \left(\bigcup_{n \in N_0} B_n(A) \right) \cup (T - T_1) = \bigcup_{n \in N_0} B_n(A \cup (T - T_1))$$

and, by Proposition 3.3 (i), it follows that Q represents a principal congruence. Hence L has PC. □

Corollary 3.10. *Let L be a double Stone algebra and X be its dual space. Then L has PC if and only if X is finite and $l(X) \leqq 3$ (if and only if L is finite and $l(J(L)) \leqq 3$).*

Proof. Suppose that L has PC. By Theorem 3.9, we know that $\mathrm{Mid}\, X$ is finite and $l(X) \leqq 3$. Let $x \in \mathrm{Min}\, X$. The set $\uparrow x$ is closed. Consider the open set $X - \uparrow x$. It is easy to prove that $X - \uparrow x$ is a double p-subset. Thus $X - \uparrow x$ represents a congruence that must be principal. By Proposition 3.2 (ii), the set $X - \uparrow x$ is clopen and so $\uparrow x$ is open and $\{x\} = \uparrow x \cap \mathrm{Min}\, X$ is open in $\mathrm{Min}\, X$. Now, every subset of the compact Hausdorff space $\mathrm{Min}\, X$ is open and, by [**5**, Lemma 10.9A], the set $\mathrm{Min}\, X$ is finite. In a similar way we prove that $\mathrm{Max}\, X$ is also finite. Therefore, $X = \mathrm{Min}\, X \cup \mathrm{Max}\, X \cup \mathrm{Mid}\, X$ is finite.

The converse is immediate, by Theorem 3.9. □

The following examples prove that conditions (i), (ii) and (iii) of Theorem 3.9 are independent.

1. Consider L the 6-element chain with $1^+ = 0$ and $x^+ = 1$, for every $x \neq 1$; $0^* = 1$ and $x^* = 0$, for every $x \neq 0$. Its dual space X is the five element chain: $x_0 < x_1 < x_2 < x_3 < x_4$. Now, X satisfies conditions (i) and (iii), but it does not satisfy (ii). Notice that L does not have PC: $\{x_1, x_3\}$ represents a congruence which is not principal.

2. Let $\{0\}$ and $\{1\}$ be one point double p-spaces and $N\infty = \{x_n \mid n \in N \cup \{\infty\}\}$ be the double p-space defined by the partial order given by $x_i \| x_j$, for all $i, j \in N \cup \{\infty\}$, $i \neq j$, and the one-point compactification of a countable discrete space, [5, Example 10.11]. Consider $X = \{0\} \oplus N\infty \oplus \{1\}$, where $\oplus$ denotes the usual linear sum and which is endowed with the disjoint union topology [5, Exercise 10.4]. Now, X satisfies (ii) and (iii), but it does not satisfy (i). The dual double p-algebra of X does not have PC: $N\infty - \{x_\infty\}$ represents a congruence which is not principal.

3. Let $N\infty$ be the double p-space defined in the previous example. Obviously, $N\infty$ satisfies (i) and (ii), but it does not satisfy (iii): $\{x_n \mid n \in N\}$ is an open up-down subset of $N\infty$ and there is no clopen up set A of $N\infty$ such that $\{x_n \mid n \in N\} = \bigcup_{n \in N} (\uparrow\downarrow)^n(A)$. The double p-algebra L, dual of $N\infty$, does not have PC: the underlying lattice is a Boolean lattice and every lattice congruence is a congruence of L, thus by [3, Theorem 1] L does not have PC. The set $\{x_n \mid n \in N\}$ represents a congruence which is not principal.

In [2], Beazer characterizes some classes of distributive double p-algebras that have PC. In order to show that his characterizations follow from our Theorem 3.9, we must translate some properties of distributive double p-algebras into properties of its dual space and vice-versa.

Let $L \in \mathbf{B}_\omega^\omega$. The dually dense set of L is $D^+(L) = \{x \in L \mid x^+ = 1\}$ and it is an ideal of L. The core of L is $C(L) = \{x \in L \mid x^+ = 1, x^* = 0\}$ and $\mathrm{Cen}(L)$ is the Boolean lattice of the complemented elements of L. If $a \in L$ the elements $a^{n(+*)}$, with $n \in N_0$, are defined in L, inductively, as follows: $a^{0(+*)} = a$ and $a^{(n+1)(+*)} = (a^{n(+*)})^{+*}$. We say that L has finite range if, for every $a \in L$, there is $k \in N_0$ such that $a^{(k+1)(+*)} = a^{k(+*)}$, this is equivalent to $a^{k(+*)} \in \mathrm{Cen}(L)$.

Observation. *Let $L \in \mathbf{B}_\omega^\omega$ and X its dual space.*

(i) *Let $U \in \mathcal{O}(X)$. Identifying L and $\mathcal{O}(X)$ we have that*

$U \in D^+(L)$ if and only if $U \subseteq X - \mathrm{Max}\, X$;

$U \in C(L)$ if and only if $\mathrm{Min}\, X \subseteq U \subseteq X - \mathrm{Max}\, X$;

$U \in \mathrm{Cen}(L)$ if and only if U is an up set;

for all $n \in N_0$, $U^{n(+*)} = X - (\uparrow\downarrow)^n(X - U)$.

Therefore

$D^+(L)$ is principal if and only if $X - \mathrm{Max}\, X$ is closed (therefore clopen);

$C(L) \neq \emptyset$ if and only if $\mathrm{Min}\, X \subseteq X - \mathrm{Max}\, X$;

L has finite range if and only if, for every clopen up set V of X, there exists $k \in N_0$ such that $(\uparrow\downarrow)^{k+1}(V) = (\uparrow\downarrow)^k(V)$, that is $(\uparrow\downarrow)^k(V) \in \mathrm{Cen}(L)$.

(ii) *The open up-down subsets of X are the sets of the form $\bigcup_{n \in N_0} (\uparrow\downarrow)^n(A)$ where A is a clopen up set if and only if every normal filter F of L is principal (i.e., there is $a \in L$ such that $F = \{x \in L \mid (\exists n \in N_0)\ x \geqq a^{n(+*)}\}$).*

(iii) *$l(X) \leqq 3$ if and only if the poset of all prime ideals of L contains no 5-element chain.*

Proposition 3.11. *Let $L \in \mathbf{B}_{\omega}^{\omega}$ and X be its dual space. If* Mid X *is finite, then every determination class of L is finite.*

Proof. Suppose that Mid X is finite. We identify L and $\mathcal{O}(X)$. Now, let $U_0 \in \mathcal{O}(X)$. A clopen down set U is in the determination class of U_0 if and only if $U - \operatorname{Mid} X = U_0 - \operatorname{Mid} X$. Hence, since Mid X is finite, there is a finite number of elements in the determination class of U_0. □

Proposition 3.12. *Let X be the dual space of $L \in \mathbf{B}_{\omega}^{\omega}$, Suppose that there is an open set V satisfying the following conditions*

(i) $\operatorname{Min} X \cap \operatorname{Max} X \subseteq V$;

(ii) *V is contained in every clopen down set U such that* $\operatorname{Min} X \subseteq U$.

If every determination class of L is finite, then Mid X *is finite.*

Proof. Let us identify L and $\mathcal{O}(X)$. Suppose that Mid X is infinite. Either Mid X does not satisfy, at least, one of the chain conditions or Mid X has an infinite antichain. First, suppose that Mid X does not satisfy the descending chain condition, i.e., there are $y_n \in \operatorname{Mid} X$, with $n \in N$, such that $y_{n+1} < y_n$. For each $n \in N$, consider

$$\Gamma_n = \operatorname{Min} X \cup \{y_{n+1}\} \text{ and } \Delta_n = (\operatorname{Max} X \cap (X - V)) \cup \{y_1, \dots, y_n\},$$

which are closed subsets of X. For all $x \in \Gamma_n$ and $z \in \Delta_n$, we have $z \nleq x$. The total order disconnectedness and compactness provides, for each $n \in N$, a clopen down set U_n such that $\Gamma_n \subseteq U_n \subseteq X - \Delta_n$. Now let m, n be distinct natural numbers. It is obvious that $U_n \neq U_m$. Next notice that if $x \in \operatorname{Min} X$, then $x \in U_n$ and $x \in U_m$. Also, if $x \in \operatorname{Max} X - \operatorname{Min} X$, then, by [5, Lemma 10.16], there is a clopen down set U such that $\operatorname{Min} X \subseteq U$ and $x \notin U$. Hence, by (2), we have that $x \notin V$ and then $x \notin U_n \cup U_m$. Thus, $U_n - \operatorname{Mid} X = U_m - \operatorname{Mid} X$. Therefore, the clopen down sets U_n, with $n \in N$, are in the same determination class, a contradiction.

Second, let us assume that Mid X does not satisfy the ascending chain condition or that Mid X has an infinite antichain. Then there are $y_n \in \operatorname{Mid} X$, with $n \in N$, such that $y_{n+1} \nleq y_n$. For each $n \in N$, consider $\Gamma_n = \operatorname{Min} X \cup \{y_1, \dots, y_n\}$ and $\Delta_n = (\operatorname{Max} X \cap (X - V)) \cup \{y_{n+1}\}$. Reasoning as before leads us once again to a contradiction. □

At this point we observe that if L is such that $C(L) \neq \emptyset$, then we may consider $V = \emptyset$ and so it is obvious that the sets $U_n, n \in N$, of last proof are in $C(L)$. Thus we have the following

Corollary 3.13. *Let $L \in \mathbf{B}_{\omega}^{\omega}$ be such that $C(L) \neq \emptyset$ and let X be its dual space. If $C(L)$ is finite, then* Mid X *is finite.*

Corollary 3.14. *Let $L \in \mathbf{B}_{\omega}^{\omega}$ be such that $D^+(L)$ is principal and let X be its dual space. If every determination class of L is finite, then* Mid X *is finite.*

Proof. If $D^+(L)$ is principal, then $X-\operatorname{Max} X$ is clopen and so is $\uparrow((X-\operatorname{Max} X)\cap \operatorname{Min} X)$. As $\uparrow((X-\operatorname{Max} X)\cap \operatorname{Min} X)=X-(\operatorname{Min} X\cap \operatorname{Max} X)$, we have that $\operatorname{Min} X \cap \operatorname{Max} X$ is clopen and so it is open. The result now follows from Proposition 3.12, by taking $V=\operatorname{Min} X\cap \operatorname{Max} X$. □

Corollary 3.15. *Let $L\in \mathbf{B}_\omega^\omega$ be such that L has finite range and* $\operatorname{Cen}(L)$ *is finite and let X be the dual space of L. Then if every determination class of L is finite so is* $\operatorname{Mid} X$.

Proof. We identify L and $\mathcal{O}(X)$. Let $I=\{U\in\mathcal{O}(X)\mid \operatorname{Min} X\subseteq U\}$. For every $U\in I$, there is $k(U)\in N_0$ such that $U^{k(U)(+*)}\in \operatorname{Cen}(L)$. Consider $\mathscr{A}=\{U^{k(U)(+*)}\mid U\in I\}$ and $V=\cap\mathscr{A}$. Since $\operatorname{Cen}(L)$ is finite, so is $\mathscr{A}$. Thus V is an open subset of X. If $U\in I$, then $V=\cap\mathscr{A}\subseteq U^{k(U)(+*)}\subseteq U$ and

$$U^{k(U)(+*)}=X-(\uparrow\downarrow)^{k(U)}(X-U)\supseteq X-(\uparrow\downarrow)^{k(U)}(X-\operatorname{Min} X)\supseteq \operatorname{Min} X\cap \operatorname{Max} X.$$

Therefore, $V=\cap\mathscr{A}\supseteq \operatorname{Min} X\cap\operatorname{Max} X$ and, by Proposition 3.12, the set $\operatorname{Mid} X$ is finite. □

Proposition 3.16. *Let $L\in\mathbf{B}_\omega^\omega$ be such that L has finite range. Then the following conditions are equivalent*

(i) *every normal filter of L is principal;*
(ii) *every open up-down subset of the dual space of L is closed;*
(iii) $\operatorname{Cen}(L)$ *is finite.*

Proof. Let X be the dual space of L and identify L with $\mathcal{O}(X)$.

(i)$\Rightarrow$(ii) Let Γ be an open up-down subset of X. Then, by Observation (ii), there is a clopen up set A such that $\Gamma=\bigcup_{n\in N_0}(\uparrow\downarrow)^n(A)$. Since Γ has finite range, we have $\Gamma=(\uparrow\downarrow)^k(A)$, for some $k\in N_0$. Thus Γ is closed in X.

(ii)$\Rightarrow$(i) This is obvious, by Observation (ii).

(ii)$\Rightarrow$(iii) We prove that the Boolean lattice $\operatorname{Cen}(L)$ satisfies the ascending chain condition and, that, consequently, $\operatorname{Cen}(L)$ is finite. Let $U_n, n\in N$, be clopen up-down subsets of X such that $U_n\subseteq U_{n+1}$. Consider the open up-down set $\Gamma=\bigcup_{n\in N}U_n$. The set Γ must be closed. Now, since X is compact and $\{U_n\mid n\in N\}$ is a chain, we have $\Gamma=U_s$, for some $s\in N$. Therefore, $\operatorname{Cen}(L)$ satisfies the ascending chain condition.

(iii)$\Rightarrow$(ii) Let Γ be an open up-down subset of X and $I=\{U\in\mathcal{O}(X)\mid X-U\subseteq\Gamma\}$. For each $U\in I$, there is $k(U)\in N$ such that $(\uparrow\downarrow)^{k(U)}(X-U)\in\operatorname{Cen}(L)$. Let $V_U=(\uparrow\downarrow)^{k(U)}(X-U)$. The centre of L contains $\{V_U\mid U\in I\}$, which must be finite. Consider $A=\bigcup_{U\in I}V_U$. This is a clopen up-down set contained in Γ. Now let $x\in\Gamma$. The set $X-\Gamma$ is a closed down set and $x\notin X-\Gamma$. Then, by [5, Lemma 10.16], there is a clopen down set U such that $X-\Gamma\subseteq U$ and $x\notin U$, i.e., $x\in X-U$ and $U\in I$. Thus $x\in A$ and $\Gamma=A$. Therefore, Γ is closed in X, as required. □

Corollary 3.17. [2, Theorem 4.3] *Let $L \in \mathbf{B}_\omega^\omega$ be such that $D^+(L)$ is a principal ideal. Then L has PC if and only if*

(i) *every normal filter of L is principal;*
(ii) *every determination class of L is finite;*
(iii) *there is no 5-element chain in the poset of prime ideals of L.*

Proof. This is an immediate consequence of Theorem 3.9, Observation (ii) and (iii), Proposition 3.11 and Corollary 3.14. □

In [2, Corollary 4.4], Beazer characterizes the distributive double p-algebras with finite range and $D^+(L)$ principal that have PC. In the next corollary, we show that requiring $D^+(L)$ principal is not necessary. In fact we are able to describe, in the same algebraical way, all the distributive double p-algebras with the finite range that have PC.

Corollary 3.18. *Let $L \in \mathbf{B}_\omega^\omega$ be such that L has finite range. Then L has PC if and only if*

(i) Cen(L) *and every determination class are finite;*
(ii) *there is no 5-element chain in the poset of all prime ideals of L.*

Proof. It follows from Theorem 3.9, Observation (ii) and (iii), Propositions 3.11 and 3.16 and Corollary 3.15. □

Corollary 3.19. *Let $L \in \mathbf{B}_\omega^\omega$ be such that its p-algebra reduct is Stone. Then L has PC if and only if conditions* (*i*) *and* (*ii*) *in the statement of Corollary 3.18 hold.*

Proof. We start by noticing that if the p-algebra reduct of L is Stone, then L has finite range and then we apply Corollary 3.18. □

In [2, Corollary 4.5] not only conditions (i) and (ii) are required but also is the existence of an element $t \in D^+(L)$ such that $D^+(L) \subseteq (t^{**}]$. However, we observe that if L is a distributive double p-algebra whose p-algebra reduct is Stone and Cen(L) is finite, there is always an element $t \in D^+(L)$ such that $D^+(L) \subseteq (t^{**}]$. In fact, if we identify L and $\mathcal{O}(X)$, then $I = \{U^{**} \mid U \in D^+(L)\}$ is contained in Cen(L) and so I is finite. Next, for each $\Gamma \in I$ we choose $U_\Gamma \in D^+(L)$ such that $U_\Gamma^{**} = \Gamma$. Hence $V = \bigcup_{\Gamma \in I} U_\Gamma \in D^+(L)$ and, for every $U \in D^+(L)$, we have that $U \subseteq U^{**} \subseteq V^{**}$. Therefore $D^+(L) \subseteq (V^{**}]$, as required.

In [3], Blyth and Varlet showed that a distributive lattice L has PC if and only if L is finite and $l(J(L)) \leqq 1$. Therefore the next corollary corresponds to Theorem 4.9 of [2].

Corollary 3.20. [2, Theorem 4.9] *Let $L \in \mathbf{B}_\omega^\omega$ be such that $C(L) = \emptyset$. Then L has PC if and only if*

(i) *every normal filter of L is principal;*
(ii) *$C(L)$ is finite and $l(J(C(L))) \leqq 1$.*

Proof. First we prove that Mid X is finite and $l(X)\leqq 3$ if and only if $C(L)$ is finite and $l(J(C(L)))\leqq 1$. Let us assume that Mid X is finite and $l(X)\leqq 3$. By Proposition 3.11, we have that $C(L)$ is finite as $C(L)$ is a determination class. Suppose that $l(J(C(L)))>1$. Let U_n, with $1\leqq n\leqq 3$, be elements of $J(C(L))$ such that $U_1\subset U_2\subset U_3$. For $n=2,3$, consider the closed down set $V_n=\downarrow(U_n-U_{n-1})\cup \text{Min}\,X$. Each V_n is open, since $V_n=X-(\text{Max}\,X\cup Y_n)$ for some $Y_n\subseteq \text{Mid}\,X$ and Mid X is finite and so is Y_n. Thus $V_n\in C(L)$. Obviously, $U_n=V_n\cup U_{n-1}$, for $n=2,3$. As $U_n\in J(C(L))$, for $1\leqq n\leqq 3$, we have

$$\text{Min}\,X\subset U_1, U_3\subseteq X-\text{Max}\,X, U_2=V_2 \quad\text{and}\quad U_3=V_3.$$

Next let $x_1\in U_1-\text{Min}\,X$. Since $U_1\subset U_2=V_2=(\downarrow(U_2-U_1)\cup \text{Min}\,X)$, there is $x_2\in U_2-U_1$ such that $x_1<x_2$. Similarly, there is $x_3\in U_3-U_2$ such that $x_2<x_3$. On the other hand, every element of X contains a minimal one and it is contained in a maximal one. Therefore, $l(X)\geqq 4$, which is a contradiction.

Conversely, suppose that $C(L)$ is finite and $l(J(C(L)))\leqq 1$. By Corollary 3.13, it follows that Mid X is finite. Suppose that $l(X)>3$. Let x_n, with $1\leqq n\leqq 5$, be elements of X such that $x_1<x_2<x_3<x_4<x_5$. For $n\in\{2,3,4\}$, consider the closed down set $U_n=(\downarrow x_n)\cup \text{Min}\,X$. Now each U_n is open, since $U_n=X-(\text{Max}\,X\cup Y_n)$ for some $Y_n\subseteq \text{Mid}\,X$. Thus, it is obvious that $U_n\in J(C(L))$, for $n\in\{2,3,4\}$, and $U_2\subset U_3\subset U_4$, a contradiction.

Finally, the result follows from Theorem 3.9 and Observation (ii). □

Corollary 3.21. [2, Corollary 4.10] *Let $L\in \mathbf{B}_\omega^\omega$ be such that L has finite range and $C(L)\neq\emptyset$. Then L has PC if and only if*

(i) Cen(L) *is finite;*

(ii) $C(L)$ *is finite and* $l(J(C(L)))\leqq 1$.

Proof. This is an immediate consequence of Corollary 3.20 and Proposition 3.16. □

REFERENCES

1. M. E. Adams, Principal congruences in de Morgan algebras, *Proc. Edinburgh Math. Soc.* **30** (1987), 415–421.

2. R. Beazer, Some p-algebras and double p-algebras having only principal congruences, *Glasgow Math. J.* **34** (1992), 157–164.

3. T. S. Blyth and J. C. Varlet, Principal congruences on some lattice-ordered algebras, *Discrete Math.* **81** (1990), 323–329.

4. B. A. Davey, Subdirectly irreducible distributive double p-algebras, *Algebra Universalis* **8** (1978), 73–88.

5. B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order* (Cambridge University Press, Cambridge, 1990).

6. M. S. Goldberg, *Distributive p-algebras and Ockham algebras: a Topological Approach* (Ph.D. thesis, La Trobe University, Bundoora, Australia, 1979).

7. H. A. Priestley, Stone lattices: a topological approach, *Fund. Math.* **84** (1974), 127–143.

8. H. A. PRIESTLEY, The construction of spaces dual to pseudocomplemented distributive lattices, *Quart. J. Oxford* **26** (1975), 215–228.

9. J. VAZ DE CARVALHO, Congruências principais em álgebras de Stone duplas, *Actas XV Jornadas Luso-Espanholas de Matemática* **1** (1990), 49–54.

CENTRO DE ÁLGEBRA DA UNIVERSIDADE DE LISBOA
AV. PROF. GAMA PINTO, 2
1699 LISBOA CODEX
PORTUGAL

https://doi.org/10.1017/S0013091500023257 Published online by Cambridge University Press