AN EXTENSION OF A THEOREM OF GORDON

by E. M. WRIGHT†

(Received 7 September, 1964)

In what follows all small Latin letters denote non-negative integers or functions whose values are non-negative integers. Let $N = (n_1, \ldots, n_j)$ be a j-dimensional vector and let $q = q(k;\ N) = q(k;\ n_1, \ldots, n_j)$ be the number of partitions of N into just k parts, each part being a vector whose components are non-negative integers. We write

$$Q_j(k) = Q_j(k;\ X_1, \ldots, X_j) = \sum_{n_1, \ldots, n_j = 0}^{\infty} q(k;\ n_1, \ldots, n_j) X_1^{n_1} \ldots X_j^{n_j}$$

for the generating function of q. We have

$$F_j = \prod_{h_1, \ldots, h_j = 0}^{\infty} (1 - X_1^{h_1} \ldots X_j^{h_j} Y)^{-1} = 1 + \sum_{k=1}^{\infty} Q_j(k) Y^k.$$

It is well known [3] that

$$F_1 = \prod_{h=0}^{\infty} (1 - X_1^h Y)^{-1} = 1 + \sum_{k=1}^{\infty} Y^k \prod_{s=1}^{k} (1 - X_1^s)^{-1},$$

so that

$$Q_1(k) = \prod_{s=1}^{k} (1 - X_1^s)^{-1} = U(X_1)$$

(say), but until 1956 the form of $Q_j(k)$ for $j > 1$ was not known. Carlitz [1] and I [4] showed independently that

$$Q_j(k) = P_j(k;\ X_1, \ldots, X_j) \prod_{i=1}^{j} U(X_i). \tag{1}$$

(Carlitz dealt only with $j = 2$ but this case presents the essential difficulties.) Here $P = P_j = P_j(k)$ is a polynomial in the X, in which no term consists of a power of a single X_i only. Thus $P_1 = 1$ but, when $j > 1$, P_j is of degree $g = \frac{1}{2}k(k-1)$ in each X_i, so that

$$P_j = \sum_{h_1, \ldots, h_j = 0}^{g} \lambda(h_1, \ldots, h_j) X_1^{h_1} \ldots X_j^{h_j}.$$

Hence, by (1),

$$q(k;\ n_1, \ldots, n_j) = \sum_{h_1, \ldots, h_j = 0}^{g} \lambda(h_1, \ldots, h_j) \prod_{i=1}^{j} q(1;\ n_i - h_i). \tag{2}$$

† The research reported herein has been sponsored by the Office, Chief of Research and Development, U.S. Department of the Army, through its European Research Office under contract DA-91-591-EUC-3256.

In [4] I conjectured that the λ are non-negative. Recently Gordon [2] proved this conjecture, essentially by finding the combinatorial interpretation of (2). I have nothing to add to his elegant proof of this result. But he goes on (by a quite different argument) to prove that

$$P_j(k;\ \xi, \eta, X_3, \ldots, X_j) = 0, \tag{3}$$

where ξ, η are primitive uth and tth roots of unity respectively and $1 \leqq u < t \leqq k$. For this purpose he uses a recurrence relation for the $P_j(k)$, which both Carlitz [1] and I [4] found.

There is another expression for $P_j(k)$, which I found in [4] and which appears at first sight to be rather unpromising. In fact, however, it has proved [5, 7] unexpectedly useful to calculate explicit formulae for $q_j(k)$ for general j and not too large k and also asymptotic formulae for large n_i and all k. Recently [6] I found the combinatorial explanation of this expression. Here I use the expression to give an alternative proof of Gordon's result (3) and to take this particular approach to the problem of the form of $P_j(k)$ somewhat further.

We write

$$\beta(m) = \prod_{i=1}^{j} (1 - X_i^m), \qquad \gamma(m) = \prod_{i=1}^{j} \prod_{\rho} (1 - \rho X_i),$$

where ρ runs through all primitive mth roots of unity. Thus

$$\beta(m) = \prod_{d \mid m} \gamma(d).$$

Again $\pi = \pi(k)$ denotes the partition of k into $h(1)$ parts 1, $h(2)$ parts 2, and so on, and $\sum_{\pi(k)}$ denotes summation over all partitions π of k. Then (6) and (9) of [4] give us

$$P_j(k) = \sum_{\pi(k)} \Omega(\pi),$$

where

$$\Omega(\pi) = \left\{ \prod_{h=1}^{k} \beta(h) \right\} \Big/ \prod_{m}^{k} \{ h(m)!\, (m\beta(m))^{h(m)} \},$$

a polynomial in the X.

Let $1 \leqq u \leqq k$ and write $v = [k/u]$ and $k = uv + w$, so that $0 \leqq w < u$. We consider separately those partitions π_1 of k which have v parts u and the remaining partitions π_2 in which there are at most $v-1$ parts u. We have

$$P_j(k) = \sum_{\pi_1} \Omega(\pi_1) + \sum_{\pi_2} \Omega(\pi_2) = S_1 + S_2$$

(say). In the numerator of $\Omega(\pi_2)$, the factor $\gamma(u)$ occurs just v times (once in $\beta(h)$ for $h = u$, $2u$, $3u, \ldots, vu$), while it occurs at most $v-1$ times in the denominator. Hence $\Omega(\pi_2)$ has the factor $\gamma(u)$. Thus

$$S_2 = \sum_{\pi_1} \Omega(\pi_2) = \gamma(u) T_2,$$

https://doi.org/10.1017/S2040618500035152 Published online by Cambridge University Press

where T_2 is a polynomial in the X. Again

$$S_1 = \sum_{\pi_1} \Omega(\pi_1)$$
$$= (v!)^{-1}\{u\beta(u)\}^{-v} \prod_{h=1}^{k} \beta(h) \sum_{\pi(w)} \prod_{m} \{h(m)!\}^{-1}\{m\beta(m)\}^{-h(m)}$$
$$= (v!)^{-1}\{u\beta(u)\}^{-v} P_j(w) \prod_{h=w+1}^{k} \beta(h) = T_1 P_j(w),$$

where T_1 is a polynomial in the X. If $u < t \leqq k$, then $\gamma(t)$ is a factor of $\prod_{h=w+1}^{k} \beta(h)$, but not of $\beta(u)$. Hence $\gamma(t)$ is a factor of T_1. Thus, if ξ is a root of $\gamma(u)$ and η a root of $\gamma(t)$, we have

$$S_2(\xi, X_2, \ldots) = 0, \quad S_1(X_1, \eta, X_3, \ldots) = 0, \quad P_j(k;\ \xi, \eta, X_3, \ldots) = 0,$$

which is Gordon's result.

By a fairly obvious extension of our argument, we find more generally that, if

$$1 \leqq u_1 < u_2 < \ldots < u_a \leqq k, \quad v_b = [k/u_b], \quad w_b = k - u_b v_b,$$

then

$$P_j(k) = \sum_{b=1}^{a} \frac{P_j(w_b) \prod_{h=w_b+1}^{k} \beta(h)}{v_b! \{u_b \beta(u_b)\}^{v_b}} + T \prod_{b=1}^{a} \gamma(u_b),$$

where T is a polynomial in the X.

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UNIVERSITY OF ABERDEEN