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THE GROWTH OF THE POSITIVE SOLUTIONS OF $Lu=0$ NEAR THE BOUNDARY OF AN INNER NTA DOMAIN

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§ 1. Introduction

Let D be a bounded domain in the Euclidean space $\boldsymbol{R}^n$ $(n \geqq 2)$ and L a uniformly elliptic partial differential operator of second order with α-Hölder continuous coefficients $(0 < \alpha \leqq 1)$ on D.

According to N. Suzuki [3], D is said to be associated with the cone of angle $\theta < \pi/2$ if there exist positive constants h, d_0 and $K_0 \geqq 1$ such that:

(i) For any $z \in \partial D$, there exists $e_z \in \boldsymbol{R}^n$ with $|e_z| = 1$ such that $\Gamma_\theta(z, e_z) \subset D$, where $\Gamma_\theta(z, e_z)$ is the half cone obtained from $\{x \in \boldsymbol{R}^n; \sqrt{x_2^2 + \cdots + x_n^2} < x_1 \tan\theta,\ 0 < x_1 < h\}$ by the translation z and the rotation e_z.

(ii) Put $A_D = \{y = z + te_z \in \boldsymbol{R}^n;\ z \in \partial D,\ 0 < t < h/2\}$. Then for any $x \in D$ with $d(x) \leqq d_0$, there exist $y_x \in A_D$ and a polygonal line L_x from x to y_x such that $d(x) \leqq d(y_x)$ and the length of L_x is $\leqq K_0 d(L_x, \partial D)$.

In [4] he proved the following result:

If D is associated with a cone, there exist constants $m, m' \geqq 1$ such that for any positive solution of $Lu = 0$ in D,

$$C_u^{-1}(d(x))^m \leqq u(x) \leqq C_u(d(x))^{-m'} \tag{1}$$

with some constant $C_u \geqq 1$ depending on u, where $d(x)$ denotes the distance between x and ∂D, the boundary of D. In this paper, we shall define inner NTA (non-tangentially accessible) domains and show that for an inner NTA domain, we can choose two positive constants $m, m' \geqq 1$ satisfying (1) for all positive solutions of $Lu = 0$ in D. This is a direct extension of N. Suzuki's result. As applications of our main result, we shall establish the uniqueness theorem for L-superharmonic functions on an inner NTA domain and the Harnack inequality for inner NTA domains.

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§ 2. Preliminaries

Let D be a domain in $\boldsymbol{R}^n$. For three numbers $0 < \alpha \leqq 1$, $\lambda \geqq 1$ and $\eta \geqq 0$, we denote by $\mathscr{L}(\alpha, \lambda, \eta; D)$ the set of all uniformly elliptic differential operators L of the form

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

with

$$\lambda^{-1}|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqq \lambda|\xi|^2 ,$$

$$\sum_{i,j=1}^{n} |a_{ij}(x) - a_{ij}(y)| + \sum_{i=1}^{n} |b_i(x) - b_i(y)| + |c(x) - c(y)| \leqq \eta|x - y|^\alpha ,$$

$$\sum_{i=1}^{n} |b_i(x)| \leqq \eta \quad \text{and} \quad -\eta \leqq c(x) \leqq 0$$

for all $x, y \in D$ and $\xi \in \boldsymbol{R}^n$, where $|x - y|$ is the distance between x and y. For $L \in \mathscr{L}(\alpha, \lambda, \eta; D)$, a function u of class C^2 on D is said to be L-harmonic in D if $Lu = 0$ on D. We denote by $H_L(D)$ the set of all L-harmonic functions on D and put $H_L^+(D) = \{u \in H_L(D);\ u > 0 \text{ on } D\}$.

A lower semi-continuous function u on D is said to be L-superharmonic if u satisfies the following conditions:

(i) $-\infty < u \leqq +\infty$, $u \not\equiv +\infty$.

(ii) For any open ball B with $\bar{B} \subset D$ and any $v \in H_L(B)$ which is continuous on $\bar{B}$, we have

$$u \geqq v \text{ on } \partial B \Longrightarrow u \geqq v \text{ in } B .$$

For $x \in \boldsymbol{R}^n$ and $r > 0$, $B(x, r)$ (resp. $\mathring{B}(x, r)$) denotes the closed (resp. open) ball with center x and radius r. For an open or closed ball B, $r(B)$ denotes the radius of B.

The following Harnack inequality for L-harmonic functions plays an essential role in this paper.

Proposition 1 ([1], p. 109). *For given $\lambda \geqq 1$, $0 < \alpha \leqq 1$ and $\eta \geqq 0$, there exists a constant $K \geqq 1$ depending only on λ, α and η such that for any $x \in \boldsymbol{R}^n$, $0 < r < 1$, $L \in \mathscr{L}(\alpha, \lambda, \eta; \mathring{B}(x, r))$, $u \in H_L^+(\mathring{B}(x, r))$ and any $0 < s < 1$, we have*

$$K^{-1}(1 - s)(1 + s)^{1-n}u(x) \leqq u(y) \leqq K(1 - s)^{1-n}(1 + s)u(x) \tag{2}$$

for all $y \in B(x, sr)$.

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For a bounded domain D and $x \in D$, we denote by $d(x) = d_D(x)$ the distance between x and ∂D.

Definition 1. Let D be a bounded domain in $\boldsymbol{R}^n$, M a constant > 1 and N a positive integer. An M-Harnack chain of the length N in D is a finite sequence of closed balls $(B_j)_{j=1}^N$ contained in D such that $B_j \cap B_{j+1} \neq \emptyset$ $(j = 1, \cdots, N-1)$ and

$$M^{-1} \leqq r(B_j)/d(B_j, \partial D) \leqq M\,,$$

where $d(B_j, \partial D)$ denotes the distance between B_j and ∂D.

Let $x, y \in D$. We say that y can be connected with x by an M-Harnack chain $(B_j)_{j=1}^N$ of the length N in D if x is the center of B_1 and $y \in B_N$. For $x \in D$, we denote by $H_{M,N}(x)$ the set of the points which can be connected with x by an M-Harnack chain of the length N in D.

Definition 2. Let $M > 1$ be a constant, N a positive integer and $0 < \nu < 1$ a constant. A bounded domain D in $\boldsymbol{R}^n$ is called an (M, N, ν)-inner NTA domain if there exist a constant $r_0 > 0$ and a mapping $\Phi(z) = (z_j(z))_{j=1}^\infty$ from ∂D to sequences in D with $d(z_1(z)) \geqq r_0$ and $\lim_{j\to\infty} z_j(z) = z$ satisfying the following two conditions:

(I) For any $z \in \partial D$,

$$z_{j+1}(z) \in H_{M,N}(z_j(z)) \qquad (j = 1, 2, \cdots)$$

and

$$\sup_{z\in\partial D}\ \sup_{1\leqq j<\infty} d(z_j(z))/\nu^j < +\infty\,. \tag{3}$$

(II) For each $x \in D$, we put $R_x = \bigcup_{z\in\partial D}\{z_j(z);\ d(x) \leqq d(z_j(z))\}$. Then

$$\sup_{\substack{x\in D\\ d(x)\leqq r_0}}\ \inf_{H_{P,Q}(x)\cap R_x\neq\phi} P + Q < +\infty\,.$$

A bounded domain D in $\boldsymbol{R}^n$ is simply called an inner NTA domain if there exist $M > 1$, $0 < \nu < 1$ and a positive integer N such that D is an (M, N, ν)-inner NTA domain.

Remark 1. NTA domains (cf. [2], p. 93) are inner NTA domains. Here an NTA domain is a bounded domain in $\boldsymbol{R}^n$ such that there exist $M > 1$ and $r_0 > 0$ satisfying the following conditions:

(i) For any $z \in \partial D$ and any $r \leqq r_0$, there exists $a = a_r(z) \in D$ such that $d(a) \geqq M^{-1}r$ and $M^{-1}r \leqq |a - z| \leqq r$.

(ii) The complement of $\overline{D}$ also satisfies the condition (i).

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(iii) For any $\varepsilon > 0$ and any $x, y \in D$ such that $d(x) \geqq \varepsilon$, $d(y) \geqq \varepsilon$ and $|x-y| \leqq \delta$, there exists an M-Harnack chain from x to y whose length depends only on δ/ε.

We remark that there are inner NTA domains which are not NTA domains. For example, $D = \{(r, \theta) \in \boldsymbol{R}^2 \setminus \{0\};\ r \neq e^\theta,\ \theta < 0,\ r < 1\}$ is such a domain.

Remark 2. Put $M = \sin\theta/(1-\sin\theta)$, $N = 1$ and $\nu = 1 - \sin^2\theta$. Then the domain being associated with the cone of angle θ is an (M, N, ν)-inner NTA domain.

According to N. Suzuki [4], a bounded domain in $\boldsymbol{R}^n$ is said to be associated with the ball of radius $r > 0$ if there exist positive constants r, d_0 and $K_0 \geqq 1$ such that:

(i) For any $z \in \partial D$, there exists $e_z \in D$ with $d(e_z, z) = r$ such that $\mathring{B}(e_z, r) \subset D$.

(ii) Put $A_D = \{y = z + t(e_z - z) \in \boldsymbol{R}^n;\ z \in \partial D,\ 0 < t \leqq 2\}$. Then for any $x \in D$ with $d(x) \leqq d_0$, there exist $y_x \in A_D$ and a polygonal line L_x from x to y_x such that $d(x) \leqq d(y_x)$ and the length of L_x is $\leqq K_0 d(L_x, \partial D)$.

Remark 3. The above domain being associated with a ball is an $(M, 1, 1/(M+1))$-inner NTA domain for all $M > 1$.

§ 3. Main result

THEOREM 1. *Let $M > 1$ be a constant, N a positive integer, $0 < \nu < 1$ a constant and D an (M, N, ν)-inner NTA domain in $\boldsymbol{R}^n$. For a fixed $x_0 \in D$, we set $H_L^0(D) = \{u \in H_L^+(D);\ u(x_0) = 1\}$. Put*

$$m = m(M, N, \nu) = \frac{(2N-1)\log K^{-1}(M+1)^{n-2}(2M+1)^{1-n}}{\log \nu} \tag{4}$$

and

$$m' = m'(M, N, \nu) = \frac{(2N-1)\log K(M+1)^{n-2}(2M+1)}{-\log \nu}, \tag{5}$$

where K is the constant in Proposition 1. Then there exist positive constants C and C' such that for any $u \in H_L^0(D)$,

$$C(d(x))^m \leqq u(x) \leqq C'(d(x))^{-m'} \tag{6}$$

on D.

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Remark 4. (1) If a domain is associated with the cone of angle $\theta < \pi/2$, then $m = \log\{K^{-1}(1-\sin\theta)(1+\sin\theta)^{1-n}\}/(2\cdot\log\cos\theta)$ and $m' = -\log\{K(1-\sin\theta)^{1-n}(1+\sin\theta)\}/(2\cdot\log\cos\theta)$, which are also obtained by N. Suzuki [4].

(2) If the domain D is associated with a ball, we can choose $m = 1$ and $m' = n - 1$.

(3) In the case $n = 2$ and $L = \Delta$, Kuran-Schiff [3] obtained a more precise estimate for rather specific domains.

Proof of Theorem 1. Put $F = \{x \in D;\ d(x) \geqq r_0\}$, then F is compact in D. From Proposition 1, it follows that there exist two positive constants A_1 and A_2 depending only on D and x_0 such that for any $u \in H_L^0(D)$ and any $x \in F$,

$$A_1 \leqq u(x) \leqq A_2 .$$

For any $z \in \partial D$, we have $z_1(z) \in F$, so

$$A_1 \leqq u(z_1(z)) \leqq A_2 . \tag{7}$$

Let $z \in \partial D$. Then for any k, there exists an M-Harnack chain $(B_j)_{j=1}^N$ from $z_k(z)$ to $z_{k+1}(z)$. We choose $b_j \in B_j \cap B_{j+1}$ $(1 \leqq j \leqq N-1)$ and γ_k the polygonal line $\cup_{j=0}^{N-1} \overline{b_j b_{j+1}}$, where $b_0 = z_k(z)$, $b_N = z_{k+1}(z)$ and $\overline{b_j b_{j+1}}$ is the closed segment between b_j and b_{j+1}. Put $\gamma = \{z\} \cup (\cup_{k=1}^{\infty} \gamma_k)$. Then γ is a rectifiable curve from z to $z_1(z)$. Put $C_0 = K^{-1}(M+1)^{n-2}(2M+1)^{1-n}$ and $\tilde{C}_0 = K(M+1)^{n-2}(2M+1)$. Proposition 1 shows that for any $x \in \gamma_k$,

$$C_0^{2N-1} u(z_k(z)) \leqq u(x) \leqq \tilde{C}_0^{2N-1} u(z_k(z))$$
$$C_0^{(2N-1)k} u(z_1(z)) \leqq u(x) \leqq \tilde{C}_0^{(2N-1)k} u(z_1(z)) .$$

By (7), we have

$$A_1 C_0^{(2N-1)k} \leqq u(x) \leqq A_2 \tilde{C}_0^{(2N-1)k} .$$

By (3), there exists a positive constant β such that for all $k \geqq 1$,

$$d(z_k(z)) \leqq \beta \nu^k .$$

Then for any $x \in \gamma_k$, we have

$$d(x) \leqq C_2^{N-1} d(z_k(z)) \leqq C_2^{N-1} \beta \nu^k ,$$

where $C_2 = (2M+1)^2$. Putting $C_3 = A_1(\beta^{-1} C_2^{1-N})^m$ and $\tilde{C}_3 = A_2(\beta^{-1} C_2^{1-N})^{-m'}$, we have

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(8) $$C_3(d(x))^m \leqq u(x) \leqq \tilde{C}_3(d(x))^{-m'}$$

for all $x \in \gamma \cap D$.

Let $x \in D \backslash F$. By the condition (II) in Definition 2, there exist a constant $P > 1$, a positive integer Q and $z_k(z) \in R_x$ such that x can be connected to $z_k(z)$ by a P-Harnack chain of the length $\leqq Q$. From Proposition 1, it also follows that

(9) $$C_4^{2Q}u(z_k(z)) \leqq u(x) \leqq \tilde{C}_4^{2Q}u(z_k(z)),$$

where $C_4 = K^{-1}(P+1)^{n-2}(2P+1)^{1-n}$ and $\tilde{C}_4 = K(P+1)^{n-2}(2P+1)$.

Combining (8) and (9), we have

(10) $$C_3C_4^{2Q}(d(x))^m \leqq u(x) \leqq \tilde{C}_3\tilde{C}_4^{2Q}(d(x))^{-m'}$$

for all $x \in D \backslash F$. Put $C = C_3C_4^{2Q}$ and $C' = \tilde{C}_3\tilde{C}_4^{2Q}$. Then we have

$$C(d(x))^m \leqq u(x) \leqq C'(d(x))^{-m'}$$

for all $x \in D$, which completes the proof of our theorem.

§ 4. Applications

We apply our main result to the following uniqueness theorem for L-superharmonic functions.

THEOREM 2. *Let D be an (M, N, ν)-inner NTA domain, $L \in \mathscr{L}(\alpha, \lambda, \eta; D)$ and let m be the constant obtained in* (4). *If a non-negative L-superharmonic function u in D satisfies*

$$\liminf_{x \to z} u(x)/(d(x))^m = 0$$

for some $z \in \partial D$, then u is identically equal to 0.

Proof. Let G be the Green function on D with respect to L. Assume that there exists $x_0 \in D$ such that $u(x_0) > 0$. We can choose $r > 0$ such that $B(x_0, r) \subset D$ and $u(x) > 0$ on $B(x_0, r)$. There exists a positive measure $\mu \neq 0$ supported by $B(x_0, r/2)$ such that $G\mu(x)$ is finite continuous on D and $G\mu(x) \leqq u(x)$ on $B(x_0, r)$, where $G\mu(x) = \int G(x, y)d\mu(y)$. By the maximum principle, we have $G\mu(x) \leqq u(x)$ on D. Put $D' = D \backslash B(x_0, r)$; D' is an (M, N, ν)-inner NTA domain and the restriction of $G\mu$ to D' is L-harmonic in D'. Since $G\mu > 0$ on D', Theorem 1 shows that for any $x \in D'$, $G\mu(x) \geqq C(d'(x))^m$ with some $C > 0$, where $d'(x) = d(x, \partial D')$. Hence $u(x) \geqq C(d'(x))^m$ for all $x \in D'$, which contradicts our assumption. Thus Theorem

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2 is proved.

The following theorem is a generalization of the Harnack inequality on a ball.

THEOREM 3. *Let D and L be the same as in Theorem* 1 *and let m and m' be the constants obtained in* (4) *and* (5). *Then there exist positive constants C and C' such that for any $u \in H_L^+(D)$ and any relatively compact open subset Ω of D,*

$$C(d_\Omega)^m \leqq u(y)/u(x) \leqq C'(d_\Omega)^{-m'}$$

for all $x, y \in \Omega$, where $d_\Omega = d(\Omega, \partial D)$.

The above theorem immediately follows from Theorem 1.

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