SOME RESULTS ON THE ASYMPTOTIC BEHAVIOR OF LINEAR SYSTEMS

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1. Introduction. We consider first in §2 the asymptotic behavior as $t \to \infty$ of the solutions of the vector-matrix differential equation

$$\dot{x} = \{A + B(t)\}x, \tag{1.1}$$

where A is a constant n-square complex matrix, $B(t)$ a continuous complex valued n-square matrix defined on $[0, \infty)$, and x a complex n-vector.

It is readily shown **(4)** that the asymptotic behavior of solutions to (1.1) can be made to depend on the functions $\lambda_M\{A + A^* + B(t) + B^*(t)\}$ and $\lambda_m\{A + A^* + B(t) + B^*(t)\}$ where $A^* = \bar{A}'$ and λ_M, λ_m are respectively the maximum and minimum eigenvalues of the indicated Hermitian matrix. We recapitulate this brief calculation in §2.

There are two types of theorems concerning (1.1) in the sequel: (i) A arbitrary with hypotheses on the eigenvalues of $(A + A^*)$; (ii) A triangular with hypotheses on the real parts of the eigenvalues of A. In both (i) and (ii) less than the absolute integrability of the functions $B_{ij}(t)$ is required **(1**, pp. 32–63**)**.

In §3 we discuss the behavior as $t \to \infty$ of solutions to the equation

$$\dot{x} = \{A(t) + B(t)\}x, \tag{1.2}$$

in which the entries of $A(t)$ are continuous complex-valued almost-periodic functions. The main result concerning (1.2) depends on a theorem of Favard which will be stated. In this case, however, it becomes necessary to assume the absolute integrability of all entries of $B(t)$.

We set

$$||x||^2 = \sum_{i=1}^{n} |x_i|^2;$$

boundedness refers to this norm. Also let $\Re(X) = (X + \bar{X})/2$ and $\Im(X) = (X - \bar{X})/2i$. $||X||^2 = \text{trace}\,(XX^*)$, $\alpha' =$ transpose of α. We note the following two elementary results that are subsequently used:

I. *If X and Y are Hermitian n-square matrices then*

$$\lambda_M(X + Y) \leqslant \lambda_M(X) + \lambda_M(Y), \tag{1.3}$$

$$\lambda_m(X + Y) \geqslant \lambda_m(X) + \lambda_m(Y). \tag{1.4}$$

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This follows immediately upon noting that $X + Y$ is Hermitian and

$$\begin{aligned}\lambda_M(X + Y) &= \max_{t=1} z^*(X + Y)z \leqslant \max_{t=1} z^*Xz + \max_{t=1} z^*Yz \\ &= \lambda_M(X) + \lambda_M(Y),\end{aligned}$$

where $t = ||z||^2$. Similarly for (1.4).

The following well-known device is due to O. Perron.

II. *If X has eigenvalues $\lambda_1, \ldots, \lambda_n$ then for any $\epsilon > 0$ there exists a matrix $D(\epsilon)$ similar to X such that $D_{ii}(\epsilon) = \lambda_i$ and $|D_{ij}(\epsilon)| < \epsilon$ for $i \neq j$.*

For assume X is in Jordan form, set

$$H = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & \epsilon & & & 0 \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdots & & 0 & \epsilon^{n-1} \end{pmatrix}$$

and note that

$$H^{-1}XH = \begin{pmatrix} \lambda_1 & \epsilon & 0 & \cdots & 0 \\ 0 & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & 0 \\ \cdot & & & \cdot & \epsilon \\ 0 & \cdot & & 0 & \lambda_n \end{pmatrix}$$

2. The equation (1.1). In discussing (1.1) we, of course, omit the trivial solution $x(t) = 0$. We assume that the starting time of every solution is $t_0 = 0$ since any solution $x(t)$ with starting time $t_0 > 0$ may be continued over $[0, t_0]$.

THEOREM 1. *Consider* (1.1) *with A arbitrary. Assume*

$$\lambda_M(A + A^*) = \omega \tag{2.1}$$

and there exists L such that $t \geqslant L$ implies either (a):

$$\frac{1}{t}\int_0^t \max_i \Re\{B_{ii}(s)\}ds \leqslant -\tfrac{1}{2}\omega \tag{2.2}$$

$$\int_0^\infty |\Re\{B_{ij}(s)\}|ds < \infty, \quad \int_0^\infty |\Im\{B(s) - B'(s)\}_{ij}|ds < \infty \tag{2.3}$$

for $i \neq j$, or (b):

$$\frac{1}{t}\int_0^t\Big(\max_i \Re\{B_{ii}(s)\} + \sum_{i\neq j} |\Re\{B_{ij}(s)\}| + |\Im\{B(s) - B'(s)\}_{ij}|\Big)ds \leqslant -\tfrac{1}{2}\omega; \tag{2.4}$$

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then in both cases every solution of (1.1) *is uniformly bounded as* $t \to \infty$. *If in either* (2.2) *or* (2.4) *the left sides are bounded strictly below* $-\frac{1}{2}\omega$, *then every solution converges to* 0 *as* $t \to \infty$.

Proof. Taking the inner product on the left with x^* in (1.1) we obtain

$$x^*\dot{x} = x^*\{A + B(t)\}x \tag{2.5}$$

and

$$\frac{d}{dt}||x||^2 = x^*\dot{x} + \dot{x}^*x = x^*\{A + A^* + B(t) + B^*(t)\}x. \tag{2.6}$$

The matrix on the right in (2.6) is Hermitian for all t and hence let $U(t)$ be a unitary matrix reducing it to canonical form. The substitution $x = U(t)z$ then yields

$$\begin{aligned}\frac{d}{dt}||z||^2 &= z^*\text{diagonal } \lambda_i\{A + A^* + B(t) + B^*(t)\}z\\ &= \sum_{i=1}^{n} \lambda_i|z_i|^2 = ||z||^2 \sum_{i=1}^{n} \lambda_i\delta_i\end{aligned}$$

where $\delta_i = |z_i|^2/||z||^2$, $\sum\delta_i = 1$, $0 \leqslant \delta_i \leqslant 1$. Integrating we obtain

$$||z(t)||^2 = ||z_0||^2 \exp\left(\int_0^t \sum_{i=1}^{n} \lambda_i\delta_i ds\right). \tag{2.7}$$

We use (1.2) to obtain

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i\{A + A^* + B(t) + B^*(t)\}\delta_i &\leqslant \lambda_M\{A + A^* + B(t) + B^*(t)\}\\ &\leqslant \lambda_M(A + A^*) + \lambda_M\{B(t) + B^*(t)\}.\end{aligned} \tag{2.8}$$

Now let $m(s)$ be the unit eigenvector of $\{B(s) + B^*(s)\}$ such that

$$\lambda_M\{B(s) + B^*(s)\} = m^*(s)\{B(s) + B^*(s)\}m(s) \tag{2.9}$$

for $0 \leqslant s \leqslant \infty$. Also, setting $B(s) = U(s) + iV(s)$ and $m(s) = \alpha(s) + i\phi(s)$, (2.9) becomes

$$\begin{aligned}\lambda_M(B(s) + B^*(s)) &= \Re(m^*(s)\{B(s) + B^*(s)\}m(s))\\ &= \Re[\{\alpha'(s) - i\phi'(s)\}(U(s) + U'(s) + i\{V(s) - V'(s)\})(\alpha(s) + i\phi(s))]\\ &= 2(\alpha'(s)U(s)\alpha(s) + \phi'(s)U(s)\phi(s) + \alpha'(s)\{V'(s) - V(s)\}\phi(s))\\ &= 2\sum_{i=1}^{n} U_{ii}(s)\{\alpha_i^2(s) + \phi_i^2(s)\} + 2\sum_{i\neq j} U_{ij}(s)\{\alpha_i(s)\alpha_j(s) + \phi_i(s)\phi_j(s)\}\\ &\quad + 2\sum_{i\neq j}\{V_{ji}(s) - V_{ij}(s)\}\alpha_i(s)\phi_j(s).\end{aligned} \tag{2.10}$$

Now

$$m^*(s)m(s) = \sum_{i=1}^{n}\{\alpha_i^2(s) + \phi_i^2(s)\} = 1$$

and we obtain

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$$(2.11)\quad \lambda_M\{B(s) + B^*(s)\} \leqslant 2 \max U_{ii}(s) + 4\sum_{i\neq j} |U_{ij}(s)| + 2\sum_{i\neq j} |V_{ij}(s) - V_{ji}(s)|.$$

We conclude from (2.7), (2.8) and (2.11) that

$$(2.12)\quad ||z(t)||^2 \leqslant ||z_0||^2 \exp\biggl(\omega t + 2\int_0^t \max U_{ii}(s)ds + 4\int_0^t \sum_{i\neq j} |U_{ij}(s)|ds + 2\int_0^t \sum_{i\neq j} |V_{ij}(s) - V_{ji}(s)|ds\biggr).$$

In case (a), by (2.3), we select $K > 0$ such that

$$(2.13)\qquad ||z(t)||^2 \leqslant K||z_0||^2 \exp\biggl(t\{\omega + \frac{2}{t}\int_0^t \max U_{ii}(s)ds\}\biggr),$$

and the result follows from (2.2). Case (b) is analogous with the use of (2.4) and (2.12).

THEOREM 2. *Consider*

$$(2.14)\qquad \dot{x} = \{T + B(t)\}x$$

and assume

(2.15) *T is triangular, $T_{ij} = 0$ for $j < i$, and* $\max \Re\{\lambda_i(T)\} = \omega$,

$$(2.16)\qquad \int_0^\infty |B_{ij}(t)|dt < \infty, \qquad i \neq j,$$

$$(2.17)\qquad \limsup_{t\to\infty} \frac{1}{t}\int_0^t \max_i \Re\{B_{ii}(s)\}ds < -\omega;$$

then every solution of (2.14) *converges to* 0 *as* $t \to \infty$.

Proof. Let $x = Hy$ where

$$H = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \epsilon & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & 0 \\ 0 & \cdot & \cdot\ 0 & \epsilon^{n-1} \end{pmatrix},$$

$\epsilon \neq 0$ then (2.14) becomes $\dot{y} = \{D + C(t)\}y$ with $D = H^{-1}TH$, $C(t) = H^{-1}B(t)H$. Proceeding as above we obtain

$$(2.18)\quad ||z(t)||^2 = ||z_0||^2 \exp\biggl(\int_0^t \sum_{i=1}^n \lambda_i\{D + D^* + C(s) + C^*(s)\}\delta_i ds\biggr)$$

where z is the unitary transform of y. Now

$$(2.19)\qquad (D + D^*)_{ij} = \begin{cases} 2\Re\{\lambda_i(T)\}, & i = j, \\ \epsilon^{j-i}T_{ij}, & i < j, \\ \epsilon^{i-j}\bar{T}_{ji}, & j < i, \end{cases}$$

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and $C_{ij}(s) = \epsilon^{i-j}B_{ij}(s)$. By (2.16) and (2.18) there exists such a constant K that

$$(2.20)\qquad ||x(t)||^2 \leqslant K||x_0||^2 \exp\left(t\lambda_M(D + D^*) + 2\int_0^t \max \Re\{B_{ii}(s)\}ds\right);$$

but since the eigenvalues of a matrix are continuous functions of the entries, $\lambda_M(D + D^*)$ can be made to differ arbitrarily little from $2 \max \Re(\lambda_i(T)) = 2\omega$ by choosing ϵ sufficiently small. (2.17) completes the argument.

The divergence theorems follow analogously. We omit proofs.

THEOREM 3. *Consider* (1.1). *Assume*

$$(2.21)\qquad \lambda_m(A + A^*) = \omega$$

and

$$(2.22)\qquad \liminf_{t\to\infty} \frac{1}{t}\int_0^t (\min \Re\{B_{ii}(s)\} - 2\sum_{i\neq j} |\Re\{B_{ij}(s)\}| - \sum_{i\neq j} |\Im\{B(s) - B'(s)\}_{ij}|)ds > -\tfrac{1}{2}\omega;$$

then every solution of (2.1) *diverges to* ∞ *as* $t \to \infty$.

THEOREM 4. *Consider* (2.14). *Assume* (2.16), *T triangular*, $\min \Re(\lambda_i(T)) = \omega$ *and*

$$(2.23)\qquad \liminf_{t\to\infty} \frac{1}{t}\int_0^t \min \Re(B_{ii}(s))ds > -\omega;$$

then every solution of (2.14) *diverges to* ∞ *as* $t \to \infty$.

Theorems 2 and 4 provide a simple proof of the following familiar statement: *If*

$$\int_0^\infty ||B(t)||dt < \infty$$

and all solutions of $\dot{x} = Ax$ *either* (a) *converge to* 0 *or are bounded or* (b) *diverge to* ∞ *as* $t \to \infty$, *then the same is true of* (1.1). For (b) implies $\min \Re\{\lambda_i(A)\} > 0$. By a change of variable assume (1.1) is in the form $\dot{x} = \{T + SB(t)S^{-1}\}x$ with T triangular,

$$\int_0^\infty |\{SB(t)S^{-1}\}_{ij}|dt < \infty$$

for all (i, j) and (2.16) holds,

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \min \Re(\{SB(s)S^{-1}\}_{ii})ds = 0,$$

and (2.23) holds. Case (b) follows by Theorem 4. Case (a) is similar.

3. The equation (1.2). If $f(t)$ is a continuous complex-valued almost-periodic (a.p.) function on $[0, \infty)$ set

$$M\{f(t)\} = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)ds.$$

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A minor modification of an argument due to Favard **(2)** proves the following:

THEOREM 5. *If* $f(t)$ *is real-valued,* $M\{f(t)\} \geqslant 0$, *and*

$$\int_0^t f(s)ds$$

is not bounded on $[0, \infty)$, *then there exists a sequence of intervals* $[a_n, b_n]$ *with* $b_n \geqslant a_n \geqslant 0$, $a_n < a_{n+1}$ $(n = 0, 1, \ldots)$, $\lim a_n = \infty$, *such that*

$$\int_{a_n}^{b_n} f(s)ds \geqslant n.$$

We show in Theorem 6 that Theorem 5 is easily applied to obtain some sufficient conditions that imply the stability of (1.2) assuming the boundedness on $[0, \infty)$ of solutions to

$$\dot{x} = A(t)x. \tag{3.1}$$

For any finite collection of a.p. functions and any $\epsilon > 0$ there exists a common relatively dense set of translation numbers with respect to ϵ. Hence we may consider $A(t)$ an a.p. matrix function.

Denote by $X(t)$ the fundamental matrix of solutions (f.m.s.) of (3.1) with

$$X(0) = I.$$

Note that

$$\limsup_{t\to\infty} ||X(t)|| < \infty \tag{3.2}$$

and

$$|X(t)| = \exp\left\{\int_0^t \operatorname{tr} A(s)ds\right\} \tag{3.3}$$

together with the Hadamard determinant theorem imply that

$$m(h) = \limsup_{t\to\infty} \int_0^t \Re \operatorname{tr} A(s+h)ds < \infty \tag{3.4}$$

for any $h \geqslant 0$. We have

THEOREM 6. *Assume*

(i) $$M\{\Re \operatorname{tr} A(t)\} \geqslant 0,$$

(ii) (3.2) holds,

(iii) $$\limsup_{h\to\infty} m(h) < \infty,$$

(iv) $$\int_0^\infty ||B(s)||ds < \infty;$$

then all solutions of (1.2) *are uniformly bounded on* $[0, \infty)$.

Before proceeding, note that (i) and (ii) imply $M\{\Re \operatorname{tr} A(t)\} = 0$.

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Proof. First consider the translated equation

$$\dot{x} = A(t+h)x \qquad (h \geqslant 0). \tag{3.5}$$

(ii) clearly implies that all solutions of (3.5) are bounded on $[0, \infty)$ for each h.

Let $X(t; h)$ be the f.m.s. of (3.5). Suppose there exists $h_n \to \infty$ such that

$$\lim_{n\to\infty} \limsup_{t\to\infty} X(t; h_n) = \infty. \tag{3.6}$$

Then

$$\lim_{n\to\infty} \limsup_{t\to\infty} ||X(t+h_n) \operatorname{adj} X(h_n)|| \exp\left\{-\int_0^{h_n} \Re \operatorname{tr} A(s)ds\right\} = \infty,$$

and we conclude from (ii) that

$$\int_0^t \Re \operatorname{tr} A(s)ds$$

is not bounded on $[0, \infty)$. By Theorem 5 there exists a sequence of intervals $[a_n, b_n]$ such that

$$\int_{a_n}^{b_n} \Re \operatorname{tr} A(s)ds \geqslant n. \tag{3.7}$$

Setting $l_n = b_n - a_n$ and $s = a_n + t$, (3.7) becomes

$$\int_0^{l_n} \Re \operatorname{tr} A(t+a_n)dt \geqslant n,$$

and we conclude that $m(a_n) \geqslant n$, contradicting (iii). Hence there exists $K \geqslant 0$ such that

$$\limsup_{h\to\infty} \limsup_{t\to\infty} X(t; h) = K < \infty. \tag{3.8}$$

Let $u(b, t)$ be a solution of (1.2) with $u(b, 0) = b$. Using the variation of parameters formula and taking norms on both sides, we have

$$||u(b,t)|| \leqslant ||X(t)|| \, ||b|| + \int_0^t ||X(t)X^{-1}(s)|| \, ||B(s)|| \, ||u(b,s)||ds. \tag{3.9}$$

In (3.9) $t \geqslant s$, $t - s = h \geqslant 0$,

$$\frac{d}{ds}\{X(s+h)X^{-1}(h)\} = A(s+h)X(s+h)X^{-1}(h)$$

and it is obvious that $X(s+h)X^{-1}(h)$ is the f.m.s. of (3.5). By (3.8) we conclude that

$$\limsup_{t \geqslant s \geqslant 0} ||X(t)X^{-1}(s)|| = K < \infty.$$

Using an inequality due to Gronwall **(3)**, we have

$$||u(b,t)|| \leqslant K||b|| \exp\left(K\int_0^t ||B(s)||ds\right)$$

and (iv) completes the proof.

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We may remark that the argument applied to (3.5) will yield the usual stability theorem in case $A(t)$ is purely periodic without use of the Floquet representation of the f.m.s. as a product of exponential and periodic matrix functions.

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