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SUPERCRITICAL FLOW OF AN IDEAL FLUID OVER A SPILLWAY

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Abstract

The irrotational flow of an incompressible, inviscid fluid over a spillway is considered. The reciprocal ε of the Froude number is taken to be small and the method of matched asymptotic expansions is applied. The bed of the spillway is horizontal far upstream and makes an angle α with the horizontal far downstream. The inner expansion is valid upstream and over the spillway, but is invalid far downstream. The outer expansion which is valid downstream fails to satisfy the upstream conditions. Unknown constants in the outer expansion are determined by the matching and composite expansions obtained.

1. Introduction and formulation

Flows past polygonal obstacles or with polygonal boundaries and jet flows have been widely examined since 1868 (Helmholtz and Kirchoff), the effects of gravity being neglected. Both Birkhoff and Zarantonello [2] and Gurevich [6] provide a very broad coverage with extensive bibliographies. It is only since 1960 that there has been any significant progress in the solution of free surface flows of this type when gravity is not neglected. Prior to this, either inverse or *ad hoc* approximate methods had been used.

The method of matched asymptotic expansions has been used by Clarke [4] to solve the two-dimensional flow over a waterfall, by Keady [7] to solve the flow of a jet from a horizontal slot and by Ackerberg [1] to determine the flow down an inclined plane, the flow being introduced at the leading edge of the plane. Recently Keady and Norbury [8], and Budden and Norbury [3], have established this method rigorously for a number of problems of this type. Here we shall use the method to examine the flow of an infinite stream which flows horizontally prior to its motion down a spillway. The complex potential is taken as the independent variable in order to overcome the difficulty that the geometry of the free surface is not known in advance. The effects of surface tension are ignored.

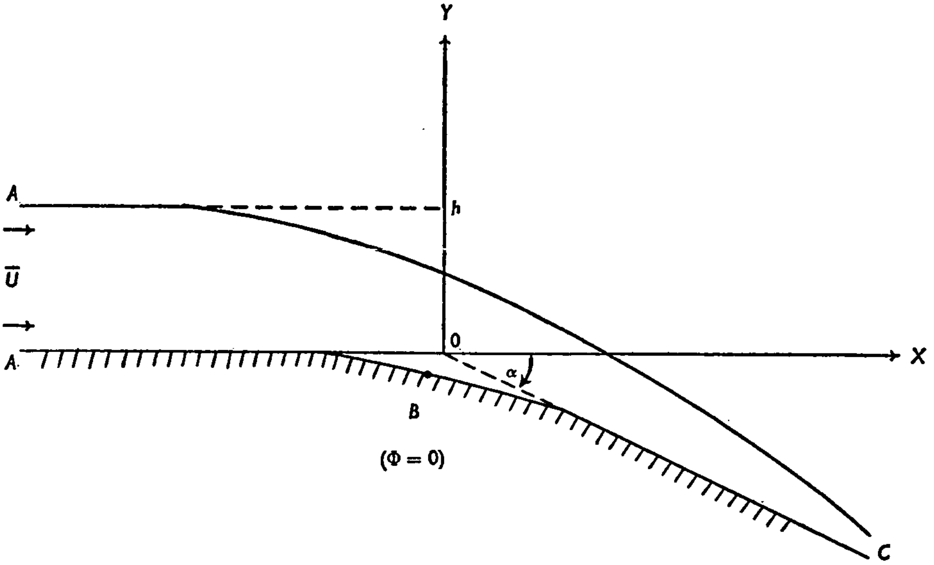

Fig. 1. Coordinate system in X–Y plane.

We choose a coordinate system $Z = X + iY$ such that the tangents to the bed at A and C intersect to give the origin in the physical Z plane (Fig. 1). Gravity acts in the negative Y direction. A complex potential $W(Z) = \Phi + i\Psi$ and a complex velocity $dW/dZ = U - iV$ must be found such that the pressure on the free surface AC is constant, and at A the fluid speed is $\bar{U}$ for $0 \leqslant Y \leqslant h$. At C the fluid speed is infinite, and the appropriate condition is deferred to later in this section. The origin in the W plane is chosen to correspond to a fixed point in the Z plane. We non-dimensionalize the coordinates, the complex potential and the complex velocity as follows:

$$\left.\begin{aligned} z &= x + iy = \frac{Z}{h}, \quad w = \phi + i\psi = \frac{W}{h\bar{U}}, \\ \frac{dw}{dz} &= u - iv = q\,e^{-i\theta} = \bar{U}^{-1}\frac{dW}{dZ}. \end{aligned}\right\} \tag{1.1}$$

As the location of the free surface AC in the physical plane is unknown, we formulate the problem in the potential w-plane (Fig. 2). Thus we let

$$\ln\frac{dw}{dz} = \ln q - i\theta = Q - i\theta, \tag{1.2}$$

where Q and $(-\theta)$ are conjugate harmonic functions of the variables (ϕ, ψ) and so satisfy the corresponding Cauchy–Riemann equations. On the free surface, the pressure p is constant. Now Bernoulli's equation when differentiated with respect

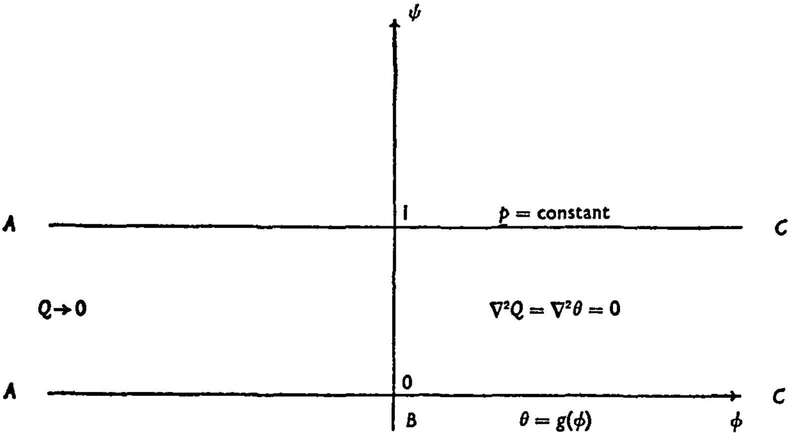

Fig. 2. Coordinate system in w-plane.

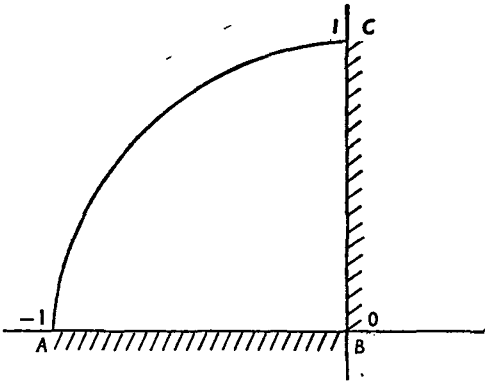

Fig. 3. Coordinate system in t-plane.

to ϕ is

$$\left.\begin{aligned} \frac{1}{\rho}\frac{\partial p}{\partial \phi}+q\frac{\partial q}{\partial \phi}+\varepsilon\frac{\partial y}{\partial \phi}=0, \\ \varepsilon = gh/\bar{U}^2. \end{aligned}\right\} \tag{1.3}$$

where

Here ε is the inverse Froude number, and we shall assume that ε is small. On the free surface $\partial p/\partial \phi$ is zero, and using the fact that

$$q=\exp\{Q\} \quad \text{and} \quad dz/dw = q^{-1}\exp\{i\theta\},$$

it follows that

$$\frac{\partial Q}{\partial \phi}+\varepsilon \exp\{-3Q\}\sin\theta=0, \quad \text{on } \psi=1, \quad -\infty<\phi<\infty. \tag{1.4}$$

The boundary condition far upstream is $q = 1$, and hence

$$Q(\phi,\psi)\to 0^{+} \quad \text{as } \phi\to-\infty, \quad 0<\psi<1. \tag{1.5}$$

Far downstream, the boundary condition is

$$\theta(\phi,\psi)\to-\alpha \quad \text{as } \phi\to\infty, \quad 0<\psi<1. \tag{1.6}$$

The final boundary condition is the specification of the slope on the bed of the spillway, and we put

$$\theta(\phi,0)=g(\phi), \quad -\infty<\phi<\infty. \tag{1.7}$$

We shall assume that $g(\phi)$ is a piecewise smooth function with at most a finite number of simple jump discontinuities, and

$$g(\phi)=\begin{cases} O(\exp(k\phi)) & \text{as } \phi\to-\infty, \\ -\alpha+O(\exp(-k\phi)) & \text{as } \phi\to\infty. \end{cases} \tag{1.8}$$

The problem specified by the boundary conditions (1.4), (1.5), (1.6) and (1.7) is an inverse problem, as the physical problem requires the specification of the slope on the bed of the spillway as a function of x, say $\theta = \theta_0(x)$, whereas (1.7) specifies the slope as $g(\phi)$, a function of ϕ. These two are related by the expression,

$$\frac{d\theta_0}{dx}\cos\{g(\phi)\}=q\frac{dg}{d\phi}. \tag{1.9}$$

Our procedure determines q (asymptotically for small ε) as a functional of $g(\phi)$, and (1.9) is then a functional equation for $g(\phi)$. However, there is one special case when the inverse problem admits a trivial solution; viz. the bed has a single abrupt transition of slope (Fig. 4), and so

$$g(\phi)=\begin{cases} 0 & \text{for } -\infty<\phi<0, \\ -\alpha & \text{for } 0<\phi<\infty. \end{cases} \tag{2.0}$$

We note that the origin of ϕ may be selected arbitrarily, and we have put $\phi = 0$ at the corner. We do not propose to examine the general inverse problem (1.9) in this paper, and so our results relate principally to the special case when $g(\phi)$ is given by (2.0). However, it should be noted that when the bed consists of a finite number of straight line segments, the functional equation (1.9) reduces to a finite system of algebraic equations, and we shall examine this case briefly in Section 5. Finally we comment that the presence of a jump discontinuity in $\theta_0(x)$, or $g(\phi)$, will cause the presence of singularities in Q and θ; we shall require that these be the "weakest" possible singularities, that is they should be such that dw/dz is $O(|z-z_0|^{-\beta/(\pi+\beta)})$ as $z \to z_0$, where z_0 is the location of the corner, and $\pi+\beta$ is the exterior angle at the corner.

We seek to determine harmonic functions Q and θ that satisfy equations (1.4), (1.5), (1.6) and (1.7). The main difficulty is that (1.4) is non-linear, and for small ε, the non-linear terms become significant as $\phi \to \infty$. When $\varepsilon \ll 1$, we shall obtain an asymptotic solution by the method of matched expansions. The inner expansion is an expansion in powers of ε, and assumes that the effects of gravity are small. This expansion fails as $\phi \to \infty$ and is complemented by an outer expansion in which the ϕ coordinate is stretched and replaced by $\phi^* = \varepsilon\phi$. The outer expansion takes account of the dominant effects of gravity far downstream.

2. The inner expansion

We assume that

$$\left.\begin{aligned} Q(\phi,\psi;\,\varepsilon) &= Q_0(\phi,\psi)+\varepsilon Q_1(\phi,\psi)+\varepsilon^2 Q_2(\phi,\psi)+\dots \\ \text{and}\qquad \theta(\phi,\psi;\,\varepsilon) &= \theta_0(\phi,\psi)+\varepsilon\theta_1(\phi,\psi)+\varepsilon^2\theta_2(\phi,\psi)+\dots. \end{aligned}\right\} \tag{2.1}$$

Substituting equations (2.1) into (1.4) and equating coefficients of each power of ε to zero, we obtain on $\psi = 1$

$$\frac{\partial Q_0}{\partial \phi} = 0, \tag{2.2}$$

$$\frac{\partial Q_1}{\partial \phi} + e^{-3Q_0}\sin\theta_0 = 0, \tag{2.3}$$

and

$$\frac{\partial Q_2}{\partial \phi} + e^{-3Q_0}(\theta_1\cos\theta_0 - 3Q_1\sin\theta_0) = 0. \tag{2.4}$$

Using the appropriate Cauchy-Riemann equations, the problem for $\theta_0(\phi,\psi)$ becomes

$$\left.\begin{aligned} &\nabla^2\theta_0 = 0; \quad -\infty<\phi<\infty; \quad 0\leqslant\psi\leqslant 1. \\ &\theta_0(\phi,0) = g(\phi); \quad -\infty<\phi<\infty. \\ &\frac{\partial\theta_0}{\partial\psi} = 0; \quad -\infty<\phi<\infty; \quad \psi = 1. \end{aligned}\right\}(\mathrm{P}_0) \tag{2.5}$$

The solution may be obtained by using a (complex) Fourier integral transform in ϕ. The result is

$$\theta_0(\phi,\psi) = \sin\frac{\pi\psi}{2}\int_{-\infty}^{\infty}\frac{g(\phi')\cosh\pi(\phi-\phi')/2}{\cosh\pi(\phi-\phi')-\cos\pi\psi}\,d\phi'. \tag{2.6}$$

Hence, putting $\phi' = \phi + 2\pi^{-1}\operatorname{arsinh} t$, we get

$$\theta_0(\phi,\psi) = \frac{1}{\pi}\sin\frac{\pi\psi}{2}\int_{-\infty}^{\infty}\frac{g(\phi')}{t^2+\sin^2(\pi\psi/2)}\,dt, \tag{2.7}$$

from which $Q_0(\phi,\psi)$ may be found using the Cauchy–Riemann equations and the boundary condition $Q_0(\phi,1)=0$.

The boundary value problem for $Q_1(\phi,\psi)$ is such that the substitution of $\theta_1(\phi,\psi)$ into the Cauchy–Riemann equations will yield Q_1 only up to an arbitrary constant for $\phi>0$. It is therefore necessary to solve first for $Q_1(\phi,\psi)$ from which $\theta_1(\phi,\psi)$ may be determined. The boundary value problem for $Q_1(\phi,\psi)$ is

$$\left.\begin{aligned}
&\nabla^2 Q_1 = 0; \quad -\infty<\phi<\infty; \quad 0\leqslant\psi\leqslant 1.\\
&Q_1\to 0 \quad \text{as } \phi\to-\infty; \quad 0\leqslant\psi\leqslant 1.\\
&\frac{\partial Q_1}{\partial\psi} = 0; \quad -\infty<\phi<\infty; \quad \psi=0.\\
&\frac{\partial Q_1}{\partial\phi} = -e^{-3Q_0}\sin\theta_0 = -f(\phi); \quad -\infty<\phi<\infty; \quad \psi=1.
\end{aligned}\right\}(\mathrm{P}_1) \tag{2.8}$$

On the free surface, $Q_0=0$ and $\psi=1$ whence $\exp(-3Q_0)=1$ and, from equation (2.7), it may be shown that

$$f(\phi) = \sin\theta_0(\phi,1) = \begin{cases} O(e^{k^*\phi}) & \text{as } \phi\to-\infty,\\ -\sin\alpha + O(e^{-k^*\phi}) & \text{as } \phi\to+\infty.\end{cases} \tag{2.9}$$

Here $k^* = \min(k,\pi/2)$. Hence, again using a (complex) Fourier integral transform, the solution of Problem (P_1) is

$$Q_1(\phi,\psi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{F(s)\cosh s\psi\, e^{-is\phi}}{is\cosh s}\,ds, \tag{2.10}$$

where $s=\sigma+i\tau$ and

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\phi)\,e^{is\phi}\,d\phi, \quad 0<\tau<k^*. \tag{2.11}$$

Integrating equation (2.11) by parts, it follows that

$$F(s) = -\frac{K(s)}{is},$$

where (2.12)

$$K(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(\phi)\,e^{is\phi}\,d\phi.$$

$K(s)$ is analytic in $-k^*<\tau<k^*$, and so $F(s)$ has a simple pole at $s=0$.

It may be shown that

$$Q_1(\phi,\psi) = \begin{cases} O(e^{k^*\phi}), & \phi < 0, \\ \phi \sin\alpha + K_1 + O(e^{-k^*\phi}), & \phi > 0, \end{cases} \tag{2.13}$$

where

$$K_1 = \int_{-\infty}^{\infty} \phi f'(\phi)\, d\phi. \tag{2.14}$$

The Cauchy–Riemann equations and the boundary condition $\theta_1(\phi, 0) = 0$, $-\infty < \phi < \infty$, give

$$\theta_1(\phi,\psi) = \begin{cases} O(e^{k^*\phi}), & \phi < 0, \\ -\psi \sin\alpha + O(e^{-k^*\phi}), & \phi > 0. \end{cases} \tag{2.15}$$

We proceed in an identical way to determine $Q_2(\phi,\psi)$. Let

$$f_1(\phi) = \theta_1 \cos\theta_0 - 3Q_1 \sin\theta_0, \quad \text{on } \psi = 1. \tag{2.16}$$

We may show that the Fourier transform $F_1(s)$ of $f_1(\phi)$ has a double pole at $s = 0$. $Q_2(\phi,\psi)$ is given by equation (2.10) with $F(s)$ replaced by $F_1(s)$. It may be shown that

$$Q_2(\phi,\psi) = \begin{cases} O(e^{k^*\phi}), & \phi < 0, \\ \tfrac{3}{2}(\psi^2 - \phi^2)\sin^2\alpha + \phi(\sin\alpha\cos\alpha - 3K_1 \sin\alpha) & \\ -\tfrac{3}{2}\sin^2\alpha + K_2 + O(e^{-k^*\phi}), & \phi > 0, \end{cases} \tag{2.17}$$

where

$$K_2 = -\int_{-\infty}^{\infty} \phi^2 f_1''(\phi)\, d\phi. \tag{2.18}$$

The Cauchy–Riemann equations give

$$\theta_2(\phi,\psi) = \begin{cases} O(e^{k^*\phi}), & \phi < 0, \\ 3\phi\psi\sin^2\alpha - \psi(\sin\alpha\cos\alpha - 3K_1\sin\alpha) + O(e^{-k^*\phi}), & \phi > 0. \end{cases} \tag{2.19}$$

3. The outer expansion and matching

Far downstream, the free fall approximation ($q^2 \approx -2\varepsilon y$) shows that Q will be a function of $\varepsilon\phi$ (see, for example, Ackerberg [1]). Hence we introduce a new variable $\phi^* = \varepsilon\phi$, and put

$$\left.\begin{aligned} Q(\phi,\psi;\,\varepsilon) &\sim Q_0(\phi^*,\psi) + \varepsilon Q_1(\phi^*,\psi) + \varepsilon^2 Q_2(\phi^*,\psi) + \dots, \\ \theta(\phi,\psi;\,\varepsilon) &\sim -\alpha + \varepsilon\theta_1(\phi^*,\psi) + \varepsilon^2\theta_2(\phi^*,\psi) + \dots. \end{aligned}\right\} \tag{3.1}$$

The Cauchy–Riemann equations in terms of ϕ^* and ψ are

$$\varepsilon\frac{\partial Q}{\partial\phi^*} = -\frac{\partial\theta}{\partial\psi}, \quad \frac{\partial Q}{\partial\psi} = \varepsilon\frac{\partial\theta}{\partial\phi^*}. \tag{3.2}$$

If we substitute equations (3.2) into (3.1) and equate coefficients of each power of ε to zero we obtain

$$\frac{\partial Q_0}{\partial\phi^*} = -\frac{\partial\theta_1}{\partial\psi}, \quad \frac{\partial Q_0}{\partial\psi} = 0, \tag{3.3}$$

$$\frac{\partial Q_1}{\partial\phi^*} = -\frac{\partial\theta_2}{\partial\psi}, \quad \frac{\partial Q_1}{\partial\psi} = 0, \tag{3.4}$$

$$\frac{\partial Q_2}{\partial\psi} = \frac{\partial\theta_1}{\partial\phi^*}. \tag{3.5}$$

Upon integration we obtain

$$\theta_1 = -\psi Q_0'(\phi^*), \quad \theta_2 = -\psi Q_1'(\phi^*) \tag{3.6}$$

and

$$\left.\begin{aligned} &Q_0 = Q_0(\phi^*), \quad Q_1 = Q_1(\phi^*), \\ &Q_2 = -\tfrac{1}{2}\psi^2 Q_0''(\phi^*) + a(\phi^*), \end{aligned}\right\} \tag{3.7}$$

where $a(\phi^*)$ is an unknown function to be determined. In equation (3.6) the boundary condition (1.7) has been applied, where terms of exponential order $(O(e^{-k^*\phi}))$ are neglected.

In terms of the new variables, the free surface boundary condition becomes $(\partial Q/\partial\phi^*) + e^{-3Q}\sin\theta = 0$, $\psi = 1$. On substituting equation (3.1) and equating coefficients of each power of ε to zero we have

$$\left.\begin{aligned} &Q_0' = e^{-3Q_0}\sin\alpha, \\ &Q_1' + 3\,e^{-3Q_0}\sin\alpha = e^{-3Q_0}Q_0'\cos\alpha, \\ &a'(\phi^*) + 3a\,e^{-3Q_0}\sin\alpha = \tfrac{1}{2}\{Q_0''' + e^{-3Q_0}[3Q_0''\sin\alpha \\ &\qquad + 2Q_1'\cos\alpha - (Q_0')^2\sin\alpha + 9Q_1^2\sin\alpha - 6Q_0'\,Q_1\cos\alpha]\}. \end{aligned}\right\} \tag{3.8}$$

It follows that

$$Q_0(\phi^*) = \tfrac{1}{3}\ln p, \tag{3.9}$$

$$Q_1(\phi^*) = \frac{\cos\alpha}{3p}\ln p + \frac{c_1}{p} \tag{3.10}$$

$$\begin{aligned} a(\phi^*) = -(6p^2)^{-1}\{&(8\sin^2\alpha + 2\cos^2\alpha - 12c_1\cos\alpha + 9c_1^2) \\ &+ (6c_1\cos\alpha - 4\cos^2\alpha)(1+\ln p) + \cos^2\alpha[(1+\ln p)^2 + 1]\} + (c_2/p), \end{aligned} \tag{3.11}$$

where

$$p(\phi^*) = c + 3\phi^* \sin\alpha \tag{3.12}$$

and c, c_1 and c_2 are constants of integration, the values of which will be found by matching with the inner solution. We note that Q_0 is simply the solution given by the free fall approximation.

For the matching of inner and outer expansions we adopt the procedure used by Ackerberg [1]. Thus

(i) Let $\phi \to +\infty$ in the inner expansion of Q and, neglecting exponentially small terms, express what remains in terms of ϕ^*;

(ii) expand the outer expansion of Q for $\phi^* \to 0$;

(iii) the arbitrary constants appearing in (ii) must be chosen so that for every term in (i) a corresponding term appears in (ii).

Carrying out (i) we obtain

$$\begin{aligned} Q_{\text{inner}} \to {} & \phi^* \sin\alpha + \varepsilon K_1 + \tfrac{3}{2}\varepsilon^2 \psi^2 \sin^2\alpha - \tfrac{3}{2}(\phi^*)^2 \sin^2\alpha \\ & + \varepsilon\phi^*(\sin\alpha\cos\alpha - 3K_1 \sin\alpha) - \tfrac{3}{2}\varepsilon^2 \sin^2\alpha \\ & + \varepsilon^2 K_2 + \varepsilon^3[\text{function of } \phi^*, \psi]. \end{aligned}$$

Carrying out (ii) we have

$$\begin{aligned} Q_{\text{outer}} \to {} & \tfrac{1}{3}\ln c + \frac{\phi^*}{c}\sin\alpha - \frac{3(\phi^*)^2}{2c^2}\sin^2\alpha + O(\phi^*)^3 \\ & + \varepsilon\left[\frac{c_1}{c} - \frac{3c_1\phi^*\sin\alpha}{c^2} + \frac{\phi^*\sin\alpha\cos\alpha}{c^2} + \frac{\cos\alpha(1 - 3\phi^* c^{-1}\sin\alpha)\ln c}{3c} \right. \\ & \left. + O((\phi^*)^2)\right] \\ & + \varepsilon^2\left[\frac{3\psi^2\sin^2\alpha}{2c^2} - \frac{1}{6c^2}[8\sin^2\alpha + 2\cos^2\alpha - 12c_1\cos\alpha + 9c_1^2 \right. \\ & + (1 + \ln c)(6c_1\cos\alpha - 4\cos^2\alpha) + \cos^2\alpha(2 + 2\ln c + (\ln c)^2)] \\ & \left. + \frac{c_2}{c} + O(\phi^*)\right] + O(\varepsilon^3). \end{aligned}$$

Matching is achieved if we take

$$c = 1, \quad c_1 = K_1 \quad \text{and} \quad c_2 = -\tfrac{1}{6}\sin^2\alpha + K_1\left(\frac{3K_1}{2} - \cos\alpha\right) + K_2. \tag{3.13}$$

Using the additive rule for the formation of composite expansions (see, for example, Van Dyke [9], p. 95), we obtain

$$\left.\begin{aligned} & Q \sim Q_0(\phi, \psi) + Q_0(\phi^*, \psi) + \varepsilon Q_1(\phi^*, \psi) + \varepsilon^2 Q_2(\phi^*, \psi) + o(\varepsilon^2) \\ \text{and} \quad & \theta \sim \theta_0(\phi, \psi) + \varepsilon\theta_1(\phi^*, \psi) + \varepsilon^2 \theta_2(\phi^*, \psi) + o(\varepsilon^2). \end{aligned}\right\} \tag{3.14}$$

However, as observed by Keady [7] in a similar problem, these composite expansions are singular far *upstream*, where the outer expansion breaks down when $p = 0$. Keady [7] showed that a further transformation of the complex plane overcomes this difficulty; and we shall follow his method here. Let

$$w = \frac{2}{\pi}\ln\left(\frac{1-t^2}{1+t^2}\right). \tag{3.15}$$

This maps the strip $0<\psi<1$, $-\infty<\phi<\infty$ (Fig. 2) into a quarter-circle of the t-plane, $|t|<1$ and $\pi/2<\arg t<\pi$ (Fig. 3). Then in the outer region where t is near i, we have

$$\varepsilon w = -\frac{2\varepsilon}{\pi}\ln\left(\frac{1+t^2}{2}\right) + (\text{transcendentally small terms}). \tag{3.16}$$

Since $Q_0(\phi^*, \psi)$ is independent of ψ, it follows that, on ABC,

$$Q_0(\phi^*) = \tfrac{1}{3}\ln\left[1-\frac{6\varepsilon\sin\alpha}{\pi}\ln\left(\frac{1+t^2}{2}\right)\right]. \tag{3.17}$$

This expression remains regular far upstream and so may be used in the composite expansion. A similar procedure is followed for the other terms in the composite expansion.

4. The free surface

We have now established that $c_1 = K_1$ as defined by equation (2.14). Integrating by parts it may be shown that

$$K_1 = -\int_{-\infty}^{\infty}\{f(\phi)+\sin\alpha H(\phi)\}\,d\phi, \tag{4.1}$$

where $H(\phi)$ is the Heaviside step function, and $f(\phi)$ is $\sin\theta_0(\phi, 1)$ (see equation (2.9)). If the bed is symmetrical about the point B in the physical plane (see Fig. 1), then $\theta_0(\phi, 1)$ will be symmetrical about $-\tfrac{1}{2}\alpha$, tending to zero as $\phi\to-\infty$ and $-\alpha$ as $\phi\to\infty$. Thus we may put

$$\theta_0(\phi, 1) = -\tfrac{1}{2}\alpha - u(\phi), \tag{4.2}$$

where $u(\phi)$ is an odd function of ϕ, and $u(0) = 0$, while $u(\infty) = \tfrac{1}{2}\alpha$. Substituting (4.2) into (4.1) it follows that

$$K_1 = \int_0^{\infty} 2\sin\tfrac{1}{2}\alpha\{\cos u(\phi) - \cos\tfrac{1}{2}\alpha\}\,d\phi, \tag{4.3}$$

and so K_1 is positive for a symmetrical bed. We shall show below, that when K_1 is positive, the free surface is lower (for a given point downstream) than the free fall approximation Q_0 would suggest.

Let s and n be coordinates along and normal to the bed respectively. Then

$$\left.\begin{aligned} s &= \exp(i\alpha)\int_0^{\phi} \left.\frac{dz}{dw}\right|_{\psi=0} d\phi + \operatorname{Re} h(\phi, 1), \\ n &= \operatorname{Im} h(\phi, 1), \\ \text{where}\qquad h(\phi, 1) &= i\,e^{i\alpha}\int_0^1 \frac{dz}{dw}\,d\psi. \end{aligned}\right\} \tag{4.4}$$

Now $dw/dz = q\,e^{-i\theta}$ giving $dz/dw = e^{-Q}.e^{-i\alpha}$ on $\psi = 0$. Hence it follows that

$$s = \int_0^{\phi} e^{-Q(\phi,0)}\,d\phi + \operatorname{Re} h(\phi, 1), \tag{4.5}$$

where $Q(\phi, 0)$ is obtained from the composite expansion. In determining $h(\phi, 1)$ we use the outer expansions for Q and θ. The integral in equation (4.4) may be evaluated to give (Collings [5])

$$\begin{aligned} n = \frac{1}{p^{\frac13}}\Big[1 &+ \frac{\varepsilon}{p}\{-c_1 - \tfrac13\cos\alpha\ln p\} + \frac{\varepsilon^2}{p^2}\{\tfrac23\sin^2\alpha + \cos^2\alpha\sin^2\alpha \\ &+ \tfrac29\cos^2\alpha(\ln p)^2 - \cos^2\alpha(\tfrac13 + \sin^2\alpha)\ln p \\ &+ \tfrac13 c_1\cos\alpha\ln p + \tfrac12 c_1^2 - c_2 p\}\Big] + O(\varepsilon^3) \end{aligned}$$

and

$$\begin{aligned} s = \int_0^{\phi} e^{-Q(\phi,0)}\,d\phi + \frac{1}{p^{\frac13}}\Big[\frac{\varepsilon}{p}.\tfrac12\sin\alpha &+ \frac{\varepsilon^2}{p^2}\{\tfrac12\sin\alpha\cos^3\alpha - \tfrac12\sin^3\alpha\cos\alpha \\ &- \sin\alpha\cos\alpha(\cos^2\alpha - \tfrac13)\ln p - \tfrac12 c_1\sin\alpha\}\Big] + O(\varepsilon^3), \end{aligned} \tag{4.6}$$

where $p = 1 + 3\varepsilon\phi\sin\alpha$.

It is clear then from equation (4.6) that if c_1 is positive, then the free surface is lower (for a given ϕ). Further, even when c_1 is negative, the term of $O(\varepsilon)$ in (4.6) eventually becomes negative, and the free surface is lower. Also it may be shown that if c_1 is positive, then s is increased (for a given ϕ) (Collings [5]).

As a specific example we take $g(\phi) = -\alpha H(\phi)$ the origins in the w and z planes being made to correspond by an appropriate choice of constants (that is, the point

B is placed at 0 in Fig. 1). Substituting $g(\phi)$ into equation (2.7) we obtain

$$\left.\begin{aligned}\theta_0(\phi,\psi) &= -\frac{\alpha}{\pi}\left[\frac{\pi}{2}+\operatorname{artan}\left(\frac{\sinh \pi\phi/2}{\sin \pi\psi/2}\right)\right],\\ Q_0(\phi,\psi) &= -\frac{\alpha}{2\pi}\ln\left[\frac{\cosh \pi\phi/2-\cos \pi\psi/2}{\cosh \pi\phi/2+\cos \pi\psi/2}\right].\end{aligned}\right\} \tag{4.7}$$

Comparing these equations with (4.2) and substituting the expression for $u(\phi)$ (namely $(\alpha/\pi)\operatorname{artan}(\sinh \pi\phi/2)$) into (4.3), it may be shown that

$$K_1 = \frac{4}{\pi}\sin\frac{\alpha}{2}\int_0^{\pi/2} \sec v\left(\cos\frac{\alpha v}{\pi}-\cos\frac{\alpha}{2}\right)dv. \tag{4.8}$$

In particular, when α is small,

$$K_1 = 0.136\alpha^3+O(\alpha^5). \tag{4.9}$$

Using (4.7), Q_1 and θ_1 may be determined, and then K_2 determined from (2.16) and (2.18). For example, when $\alpha = 0.0873$ corresponding to a 5° change of slope, $K_1 = 9.02\times 10^{-5}$ and $K_2 = 1.2\times 10^{-2}$. The free surface may now be determined using the composite expansion (3.14) for Q. Since this composite expansion breaks down upstream, we use the composite expansion which depends on the complex variable t (see equations (3.15) and (3.17)). The results of this calculation are shown in Fig. 4.

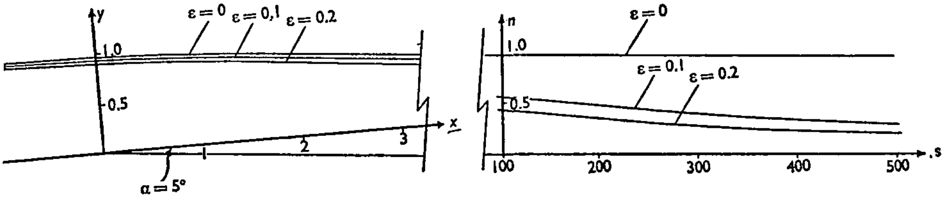

Fig. 4. The shape of the free surface for $\alpha = 5°$ and $\varepsilon = 0, 0.1, 0.2$.

5. A special case

In this section we consider the special case when the slope of the bed of the spillway is piece-wise constant with two simple jump discontinuities. Thus we let

$$\theta_0(x) = \begin{cases} 0 & \text{for } x<0,\\ -\gamma\alpha & \text{for } 0<x<x_1,\\ -\alpha & \text{for } x>x_1,\end{cases} \tag{5.1}$$

where γ is a constant. Thus $g(\phi)$ is given by

$$g(\phi) = \begin{cases} 0 & \text{for } \phi < 0, \\ -\gamma\alpha & \text{for } 0 < \phi < \phi_1, \\ -\alpha & \text{for } \phi > \phi_1. \end{cases} \tag{5.2}$$

Here we have chosen the origin of ϕ to coincide with the origin in the physical plane, while ϕ_1 depends on x_1, and will be determined once the asymptotic solution has been found. Using (5.2) in (2.7) we find that

$$\theta_0 = -\frac{\alpha}{\pi}\left[\frac{\pi}{2} + \operatorname{artan}\left(\frac{\sinh \pi(\phi-\phi_1)/2}{\sin \pi\psi/2}\right)\right] + \frac{\gamma\alpha}{\pi}\left[\operatorname{artan}\left(\frac{\sinh \pi(\phi-\phi_1)/2}{\sin \pi\psi/2}\right) - \operatorname{artan}\left(\frac{\sinh \pi\phi/2}{\sin \pi\psi/2}\right)\right] \tag{5.3}$$

and

$$Q_0 = -\frac{\alpha}{2\pi}\ln\left[\frac{\cosh \pi(\phi-\phi_1)/2 - \cos \pi\psi/2}{\cosh \pi(\phi-\phi_1)/2 + \cos \pi\psi/2}\right] + \frac{\gamma\alpha}{2\pi}\left\{\ln\left[\frac{\cosh \pi(\phi-\phi_1)/2 - \cos \pi\psi/2}{\cosh \pi(\phi-\phi_1)/2 + \cos \pi\psi/2}\right] - \ln\left[\frac{\cosh \pi\phi/2 - \cos \pi\psi/2}{\cosh \pi\phi/2 + \cos \pi\psi/2}\right]\right\}.$$

Proceeding as in Sections 2 and 3 we may determine Q_1, Q_2 and the constants c_1 and c_2. For $\phi_1 = 1$, the constant K_1 (that is, c_1) was computed numerically for a range of values of γ and α. The results are displayed in Fig. 5. When $\gamma = 1$, the case considered here reduces to the special case considered in Section 4 ($\gamma = 0$ also reduces to this special case, provided the origin for ϕ is translated to ϕ_1). When $\gamma > 1$ (the intermediate slope is steeper than the final slope), K_1 is positive and increases with γ; we recall that it was shown in Section 4 that when K_1 is positive, the free surface is lower than the free fall approximation would suggest. When $\gamma < 1$, K_1 becomes negative.

Next, on integrating (1.9) with respect to ϕ from $\phi = 0$ to ϕ_1 we find that

$$x_1 = \cos(\gamma\alpha)\int_0^{\phi_1} \exp\{-Q(\phi, 0)\}\, d\phi. \tag{5.4}$$

Since ϕ_1 is $O(1)$ with respect to ε, Q is given by the inner expansion in (5.4) and so $Q = Q_0 + O(\varepsilon)$. Substituting (5.3) into (5.4) we find that

$$x_1 = \cos(\gamma\alpha)\int_0^{\phi_1}\left\{\tanh\frac{\pi\phi}{4}\right\}^{\gamma\alpha/\pi}\left\{\tanh\frac{\pi(\phi_1-\phi)}{4}\right\}^{\alpha/\pi(1-\gamma)} d\phi + O(\varepsilon). \tag{5.5}$$

This is an algebraic relation between x_1 and ϕ_1 from which we may determine ϕ_1 for a given x_1. Clearly this procedure may be generalized to the case when the slope of the bed of the spillway is piece-wise constant, with a finite number of simple jump discontinuities.

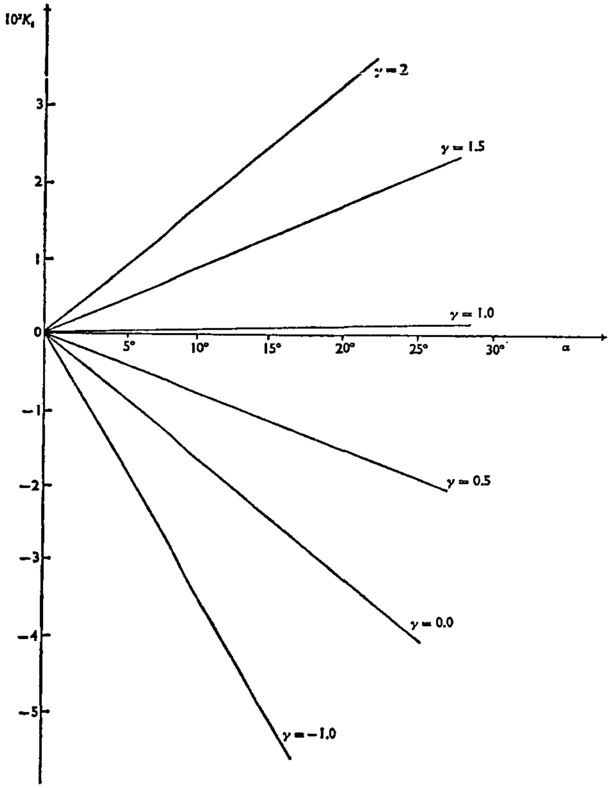

Fig. 5. 10^3 K_1 as a function of α (degrees), for $\gamma = -1\ 0$, 0.0, 0.5, 1.0, 1.5 and 2.0.

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