Note on the Polynomials which satisfy the Differential Equation

$$x\frac{d^2y}{dx^2}+(\gamma-x)\frac{dy}{dx}-\alpha y=0.$$

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§ 1. Laguerre has shown that if $\frac{1}{1-t}e^{\frac{xt}{1-t}}$ be expanded in ascending powers of t,

$$\frac{1}{1-t}e^{\frac{xt}{1-t}}=\Sigma f_n(x)\frac{t^n}{n\,!},$$

where $f_n(x)$ is a polynomial of degree n, satisfying the differential equation

$$x\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}-ny=0. \qquad \text{..................(1)}$$

A result which differs only in the substitution of $-x$ for x is given by Abel.*

Equation (1) is a special case of the equation

$$x\frac{d^2y}{dx^2}+(\gamma-x)\frac{dy}{dx}-\alpha y=0, \qquad \text{..................(2)}$$

of which one solution is $y=F(\alpha, \gamma, x)$, where

$$F(\alpha,\gamma,x)=1+\frac{a}{\gamma}x+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2\,!}+\frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3\,!}+\dots$$

This solution reduces to a polynomial if α is a negative integer, and it will be shown that the theorem of Abel and Laguerre can be extended to this polynomial.

§ 2. Let $y=\frac{1}{(1-t)^p}e^{\frac{xt}{1-t}}$, $\qquad$ ....(3)

* Laguerre, *Œuvres*, T. I., p. 436. Abel, *Œuvres*, T. II., p. 284. (1881).

then

$$\frac{\partial y}{\partial x}=\frac{t}{1-t}y,\ \frac{\partial^2 y}{\partial x^2}=\frac{t}{(1-t)^2}\,y,$$

$$\frac{\partial y}{\partial t}=\left[\frac{p}{(1-t)^{p+1}}+\frac{x}{(1-t)^{p+2}}\right]e^{\frac{xt}{1-t}}$$

$$=\frac{p(1-t)+x}{(1-t)^2}\,y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\therefore\ x\frac{\partial^2 y}{\partial x^2}+(p+x)\frac{\partial y}{\partial x}-t\frac{\partial y}{\partial t}=0.$$

Hence if $f(n, p, x)$ be the coefficient of $\frac{t^n}{n!}$ in the expansion of y in ascending powers of t, namely

$$y=\Sigma f(n, p, x)\frac{t^n}{n!}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

then $f(n, p, x)$ satisfies the differential equation

$$x\frac{d^2 f}{dx^2}+(p+x)\frac{df}{dx}-nf=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Comparing with equation (2) we have

$$f(n, p, x)=k\,F(-n, p, -x),$$

$$=k\left[1+\frac{n}{p}x+\frac{n(n-1)}{p(p+1)}\frac{x^2}{2!}+\ldots+\frac{x^n}{p(p+1)\ldots(p+n-1)}\right]^* \ (7)$$

By actually expanding the two factors of y, as given by (3), we see that the coefficient of x^n in $f(n, p, x)$ must be unity. Hence

$$k=p(p+1)\ldots(p+n-1). \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

§ 3. *Recurrence Formulae.*

From (3) and (5) we have

$$\Sigma f(n, p, x)\frac{t^n}{n!}=(1-t)\,\Sigma f(n, p+1, x)\frac{t^n}{n!},$$

whence we deduce

$$f(n, p, x)=f(n, p+1, x)-nf(n-1, p+1, x) \quad \ldots\ldots\ldots(9)$$

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* The result requires modification if p is a negative integer. In this case the *second* solution of (6) is required to express the coefficients for which $n > -p$.

In like manner if

$$f(n, 1, x) = f_n = n!\left[1 + nx + \frac{n(n-1)}{1^2 \cdot 2^2}x^2 + \frac{n(n-1)(n-2)}{1^2 \cdot 2^2 \cdot 3^2}x^3 + \dots \frac{x^n}{n!}\right], \quad \dots\dots\dots(10)$$

then

$$\Sigma f(n, p+1, x)\frac{t^n}{n!} = (1-t)^{-p}\Sigma f_n \frac{t^n}{n!},$$

$$= \left[1 + pt + p(p+1)\frac{t^2}{2!} + \dots\right]\Sigma f_n \frac{t^n}{n!},$$

whence

$$f(n, p+1, x) = f_n + npf_{n-1} + \frac{n(n-1)}{1 \cdot 2}p(p+1)f_{n-2} + \dots \quad (11)$$

A recurrence formula which affects n only can be obtained from (4),

$$(1-t)^2\frac{\partial y}{\partial t} - [p(1-t) + x]y = 0.$$

Differentiating this n times by the theorem of Leibniz and noting that

$$\left(\frac{\partial^n y}{\partial t^n}\right)_{t=0} = f(n, p, x), \text{ by } (5),$$

we find

$$f(n+1, p, x) = (2n+p+x)f(n, p, x) - n(n+p-1)f(n-1, p, x). \dots(12)$$

§4. Laguerre* has shown that the polynomial $f_n(x)$ forms the denominator of the n^{th} convergent to a continued fraction for the function

$$e^x \int_x^\infty \frac{e^{-x}}{x}\,dx.$$

It can be proved that $f(n, p+1, x)$ is similarly related to

$$x^p e^x \int_x^\infty \frac{e^{-x}}{x^{p+1}}\,dx,$$

and that the numerator of the convergent whose denominator is $f(n, p+1, x)$ also satisfies a linear differential equation of the second order. The continued fraction in question has been given

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* *Loc. Cit.*, p. 428.

by Professor Nielsen.* His method, however, does not suggest the differential equations, so that the following outline seems worth giving. The method followed is a modified form of Laguerre's.

§5. Let

$$S \equiv 1 - \frac{p}{x} + \frac{p(p+1)}{x^2} - \frac{p(p+1)(p+2)}{x^3} + \dots + \frac{p(p+1)\dots(p+2n-1)}{x^{2n}},$$

$$= \frac{\phi}{f} + \frac{1}{x^{2n}}\frac{R}{f} \qquad \text{...(13)}$$

where ϕ, f, R, are polynomials in x of degree n, n, $(n-1)$, respectively. That a unique expression of the latter form exists for S can be shown by the method of undetermined coefficients.

S satisfies the differential equation,

$$\frac{dS}{dx} - \frac{p+x}{x}S + \frac{a}{x^{2n+1}} + 1 = 0,$$

where $a = p(p+1)\dots(p+2n)$. ....(14)

Hence

$$\frac{\phi' f - \phi f'}{f^2} - \frac{2n}{x^{2n+1}}\frac{R}{f} + \frac{1}{x^{2n}}\frac{R' f - R f'}{f^2}$$
$$- \frac{p+x}{x}\left(\frac{\phi}{f} + \frac{1}{x^{2n}}\frac{R}{f}\right) + \frac{a}{x^{2n+1}} + 1 = 0.$$

When this identity is multiplied by xf^2, the part which is a polynomial in x must vanish. Taking, as we are entitled to do, the coefficient of x^n in f, and therefore in ϕ, as unity, we find in this way,

$$x(\phi' f - \phi f') - (p+x)\phi f + xf^2 + (a-b) = 0, \text{............(15)}$$

where b is the coefficient of x^{n-1} in R.

Writing (15) in the form

$$\frac{\phi' f - \phi f'}{f^2} - \frac{p+x}{x}\frac{\phi}{f} = -\frac{a-b}{xf^2} - 1,$$

we have a linear differential equation of the first order in $\frac{\phi}{f}$ whose solution, as given by the usual rule, is

$$\frac{e^{-x}}{x^p}\frac{\phi}{f} = \int_x^\infty \frac{e^{-x}}{x^p}\,dx + (ab)\int_x^\infty \frac{e^{-x}}{x^{p+1}f^2}\,dx. \text{............(16)}$$

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* *Theorie des Integrallogarithmus.* Leipzig (1906); p. 45.

$\frac{\phi}{f}$ will therefore be the n^{th} convergent to a continued fraction for

$$x^p e^x \int_x^\infty \frac{e^{-x}}{x^p} dx,$$

if we can prove that

$$r_n = \underset{n \to \infty}{\text{Lt}} (a - b) \int_x^\infty \frac{e^{-x}}{x^{p+1} f^2} dx = 0. \quad \ldots\ldots\ldots\ldots\ldots(17)$$

§ 6. *The differential equations for f and ϕ.*

The form of equation (15) shows that

(i) f has no repeated zero, since a common factor of f and f' would be a factor of $(a - b)$.

(ii) ϕ and f have no common factor.

Differentiating (15) to get rid of the unknown term b, we have

$$x(\phi'' f - \phi f'') + (\phi' f - \phi f') - (p + x)(\phi' f + \phi f') \\ - \phi f + 2x f f' + f^2 = 0,$$

or

$$[x\phi'' + (1 - p - x)\phi' - \phi + f + 2xf'] f = [xf'' + (1 + p + x) f'] \phi.$$

By virtue of (i) and (ii) this identity can only be true if

$$xf'' + (1 + p + x) f' = kf$$
$$x\phi'' + (1 - p - x)\phi' - \phi + f + 2xf' = k\phi$$

where k is a constant. Since the coefficient of x^n in f and ϕ is unity, we obtain $k = n$.

Hence the differential equation for f is

$$xf'' + (1 + p + x) f' - nf = 0 \ldots\ldots \ldots\ldots\ldots\ldots(18)$$

whence $f = f(n, p + 1, x)$, as defined by (7).

The differential equation for ϕ is then

$$x\phi'' + (1 - p - x)\phi' - (n + 1)\phi + f + 2xf' = 0 \quad \ldots\ldots\ldots(19)$$

where f has the above value.

§ 7. *Convergency.*

To establish (17) we require the value of b. This is obtained by equating the coefficients of x^{n-1} in

$$x^{2n} . S . f(n, p + 1, x) = x^{2n}\phi + R.$$

We find, using (14),

$$a-b=\frac{n!\,(p+n)!}{(p-1)!}.$$

$$r_n=\frac{n!\,(p+n)!}{(p-1)!}\int_x^\infty \frac{e^{-x}}{x^{p+1}f^2}dx.$$

From (7) it is evident that f' is positive for positive values of x, so that f is an increasing function of x for $x>0$.

Hence

$$0<r_n<\frac{n!\,(p+n)!}{p!}\cdot\frac{1}{x^{p+1}f^2}\int_x^\infty e^{-x}\,dx,$$

$$<\frac{n!\,(p+n)!}{p!\,[\,(p+1)\ldots(p+n)\,]^2}\;\frac{e^{-x}}{x^{p+1}},$$

where $f(x)$ has been replaced by $f(0)$.

$$i.e.\;\;0<r_n<\frac{n!}{(p+1)\ldots(p+n)}\;\frac{e^{-x}}{x^{p+1}},$$

$$<\frac{p!}{(n+1)\ldots(n+p)}\;\frac{e^{-x}}{x^{p+1}}.$$

and therefore $\underset{n\to\infty}{\mathrm{Lt}}\; r_n=0.$

§8. *The recurrence formulae for f and ϕ.*

We have seen, (12), that

$$f_{n+1}=(2n+p+1+x)f_n-n\,(n+p)f_{n-1},$$

when $f_n\equiv f(n,\,p+1,\,x)$.

It can be shown that

$$\phi_{n+1}=(2n+p+1+x)\,\phi_n-n(n+p)\phi_{n-1}.$$

As the proof is somewhat lengthy, in the form in which I have obtained it, it is not given here.

These relations enable us to form the continued fraction whose n^{th} convergent is $\dfrac{\phi_n}{f_n}$. We find

$$\int_x^\infty \frac{e^{-x}}{x^p}\,dx=\frac{e^{-x}}{x^p}\left[\,1-\frac{p}{x+p+1-}\,\frac{p+1}{x+p+3-}\,\frac{2\,(p+2)}{x+p+5-}\cdots\right].$$

But

$$\int_x^\infty \frac{e^{-x}}{x^p}\,dx = \frac{e^{-x}}{x^p} - p\int_x^\infty \frac{e^{-x}}{x^{p+1}}\,dx.$$

Hence

$$\int^\infty \frac{e^{-x}}{x^{p+1}}\,dx = \frac{e^{-x}}{x^p}\left[\frac{1}{x+p+1-}\,\frac{p+1}{x+p+3-}\,\frac{2(p+2)}{x+p+5-}\cdots\right]$$

which is Nielsen's result.

If the n^{th} convergent to the second continued fraction be $\frac{g_{n-1}}{f_n}$, the differential equation for g_{n-1} is

$$x\frac{d^2g}{dx^2} + (1-p-x)\frac{dg}{dx} - (n+1)g + 2f' = 0,$$

which is rather simpler than the corresponding equation for ϕ.