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PRESERVATION OF MEAN INACTIVITY TIME ORDERING FOR COHERENT SYSTEMS

T. V. RAO* ** AND
SAMEEN NAQVI,* *** *Indian Institute of Technology Hyderabad*

Abstract

Preservation of stochastic orders through the system signature has captured the attention of researchers in recent years. Signature-based comparisons have been made for the usual stochastic order, hazard rate order, and likelihood ratio orders. However, for the mean residual life (MRL) order, it has recently been proved that the preservation result does not hold true in general, but rather holds for a particular class of distributions. In this paper, we study whether or not a similar preservation result holds for the mean inactivity time (MIT) order. We prove that the MIT order is not preserved from signatures to system lifetimes with independent and identically distributed (i.i.d.) components, but holds for special classes of distributions. The relationship between these classes and the order statistics is also highlighted. Furthermore, the distribution-free comparison of the performance of coherent systems with dependent and identically distributed (d.i.d.) components is studied under the MIT ordering, using diagonal-dependent copulas and distorted distributions.

Keywords: Coherent systems; copula; distorted distribution; mean inactivity time; stochastic orders; system signature

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1. Introduction

The notion of the system signature, introduced by Samaniego [27], is a useful tool for studying coherent systems (see Barlow and Proschan [1] for a definition). For a coherent system with independent and identically distributed (i.i.d.) component lifetimes $X_1, \ldots, X_n$, having distribution function F, the system signature $\mathbf{s} = (s_1, \ldots, s_n)$ is an n-dimensional probability vector such that $s_i = \mathbb{P}(T = X_{i:n})$, where T is the system lifetime, and $X_{1:n}, \ldots, X_{n:n}$, are the order statistics of the component lifetimes. The system's life distribution $F_T(t) = \mathbb{P}(T \le t)$ can be represented explicitly as a function of the component life distribution F using the signature as follows:

$$F_T(t) = \sum_{j=1}^{n} \left(\sum_{i=1}^{j} s_i \right) \binom{n}{j} F^j(t) \overline{F}^{n-j}(t), \qquad t \ge 0. \tag{1.1}$$

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* Postal address: Indian Institute of Technology Hyderabad, Hyderabad 502285, India.
** Email address: ma19resch11001@iith.ac.in
*** Email address: sameen@math.iith.ac.in

Since the system signature is only equipped to deal with systems with a single type of component, Coolen and Coolen-Maturi [8] and Samaniego and Navarro [29] extended the concept of the system signature to that of the survival signature and the failure signature, respectively, to deal with real-world systems having multiple types of components. The system signature $\mathbf{s} = (s_1, \ldots, s_n)$ has the following relationship to the failure signature (denoted by $\mathbf{b} = (b_1, \ldots, b_n)$) and survival signature (denoted by $\overline{\mathbf{b}} = (\overline{b}_1, \ldots, \overline{b}_n)$) of an n-component coherent system with i.i.d. components (see Lindqvist *et al.* [15]):

$$b_j = \sum_{i=1}^{j} s_i, \qquad j = 1, \ldots, n, \tag{1.2}$$

$$\overline{b}_j = \sum_{i=n-j+1}^{n} s_i, \qquad j = 1, \ldots, n. \tag{1.3}$$

Preservation of stochastic orders using the system signature was first studied in Kocher *et al.* [11]. For $\mathbf{s}_1$ and $\mathbf{s}_2$ denoting the signatures of two systems whose lifetimes are T_1 and T_2, respectively, they showed that if $\mathbf{s}_1 \leq_* \mathbf{s}_2$, then $T_1 \leq_* T_2$, where $*$ represents the st, hr, or lr order (defined in Section 2); a similar preservation for the rh order (defined in Section 2) is in Navarro and Rubio [22]. Problems related to stochastic orderings under different conditions have been studied using signatures by various researchers (see Block *et al.* [6], Boland and Samaniego [7], Li and Zhang [14], Navarro *et al.* [23], and Zhang [31]). Using (1.3), Lindqvist *et al.* [15] proved that the mean residual life (MRL) ordering is not preserved from system signatures to system lifetimes for mixed coherent systems with i.i.d. components. They defined classes of distribution functions

$$\mathcal{F}_n = \left\{ F : \binom{n}{j} \int_0^\infty \overline{F}^j(u) F^{n-j}(u)\, \mathrm{d}u, \text{ is decreasing on } j = 1, \ldots, n \right\}, \qquad n \geq 2.$$

Using the stochastic ordering relation between the system signature and survival signature, Lindqvist *et al.* [15, Lemma 5] proved that for mixed coherent systems with i.i.d. components, if the component life distribution $F \in \mathcal{F}_n$, $n \geq 2$, then the MRL ordering is preserved. This gave us the motivation to check whether or not the mean inactivity time (MIT) ordering is preserved for coherent systems with i.i.d. components, as the study of the MIT ordering is equally important in reliability theory (see Kayid and Ahmad [9] and Kayid *et al.* [10]). Thus, we make use of (1.2) to prove that the MIT ordering is not preserved, in general; rather, if the component life distribution $F \in \overline{\mathcal{F}}_n$ or $\widetilde{\mathcal{F}}_n$, $n \geq 2$, then the MIT ordering is preserved for mixed coherent systems with i.i.d. components, where

$$\overline{\mathcal{F}}_n = \left\{ F : \binom{n}{j} \int_0^\infty F^j(u) \overline{F}^{n-j}(u)\, \mathrm{d}u \text{ is decreasing in } j = 1, \ldots, n \right\}, \tag{1.4}$$

and

$$\widetilde{\mathcal{F}}_n = \left\{ F : \binom{n}{j} \int_0^1 \frac{w^j (1-w)^{n-j}}{f(F^{-1}(w))} dw \text{ is decreasing in } j = 1, \ldots, n, f = F' \right\}. \tag{1.5}$$

After studying coherent systems with i.i.d. components, it is natural to study MIT comparison results for coherent systems with dependent and identically distributed (d.i.d.) components. The study of comparisons of coherent systems composed of d.i.d. components is of

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practical importance in reliability theory, since the assumption of dependent components is more intuitive (for example, the lifetimes of components in electronic devices, which are often manufactured by the same firm or produced in the same environment, tend to be d.i.d.). In the literature, several studies have been conducted to incorporate dependency among the components. For instance, Navarro *et al.* [18] studied comparisons of coherent systems with identically distributed components under different stochastic criteria (st, hr, rh, and lr orders) using domination functions. Furthermore, Navarro *et al.* [19] studied the preservation of different stochastic orders (st, hr, and rh) under the formation of generalized distorted distributions, and these results can be applied to coherent systems with identically distributed components. Navarro and Gomis [21] studied comparisons of coherent systems with identically distributed components with respect to the MRL ordering by representing the system reliability function as a dual distorted distribution. However, to the best of our knowledge, no study exists on stochastic comparisons of coherent systems with d.i.d. components with respect to the MIT ordering. Thus, to fill this gap in the literature, we make comparisons among coherent systems with 1–3 d.i.d. components using distorted distributions, when the underlying copula is a diagonal-dependent (DD) copula. The comparison results obtained are based on the structure of the system and on the properties of the underlying copula (Clayton–Oakes and Gumbel–Hougaard copulas). These comparisons are distribution-free with respect to the component life distribution. Note that these comparison results can also be applied to coherent systems with i.i.d. components.

The paper is organized as follows. In Section 2, we present preliminaries which will be helpful in proving the main results. The results on preservation of the MIT ordering for mixed coherent systems with i.i.d. components, as well as their connection with order statistics, are presented in Section 3. In Section 4, we present the MIT ordering comparisons for coherent systems with 1–3 d.i.d. components, and we discuss the connection between underlying copula properties and the MIT ordering properties of coherent systems with 1–3 d.i.d. components. Finally, we conclude the study with a discussion on coherent systems with 1–4 d.i.d. components.

2. Preliminaries

In this section, we first present the definition of the MIT, then define some stochastic orderings which we will use in the sequel. The support of the random variables, unless specified otherwise, is $(0, \infty)$. Throughout the sequel, whenever we say a function is increasing (decreasing), it means the function is non-decreasing (non-increasing), and we assume $C/0 = \infty$ for $C > 0$, $0^0 = 0$, and $0/0$ is not defined.

Definition 2.1. Let X be the lifetime of a unit, which could be a living organism or a mechanical component, with distribution function F. Ruiz and Navarro [27] defined the inactivity time of the random variable X by a conditional random variable $X_t = (t - X|X \leq t)$, which is the time elapsed from the failure of the component given that its lifetime is less than or equal to t. The conditional random variable X_t may also be called the reversed residual life. The mean inactivity time (MIT) of X is

$$\mathbb{E}(X_t) = \mathbb{E}(t - X|X \leq t) = \frac{\int_0^t F(u)du}{F(t)}, \qquad t > 0.$$

We now present the definitions of various stochastic orders, which are useful tools in comparing system lifetimes.

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Definition 2.2. Let X and Y be two random variables with survival functions $\overline{F}_X$ and $\overline{F}_Y$, distribution functions F_X and F_Y, and Lebesgue probability density functions f_X and f_Y, respectively.

(i) We say that X is smaller than Y in the *usual stochastic order* (denoted by $X \leq_{st} Y$) if, and only if, $\overline{F}_X(t) \geq \overline{F}_Y(t)$, for $t \in (0, \infty)$.

(ii) We say that X is smaller than Y in the *hazard rate order* (denoted by $X \leq_{hr} Y$) if, and only if, $\dfrac{\overline{F}_X(t)}{\overline{F}_Y(t)}$ is decreasing in $t \in (0, \infty)$.

(iii) We say that X is smaller than Y in the *reversed hazard rate order* (denoted by $X \leq_{rh} Y$) if, and only if, $\dfrac{F_X(t)}{F_Y(t)}$ is decreasing in $t \in (0, \infty)$.

(iv) We say that X is smaller than Y in the *mean residual life order* (denoted by $X \leq_{mrl} Y$) if, and only if, $\dfrac{\int_t^\infty \overline{F}_X(u)du}{\int_t^\infty \overline{F}_Y(u)du}$ is decreasing in $t \in (0, \infty)$.

(v) We say that X is smaller than Y in the *mean inactivity time order* (denoted by $X \leq_{mit} Y$) if, and only if,
$$\frac{\int_0^t F_X(u)du}{\int_0^t F_Y(u)du} \text{ is decreasing in } t \in (0, \infty).$$

(vi) We say that X is smaller than Y in the *likelihood ratio order* (denoted by $X \leq_{lr} Y$) if, and only if, $\dfrac{f_X(t)}{f_Y(t)}$ is decreasing in $t \in (0, \infty)$.

It is well known that

$$X \leq_{lr} Y \Longrightarrow X \leq_{hr} Y \Longrightarrow X \leq_{mrl} Y, \quad \text{and} \quad X \leq_{hr} Y \Longrightarrow X \leq_{st} Y;$$

$$X \leq_{lr} Y \Longrightarrow X \leq_{rh} Y \Longrightarrow X \leq_{mit} Y, \quad \text{and} \quad X \leq_{rh} Y \Longrightarrow X \leq_{st} Y.$$

For a comprehensive discussion of these orders, one may refer to Belzunce *et al.* [4], Belzunce *et al.* [3], Lai and Xie [12], Li and Li [13], Mosler and Scarsini [16], and Shaked and Shanthikumar [30].

The stochastic orders given above, in Definition 2.2, can also be used to compare the signatures of coherent systems (Kochar *et al.* [11]). One may refer to Navarro *et al.* [23], Navarro [17], and Samaniego [28] for a comprehensive treatment of the subject.

Definition 2.3. Let $\mathbf{s}_1 = (s_{11}, \ldots, s_{1n})$ and $\mathbf{s}_2 = (s_{21}, \ldots, s_{2n})$ be the signatures of two mixed n-systems having components with i.i.d. lifetimes, and let $\mathbf{b}_1 = (b_{11}, \ldots, b_{1n})$ and $\mathbf{b}_2 = (b_{21}, \ldots, b_{2n})$, respectively, be their failure signatures.

(i) The signature $\mathbf{s}_1$ is said to be smaller than $\mathbf{s}_2$ in the *usual stochastic order* (denoted by $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$) if, and only if, $b_{1j} \geq b_{2j}$ for $j = 1, \ldots, n$.

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(ii) The signature $\mathbf{s}_1$ is said to be smaller than $\mathbf{s}_2$ in the *hazard rate order* (denoted by $\mathbf{s}_1 \le_{hr} \mathbf{s}_2$) if, and only if, $\dfrac{1-b_{2n-j}}{1-b_{1n-j}}$ is decreasing in $j=1,\ldots,n$.

(iii) The signature $\mathbf{s}_1$ is said to be smaller than $\mathbf{s}_2$ in the *reversed hazard rate order* (denoted by $\mathbf{s}_1 \le_{rh} \mathbf{s}_2$) if, and only if, $\dfrac{b_{1j}}{b_{2j}}$ is decreasing in $j=1,\ldots,n$.

(iv) The signature $\mathbf{s}_1$ is said to be smaller than $\mathbf{s}_2$ in the *mean residual lifetime order* (denoted by $\mathbf{s}_1 \le_{mrl} \mathbf{s}_2$) if, and only if, $\dfrac{\sum_{i=1}^{j}(1-b_{2n-i})}{\sum_{i=1}^{j}(1-b_{1n-i})}$ is decreasing in $j=1,\ldots,n$.

(v) The signature $\mathbf{s}_1$ is said to be smaller than $\mathbf{s}_2$ in the *mean inactivity time order* (denoted by $\mathbf{s}_1 \le_{mit} \mathbf{s}_2$) if, and only if, $\dfrac{\sum_{i=1}^{j} b_{1i}}{\sum_{i=1}^{j} b_{2i}}$ is decreasing in $j=1,\ldots,n$.

(vi) The signature $\mathbf{s}_1$ is said to be smaller than $\mathbf{s}_2$ in the *likelihood ratio order* (denoted by $\mathbf{s}_1 \le_{lr} \mathbf{s}_2$) if, and only if, $\dfrac{b_{1j}-b_{1j-1}}{b_{2j}-b_{2j-1}}$ is decreasing in $j=1,\ldots,n$; with $b_{10}=b_{20}=0$.

The following implications hold:

$$\mathbf{s}_1 \le_{lr} \mathbf{s}_2 \Longrightarrow \mathbf{s}_1 \le_{hr} \mathbf{s}_2 \Longrightarrow \mathbf{s}_1 \le_{mrl} \mathbf{s}_2, \qquad \text{and} \qquad \mathbf{s}_1 \le_{hr} \mathbf{s}_2 \Longrightarrow \mathbf{s}_1 \le_{st} \mathbf{s}_2;$$

$$\mathbf{s}_1 \le_{lr} \mathbf{s}_2 \Longrightarrow \mathbf{s}_1 \le_{rh} \mathbf{s}_2 \Longrightarrow \mathbf{s}_1 \le_{mit} \mathbf{s}_2, \qquad \text{and} \qquad \mathbf{s}_1 \le_{rh} \mathbf{s}_2 \Longrightarrow \mathbf{s}_1 \le_{st} \mathbf{s}_2.$$

In addition, we know from the literature (Kocher *et al.* [11], Navarro and Rubio [22], Samaniego [28]) that $\mathbf{s}_1 \le_{*} \mathbf{s}_2 \Longrightarrow T_1 \le_{*} T_2$, where T_1 and T_2 are the lifetimes of two mixed systems and $*$ may represent st, hr, or even mrl for a specific class of distribution (see Lindqvist *et al.* [15]). Now, our interest is in seeing whether a similar implication can be obtained for the MIT ordering, i.e., whether

$$\mathbf{s}_1 \le_{mit} \mathbf{s}_2 \Longrightarrow T_1 \le_{mit} T_2. \tag{2.1}$$

We know that whenever $\mathbf{s}_1 \le_{rh} \mathbf{s}_2$ (and hence $\mathbf{s}_1 \le_{mit} \mathbf{s}_2$), we have $T_1 \le_{rh} T_2$, which further implies $T_1 \le_{mit} T_2$. This shows that (2.1) holds true when $\mathbf{s}_1 \le_{rh} \mathbf{s}_2$. However, the interesting case will be to verify (2.1), when $\mathbf{s}_1 \le_{mit} \mathbf{s}_2$ but $\mathbf{s}_1 \nleq_{rh} \mathbf{s}_2$. The following example illustrates this scenario.

Example 2.1. Consider the signatures

$$\mathbf{s}_1 = \left(\frac{1}{4},\ 0,\ \frac{3}{4}\right) \text{ and } \qquad \mathbf{s}_2 = \left(\frac{1}{8},\ \frac{2}{8},\ \frac{5}{8}\right).$$

It can be seen that

$$\mathbf{s}_1 \le_{mit} \mathbf{s}_2, \text{ but } \mathbf{s}_1 \nleq_{rh} \mathbf{s}_2.$$

Let the common component life distribution of the corresponding mixed systems be

$$F(t) = \begin{cases} 0 & \text{if } t < 0, \\ p & \text{if } 0 \le t < 1,\ 0 < p < 1, \\ 1 & \text{if } t \ge 1. \end{cases}$$

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Here the system lifetimes T_1 and T_2 can take values 0 and 1. Now, $T_1 \leq_{mit} T_2$ if and only if

$$\mathbb{E}(t_0 - T_1 | T_1 \leq t_0) \geq \mathbb{E}\,(t_0 - T_2 | T_2 \leq t_0) \qquad \text{for } t_0 > 0. \tag{2.2}$$

If $t_0 < 1$, (2.2) holds true, and if $t_0 \geq 1$, (2.2) is equivalent to

$$\mathbb{P}(T_1 \leq 0) \geq \mathbb{P}(T_2 \leq 0). \tag{2.3}$$

Taking $r = \dfrac{p}{1-p}$ and using (1.1), (2.3) can be rewritten as

$$\frac{1}{4}r + \frac{1}{4}r^2 \geq \frac{1}{8}r + \frac{3}{8}r^2,$$

which on simplification gives $r \leq 1$. Hence, $T_1 \leq_{mit} T_2$ if, and only if, $p \leq \dfrac{1}{2}$, and $T_1 \nleq_{mit} T_2$, for $p > \dfrac{1}{2}$.

Thus, from the above example, it can be concluded that (2.1) does not hold true in general, but may hold true for some specific class of distributions. We search for this specific class in the section below.

3. Preservation of MIT ordering for coherent systems with i.i.d. components

Consider a mixed coherent system of order n whose system signature and failure signature are $\mathbf{s} = (s_1, \ldots, s_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, respectively. Using the failure signature, the system life distribution (Equation (1.1)) can be rewritten as follows:

$$F_T(t) = \mathbb{P}(T \leq t) = \sum_{j=1}^{n} b_j \binom{n}{j} F^j(t)\overline{F}^{n-j}(t), \qquad t \geq 0. \tag{3.1}$$

Using (3.1), we give below a necessary and sufficient condition for two system lifetimes to be ordered with respect to the MIT ordering. The proof is omitted as it follows from Definition 2.2(v) and (3.1).

Lemma 3.1. *Let* $\boldsymbol{b}_1 = (b_{11}, \ldots, b_{1n})$ *and* $\boldsymbol{b}_2 = (b_{21}, \ldots, b_{2n})$ *be the failure signatures of two mixed coherent systems of order n. Let* T_1 *and* T_2 *be their respective lifetimes, and F be the components' life distribution. Then* $T_1 \leq_{mit} T_2$ *if, and only if,*

$$I(t) = \frac{\displaystyle\sum_{j=1}^{n} b_{1j} \binom{n}{j} \int_0^t F^j(u)\overline{F}^{n-j}(u)\, du}{\displaystyle\sum_{j=1}^{n} b_{2j} \binom{n}{j} \int_0^t F^j(u)\overline{F}^{n-j}(u)\, du}$$

is decreasing in $t \in (0, \infty)$.

In proving the main result of this section, we will make use of the following lemmas. To begin with, let us reconsider the class of distributions $\overline{\mathcal{F}}_n$ defined in (1.4):

$$\overline{\mathcal{F}}_n = \left\{ F : \binom{n}{j} \int_0^\infty F^j(u)\overline{F}^{n-j}(u)\, \mathrm{d}u \text{ is decreasing in } j = 1, \ldots, n \right\}.$$

That is, for a given $n\,(\in\mathbb{N})\geq 2$, $\overline{\mathcal{F}}_n$ contains all the distribution functions F such that $F(x)=0$ for $x<0$ and

$$\binom{n}{j}\int_0^\infty F^j(u)\overline{F}^{n-j}(u)\,\mathrm{d}u \geq \binom{n}{j+1}\int_0^\infty F^{j+1}(u)\overline{F}^{n-j-1}(u)\,\mathrm{d}u, \qquad j=1,\ldots,n-1.$$

Lemma 3.2. *For any distribution F and $n\geq 2$,*

(i) *$F\in\overline{\mathcal{F}}_n$ if, and only if, $\binom{n}{j}\int_0^t F^j(u)\overline{F}^{n-j}(u)du$ is decreasing in $j=1,\ldots,n$, for $0\leq t<\infty$;*

(ii) *$\dfrac{\int_0^s F^j(u)\overline{F}^{n-j}(u)\,du}{\int_0^t F^j(u)\overline{F}^{n-j}(u)\,du}$ is decreasing in $j=1,\ldots,n$, for $0\leq s\leq t<\infty$.*

The proof of Lemma 3.2 is omitted as it can be obtained similarly to the proofs of Lemmas 1 and 2 of Lindqvist *et al.* [15]. The next lemma establishes stochastic ordering relations between system signatures and the corresponding failure signatures. Recall that Lindqvist *et al.* [15, Lemma 5] obtained a similar result where $X\leq_{hr} Y$ if, and only if, $\mathbf{s}_1\leq_{mrl}\mathbf{s}_2$, such that the probability mass functions of the random variables X and Y are expressed in terms of survival signatures. Here, we express the probability mass functions of the random variables X and Y in terms of failure signatures and obtain corresponding results for the rh and lr orderings.

Lemma 3.3. *Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be the signatures of two mixed coherent systems of order n, and let $\boldsymbol{b}_1=(b_{11},\ldots,b_{1n})$ and $\boldsymbol{b}_2=(b_{21},\ldots,b_{2n})$ be their respective failure signatures. Let X and Y be two discrete random variables which take values from the set $\{1,\ldots,n\}$, and $\mathbb{P}(X=j)=\dfrac{b_{1j}}{\sum_{i=1}^n b_{1i}}$, $\mathbb{P}(Y=j)=\dfrac{b_{2j}}{\sum_{i=1}^n b_{2i}}$, for $j=1,\ldots,n$. Then*

(i) *$X\leq_{rh} Y$ if, and only if, $\mathbf{s}_1\leq_{mit}\mathbf{s}_2$;*

(ii) *$X\leq_{lr} Y$ if, and only if, $\mathbf{s}_1\leq_{rh}\mathbf{s}_2$.*

Proof. (i) From Definition 2.2(iii), $X\leq_{rh} Y$ if, and only if,

$$\frac{\sum_{k=1}^l b_{1k}}{\sum_{k=1}^l b_{2k}}\cdot\frac{\sum_{i=1}^n b_{2i}}{\sum_{i=1}^n b_{1i}} \text{ is decreasing in } l=1,\ldots,n$$

$$\iff \frac{\sum_{k=1}^l b_{1k}}{\sum_{k=1}^l b_{2k}} \text{ is decreasing in } l=1,\ldots,n$$

$$\iff \mathbf{s}_1\leq_{mit}\mathbf{s}_2, \text{ from Definition 2.3(v).}$$

(ii) The proof is omitted as it can be obtained similarly to that of (i). □

The following example illustrates the applicability of Lemma 3.3.

Example 3.1.

(i) Consider signatures $\mathbf{s}_1=(1/3,2/3,0,0)$ and $\mathbf{s}_2=(1/6,2/3,1/6,0)$. Then the corresponding failure signatures are $\mathbf{b}_1=(b_{11},b_{12},b_{13},b_{14})=(1/3,1,1,1)$ and

$\mathbf{b}_2 = (b_{21}, b_{22}, b_{23}, b_{24}) = (1/6, 5/6, 1, 1)$. Note that $\mathbf{s}_1 \leq_{mit} \mathbf{s}_2$. Define the discrete random variables X and Y sothat $\mathbb{P}(X = j) = \frac{b_{1j}}{\sum_{i=1}^{4} b_{1i}}$ and $\mathbb{P}(Y = j) = \frac{b_{2j}}{\sum_{i=1}^{4} b_{2i}}$, for $j = 1, 2, 3, 4$, i.e.,

$$\mathbb{P}(X = j) = \begin{cases} 1/10 & \text{if } j = 1, \\ 3/10 & \text{if } j = 2, 3, 4, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbb{P}(Y = j) = \begin{cases} 1/18 & \text{if } j = 1, \\ 5/18 & \text{if } j = 2, \\ 1/3 & \text{if } j = 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

It can be seen that $X \leq_{rh} Y$, and the same can be inferred directly from Lemma 3.3(i).

(ii) Consider signatures $\mathbf{s}_1 = (1/2, 1/2, 0, 0)$ and $\mathbf{s}_2 = (1/3, 1/3, 1/3, 0)$. Then the corresponding failure signatures are $\mathbf{b}_1 = (b_{11}, b_{12}, b_{13}, b_{14}) = (1/2, 1, 1, 1)$ and $\mathbf{b}_2 = (b_{21}, b_{22}, b_{23}, b_{24}) = (1/3, 2/3, 1, 1)$. Note that $\mathbf{s}_1 \leq_{rh} \mathbf{s}_2$. Define the discrete random variables X and Y so that $\mathbb{P}(X = j) = \frac{b_{1j}}{\sum_{i=1}^{4} b_{1i}}$ and $\mathbb{P}(Y = j) = \frac{b_{2j}}{\sum_{i=1}^{4} b_{2i}}$, for $j = 1, 2, 3, 4$, i.e.,

$$\mathbb{P}(X = j) = \begin{cases} 1/7 & \text{if } j = 1, \\ 2/7 & \text{if } j = 2, 3, 4, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbb{P}(Y = j) = \begin{cases} 1/9 & \text{if } j = 1, \\ 2/9 & \text{if } j = 2, \\ 1/3 & \text{if } j = 3, 4, \\ 0 & \text{otherwise.} \end{cases}$$

It can be seen that $X \leq_{lr} Y$, and the same can be inferred directly from Lemma 3.3(ii).

To state our next lemma for discrete positive distributions, we employ Theorem 1.B.50 of Shaked and Shanthikumar [30].

Lemma 3.4. *Let X and Y be random variables as defined in Lemma 3.3, such that $X \leq_{rh} Y$. Let $\alpha(j)$ and $\beta(j)$, for $j = 1, \ldots, n$, be numbers such that $\beta(j)$ is positive, and $\frac{\alpha(j)}{\beta(j)}$ and $\beta(j)$ are decreasing in $j = 1, \ldots, n$. Then*

$$\frac{\sum_{j=1}^{n} \alpha(j) b_{2j}}{\sum_{j=1}^{n} \beta(j) b_{2j}} \leq \frac{\sum_{j=1}^{n} \alpha(j) b_{1j}}{\sum_{j=1}^{n} \beta(j) b_{1j}}. \tag{3.2}$$

Now we are ready to prove our main result. We show that, if the component life distribution of mixed systems belongs to $\overline{\mathcal{F}}_n$, $n \geq 2$, then the MIT order is preserved from signatures to system lifetimes (i.e., (2.1) holds true). Also, we prove that these classes of distribution functions $\overline{\mathcal{F}}_n$, $n \geq 2$, are strictly nested and have a nonempty intersection.

Theorem 3.1.

(i) Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be the signatures of two mixed systems of order n, and let $\mathbf{b}_1 = (b_{11}, \ldots, b_{1n})$ and $\mathbf{b}_2 = (b_{21}, \ldots, b_{2n})$ be their respective failure signatures. Let T_1

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and T_2 be the lifetimes of these systems, respectively. If the component life distribution $F \in \overline{\mathcal{F}}_n$, $n \geq 2$, and $\mathbf{s}_1 \leq_{mit} \mathbf{s}_2$, then $T_1 \leq_{mit} T_2$.

(ii) *The classes of distribution functions $\overline{\mathcal{F}}_n$ are such that $\overline{\mathcal{F}}_n \subset \overline{\mathcal{F}}_{n-1}$, $n \geq 3$, and $\cap \overline{\mathcal{F}}_n \neq \phi$, $n \geq 2$.*

Proof.

(i) Assume $F \in \overline{\mathcal{F}}_n$ and $\mathbf{s}_1 \leq_{mit} \mathbf{s}_2$. For $j = 1, \ldots, n$, define

$$\alpha(j) = \binom{n}{j} \int_0^s F^j(u)\overline{F}^{n-j}(u)\, du, \quad \text{and}$$

$$\beta(j) = \binom{n}{j} \int_0^t F^j(u)\overline{F}^{n-j}(u)\, du.$$

Using Lemma 3.2(i), it can be seen that $\beta(j)$ is decreasing in $j = 1, \ldots, n$, and using Lemma 3.2(ii), it follows that $\dfrac{\alpha(j)}{\beta(j)}$ is decreasing in $j = 1, \ldots, n$, for $0 \leq s \leq t < \infty$. Since $X \leq_{rh} Y$ by Lemma 3.3(i), it follows that conditions of Lemma 3.4 are satisfied. Thus, on substituting $\alpha(j)$ and $\beta(j)$ in (3.2), we get

$$\frac{\sum_{j=1}^{n} b_{2j} \binom{n}{j} \int_0^s F^j(u)\overline{F}^{n-j}(u)\, du}{\sum_{j=1}^{n} b_{2j} \binom{n}{j} \int_0^t F^j(u)\overline{F}^{n-j}(u)\, du} \leq \frac{\sum_{j=1}^{n} b_{1j} \binom{n}{j} \int_0^s F^j(u)\overline{F}^{n-j}(u)\, du}{\sum_{j=1}^{n} b_{1j} \binom{n}{j} \int_0^t F^j(u)\overline{F}^{n-j}(u)\, du},$$

which implies $T_1 \leq_{mit} T_2$, from Lemma 3.1.

(ii) Suppose $F \in \overline{\mathcal{F}}_n$. Let

$$d_{n-1,j} = \binom{n-1}{j} \int_0^\infty F^j(u)\overline{F}^{n-j-1}(u)\, du.$$

Multiplying by $d_{n-1,j}$ on both sides of the identity $F(u) + \overline{F}(u) = 1$, we get

$$\begin{aligned} d_{n-1,j} &= \binom{n-1}{j} \int_0^\infty F^{j+1}(u)\, \overline{F}^{n-j-1}(u)du + \binom{n-1}{j} \int_0^\infty F^j(u)\, \overline{F}^{n-j}(u)du \\ &= \frac{j+1}{n} d_{n,j+1} + \frac{n-j}{n} d_{n,j}. \end{aligned}$$

Now, in order to show that $F \in \overline{\mathcal{F}}_{n-1}$, it suffices to show that $d_{n-1,j+1} - d_{n-1,j} \leq 0$, for $j = 1, \ldots, n-2$. Here,

$$\begin{aligned} d_{n-1,j+1} - d_{n-1,j} &= \frac{j+2}{n} d_{n,j+2} + \frac{n-j-1}{n} d_{n,j+1} - \frac{j+1}{n} d_{n,j+1} - \frac{n-j}{n} d_{n,j}, \\ &= \frac{j+2}{n}\left(d_{n,j+2} - d_{n,j+1}\right) + \frac{j+1}{n}\left(d_{n,j+1} - d_{n,j}\right), \\ &\leq 0, \end{aligned}$$

since $F \in \overline{\mathcal{F}}_n$. Thus, $F \in \overline{\mathcal{F}}_{n-1}$ also, implying that $\overline{\mathcal{F}}_n \subseteq \overline{\mathcal{F}}_{n-1}$, $n \geq 3$.

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Now, in order to show that $\overline{\mathcal{F}}_n$ is strictly contained in $\overline{\mathcal{F}}_{n-1}$, let us consider

$$F(u)=\begin{cases}0 & \text{if } u<0,\\ p & \text{if } 0\le u<1,\ 0<p<1,\\ 1 & \text{if } u\ge 1.\end{cases}$$

Here,

$$\begin{aligned}d_{n,j+1}-d_{n,j}&=\binom{n}{j+1}p^{j+1}(1-p)^{n-j-1}-\binom{n}{j}p^j(1-p)^{n-j}\\&=\binom{n}{j}\frac{p^j(1-p)^{n-j-1}}{j+1}\left[(n+1)p-(j+1)\right],\end{aligned}$$

which is less than or equal to zero if, and only if, $p\le\frac{j+1}{n+1}$, for $j=1,\ldots,n-1$. Hence, $F\in\overline{\mathcal{F}}_n$ if, and only if, $p\le\frac{2}{n+1}$ (using (1.4)), and $F\in\overline{\mathcal{F}}_{n-1}$ if, and only if, $p\le\frac{2}{n}$. So, if $\frac{2}{n+1}<p\le\frac{2}{n}$, then $F\in\overline{\mathcal{F}}_{n-1}$ but $F\notin\overline{\mathcal{F}}_n$, $n\ge 3$. Thus, $\overline{\mathcal{F}}_n\subset\overline{\mathcal{F}}_{n-1}$. Next, to show that $\cap\overline{\mathcal{F}}_n\neq\phi$, consider

$$F(u)=\begin{cases}0 & \text{if } u<0,\\ u^3 & \text{if } 0\le u\le 1,\\ 1 & \text{if } u\ge 1.\end{cases}$$

It is easy to see that

$$d_{n,j}=\binom{n}{j}\int_0^\infty F^j(u)\,\overline{F}^{n-j}(u)\,du=\frac{1}{3}\cdot\frac{n!}{\Gamma(n+\frac{2}{3})}\cdot\frac{\Gamma(j+\frac{1}{3})}{j!}$$

is decreasing in $j=1,\ldots,n$, which implies $F\in\overline{\mathcal{F}}_n$, $n\ge 2$, using (1.4). □

The following example illustrates the applicability of Theorem 3.1(i).

Example 3.2.

(i) Let $s_1=(3/8,2/8,3/8)$ and $s_2=(2/8,4/8,2/8)$ be the signatures of two mixed systems. Let T_1 and T_2 be their respective lifetimes. It can be seen that $s_1\le_{mit}s_2$ and $s_1\nleq_{rh}s_2$. Let the component life distribution be

$$F(u)=\begin{cases}0 & \text{if } u<0,\\ \left(\frac{u}{2}\right)^3 & \text{if } 0\le u\le 2,\\ 1 & \text{if } u\ge 2.\end{cases}$$

Since $\binom{n}{j}\int_0^\infty F^j(u)\,\overline{F}^{n-j}(u)\,du$ is decreasing in $j=1,\ldots,n$, for $n=3$, we see that $F\in\overline{\mathcal{F}}_3$. Thus, the conditions of Theorem 3.1(i) are satisfied, and we conclude that $T_1\le_{mit}T_2$. The same can be inferred from Figure 1, which shows that $I(u)$ (defined in Lemma 3.1) is decreasing in $u\in(0,2)$ (see Lemma 3.1).

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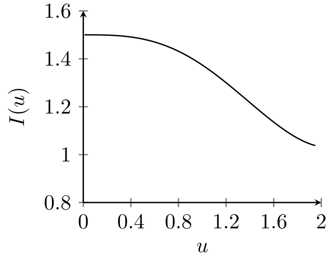

FIGURE 1. Plot of $I(u)$.

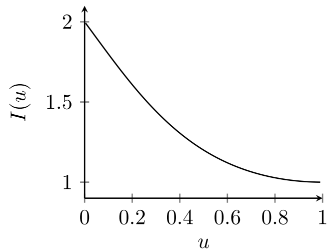

FIGURE 2. Plot of $I(u)$.

(ii) Let $s_1 = (1/3, 0, 1/2, 1/6)$ and $s_2 = (1/6, 1/6, 2/3, 0)$ be the signatures of two mixed coherent systems. Let T_1 and T_2 be their respective lifetimes. It can be seen that $s_1 \leq_{mit} s_2$ and $s_1 \nleq_{rh} s_2$. Let the component life distribution be

$$F(u) = \begin{cases} 0 & \text{for } u < 0, \\ u & \text{for } 0 \leq u \leq 1, \\ 1 & \text{for } u \geq 1. \end{cases}$$

It is easy to verify that $F \in \overline{\mathcal{F}}_4$. Hence the conditions of Theorem 3.1(i) are satisfied, and we can conclude that $T_1 \leq_{mit} T_2$. The same can be inferred from Figure 2, which shows that $I(u)$ is decreasing in $u \in (0, 1)$ (see Lemma 3.1).

Now, for the case when F is absolutely continuous, we consider the class of distributions defined in (1.5):

$$\widetilde{\mathcal{F}}_n = \left\{ F : \binom{n}{j} \int_0^1 \frac{w^j(1-w)^{n-j}}{f(F^{-1}(w))} dw \text{ is decreasing in } j = 1, \ldots, n, f = F' \right\}.$$

That is, for a given $n\ (\in \mathbb{N}) \geq 2$, $\widetilde{\mathcal{F}}_n$ contains all the absolutely continuous distribution functions F such that $F(x) = 0$ for $x < 0$ and

$$\binom{n}{j} \int_0^1 \frac{w^j(1-w)^{n-j}}{f(F^{-1}(w))} dw \geq \binom{n}{j+1} \int_0^1 \frac{w^{j+1}(1-w)^{n-j-1}}{f(F^{-1}(w))} dw, \qquad j = 1, \ldots, n-1.$$

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Clearly, $\widetilde{\mathcal{F}}_n \subseteq \overline{\mathcal{F}}_n$, $n \geq 2$. In order to provide the preservation result for the $\leq_{mit}$ ordering when $F \in \widetilde{\mathcal{F}}_n$, $n \geq 2$, we begin with the following lemma, which is a special case of Lemma 3.1, when F is absolutely continuous.

Lemma 3.5. *Let $\boldsymbol{b}_1 = (b_{11}, \ldots, b_{1n})$ and $\boldsymbol{b}_2 = (b_{21}, \ldots, b_{2n})$ be the failure signatures of two mixed coherent systems of order n. Let T_1 and T_2 be their respective lifetimes. Let F be an absolutely continuous component life distribution, and let f denote its probability density function. Then $T_1 \leq_{mit} T_2$ if, and only if,*

$$\frac{\sum_{j=1}^{n} b_{1j}\binom{n}{j}\int_0^s \left(w^j(1-w)^{n-j}/f(F^{-1}(w))\right) dw}{\sum_{j=1}^{n} b_{2j}\binom{n}{j}\int_0^s \left(w^j(1-w)^{n-j}/f(F^{-1}(w))\right) dw}$$

is decreasing in $s \in (0, 1)$.

The proof follows immediately from Lemma 3.1 when we substitute $F(u) = w$ in $\int_0^t F^j(u)\overline{F}^{n-j}(u)\, du$.

Now we are ready to present the preservation result, which is another version of Theorem 3.1, for the case when F is absolutely continuous.

Theorem 3.2.

(i) Let $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ be the signatures of two mixed systems of order n, and let $\boldsymbol{b}_1 = (b_{11}, \ldots, b_{1n})$ and $\boldsymbol{b}_2 = (b_{21}, \ldots, b_{2n})$ be their respective failure signatures. Let T_1 and T_2 be the respective lifetimes of these systems. If the absolutely continuous component life distribution $F \in \widetilde{\mathcal{F}}_n$ and $\boldsymbol{s}_1 \leq_{mit} \boldsymbol{s}_2$, then $T_1 \leq_{mit} T_2$.

(ii) If F is an absolutely continuous distribution function whose probability density function f is increasing, then $F \in \widetilde{\mathcal{F}}_n$, $n \geq 2$.

The proof of Theorem 3.2(i) follows from Theorem 3.1 and Lemma 3.5, and the proof of Theorem 3.2(ii) follows similarly to the proof of Theorem 3 of Lindqvist *et al.* [15]; hence these proofs are omitted.

Note that Theorem 3.2(ii) provides only a sufficient condition. To illustrate the applicability of Theorem 3.2, we present the following example of a parametric family of distributions.

Example 3.3. Consider a random variable X following the power function distribution with parameters $\alpha > 0$ and $\theta > 0$, having distribution function given by

$$F(t, \alpha, \theta) = \begin{cases} 0 & \text{if } t \leq 0, \\ \left(\dfrac{t}{\theta}\right)^{\alpha} & \text{if } 0 \leq t \leq \theta, \\ 1 & \text{if } t \geq \theta. \end{cases}$$

Note that, for $\alpha \geq 1$, the density function $f(t, \alpha, \theta)$ $\big(f(t, \alpha, \theta) = F'(t, \alpha, \theta)\big)$ is increasing; hence $F(t, \alpha, \theta) \in \widetilde{\mathcal{F}}_n$, $n \geq 2$ (using Theorem 3.2(ii)). Moreover, the fact that $F(t, \alpha, \theta) \in \widetilde{\mathcal{F}}_n$, $n \geq 2$, is also evident from the fact that

$$\binom{n}{j}\int_0^1 \frac{w^j(1-w)^{n-j}}{f(F^{-1}(w))}dw$$

is decreasing in $j=1, \ldots, n$. Furthermore, for $\mathbf{s}_1=(2/3, 0, 0, 1/3)$ and $\mathbf{s}_2=(1/6, 1/6, 2/3, 0)$, it is easy to verify that $\mathbf{s}_1 \leq_{mit} \mathbf{s}_2$. Thus, using Theorem 3.2(i), we can conclude that the corresponding system lifetimes T_1 and T_2 satisfy $T_1 \leq_{mit} T_2$.

Remark 3.1. Note that the proposed class of distributions $\overline{\mathcal{F}}_n$, $n \geq 2$, has a connection with the theory of order statistics. It can be understood as follows: let $X_1, \ldots, X_n$ be i.i.d. component lifetimes of a mixed coherent system. Let F be the component life distribution, and let $X_{1:n}, \ldots, X_{n:n}$ be the order statistics of the X_i. One can see that $P_t(j, n)=\binom{n}{j}F^j(t)\overline{F}^{n-j}(t)$ can be interpreted as the probability that exactly $n-j$ components are functioning at time t, and it can be expressed in terms of the order statistics as

$$P_t(j, n)=\mathbb{P}\left(X_{j:n} \leq t < X_{j+1:n}\right), \qquad n \geq 2, \ j=1, \ldots, n-1. \tag{3.3}$$

Let

$$D_{j,\,n}=X_{j+1:n}-X_{j:n}, \qquad j=1, \ldots, n-1.$$

Using (3.3), we have

$$\begin{aligned}
\int_0^\infty P_t(j, n)dt &= \int_0^\infty \mathbb{P}\left(X_{j:n} \leq t < X_{j+1:n}\right) dt \\
&= \int_0^\infty \mathbb{P}\left(X_{j+1:n} \geq t\right) dt - \int_0^\infty \mathbb{P}\left(X_{j:n} \geq t\right) dt \\
&= \mathbb{E}(X_{j+1:n}) - \mathbb{E}(X_{j:n}) \\
&= \mathbb{E}\left(D_{j,\,n}\right), \qquad j=1, \ldots, n-1.
\end{aligned} \tag{3.4}$$

However, for the case when $j=n$,

$$\int_0^\infty P_t(n, n)dt = \int_0^\infty F^n(t)dt = \int_0^\infty \mathbb{P}(X_{n:n} \leq t)dt$$

cannot be expressed using the spacings of the order statistics of the X_i. Moreover, if the support of the X_i is the interval (a, ∞), $a \geq 0$, then $\int_0^\infty P_t(n, n)dt$ diverges to $+\infty$.

Based on the above remark, we have the following proposition.

Proposition 3.1. *For any distribution F, we say that $F \in \overline{\mathcal{F}}_n$, $n \geq 2$, if, and only if,*

$$\mathbb{E}\left(D_{j+1,\,n}\right) \leq \mathbb{E}\left(D_{j,\,n}\right) \quad \textit{for } j=1, \ldots, n-2, \ \textit{and} \quad \int_0^\infty P_t(n, n)dt \leq \mathbb{E}(D_{n-1,n}).$$

The following example illustrates the applicability of Proposition 3.1.

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Example 3.4.

(i) Consider a random variable X following a binomial distribution $Bin(2, 1/2)$ having distribution function given by

$$F_1(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1/4 & \text{if } 0 \le t < 1, \\ 3/4 & \text{if } 1 \le t < 2, \\ 1 & \text{if } t \ge 2. \end{cases}$$

It can be seen that $\mathbb{E}\left(D_{1,\,3}\right) = \mathbb{E}\left(D_{2,\,3}\right) = 9/16$ and

$$\int_0^\infty P_t(3,\,3)dt = \int_0^2 F_1^3(t)dt = 7/16.$$

Thus,

$$\int_0^\infty P_t(3,\,3)dt \le \mathbb{E}(D_{2,3}) \le \mathbb{E}(D_{1,3}).$$

Hence, $F_1(t) \in \overline{\mathcal{F}}_3$ (using Proposition 3.1).

(ii) Consider a random variable X following a Pareto distribution with shape parameter $\alpha > 0$ and scale parameter $\theta > 0$, having distribution function given by

$$F_2(t,\,\alpha,\,\theta) = \begin{cases} 0 & \text{if } \ t \le \theta, \\ 1 - \left(\dfrac{\theta}{t}\right)^\alpha & \text{if } \ t \ge \theta. \end{cases}$$

It can be seen that when $\alpha = 2$ and $\theta = 2$, $\mathbb{E}\left(D_{1,\,2}\right) = 2.67$. However,

$$\int_0^\infty P_t(2,\,2)dt = \int_2^\infty F_2^2(t,\,2,\,2)dt$$

diverges to $+\infty$. Thus,

$$\int_0^\infty P_t(2,\,2)dt \nleq \mathbb{E}(D_{1,2}),$$

and consequently, $F_2(t,\,2,\,2) \notin \overline{\mathcal{F}}_2$ (using Proposition 3.1). This highlights the significance of the condition

$$\int_0^\infty P_t(n,\,n)dt \le \mathbb{E}(D_{n-1,n}), \qquad n \ge 2,$$

in Proposition 3.1.

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(iii) Consider a random variable X having distribution function $F_3(t, \alpha, \beta)$, $\alpha \geq 1$ and $\beta > 0$, given by

$$F_3(t, \alpha, \beta) = \begin{cases} 0 & \text{if } t \leq 0, \\ 2\left(1 - e^{-t^\beta}\right)^{(2^{(1/\alpha)})} - \left(1 - e^{-t^\beta}\right)^{(3^{(1/\alpha)})} & \text{if } t \geq 0. \end{cases}$$

Note that $F_3(t, \alpha, \beta)$ can be constructed using the distorted distribution $q_7(u)$ (see Table 1 in Section 4) and the Gumbel–Hougaard copula (defined in Section 4), by replacing u with $1 - e^{-t^\beta}$ and θ with α, respectively, in $q_7(u)$. It can be seen that when $\alpha = 2$ and $\beta = 4$, $\mathbb{E}\left(D_{1,\,3}\right) = 0.200$ and $\mathbb{E}\left(D_{2,\,3}\right) = 0.198$. However,

$$\int_0^\infty P_t(3, 3)dt = \int_0^\infty F_3^3(t, 2, 4)dt$$

diverges to $+\infty$. Thus,

$$\int_0^\infty P_t(3, 3)dt \nleq \mathbb{E}(D_{2,3}) \leq \mathbb{E}\left(D_{1,\,3}\right).$$

Hence, $F_3(t, 2, 4) \notin \overline{\mathcal{F}}_3$ (using Proposition 3.1), thereby signifying the need for the condition

$$\int_0^\infty P_t(n, n)dt \leq \mathbb{E}(D_{n-1,n}), \qquad n \geq 2,$$

in Proposition 3.1.

Note that Proposition 3.1 provides a necessary and sufficient condition for any distribution function F to belong to $\overline{\mathcal{F}}_n$, $n \geq 2$. One can also conclude from Proposition 3.1 that the distribution function F of any random variable X whose support is (a, ∞), $a \geq 0$, does not belong to $\overline{\mathcal{F}}_n$, for $n \geq 2$, i,e., $F \notin \overline{\mathcal{F}}_n$, $n \geq 2$ (as shown in Example 3.4(ii)–(iii) above). Since $\widetilde{\mathcal{F}}_n \subseteq \overline{\mathcal{F}}_n$, $n \geq 2$, Proposition 3.1 also holds for absolutely continuous distribution functions F. Recall that, in Theorem 3.2(ii), for a distribution F to belong to $\widetilde{\mathcal{F}}_n$ ($n \geq 2$), the probability density function f has to be increasing; however, this is only a sufficient condition. It will be interesting to see whether $F \in \widetilde{\mathcal{F}}_n$ even when the probability density function f is not necessarily increasing. To address this, we present some distributions which belong to $\widetilde{\mathcal{F}}_n$ (using Proposition 3.1), but whose densities are not necessarily increasing.

Example 3.5.

(i) Consider a random variable X following the uniform distribution $U(0, 1)$, having distribution function given by

$$F_4(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ t & \text{if } 0 \leq t \leq 1, \\ 1 & \text{if } t \geq 1. \end{cases}$$

It can be seen that $\mathbb{E}\left(D_{1,\,2}\right) = 1/3$ and

$$\int_0^\infty P_t(2, 2)dt = \int_0^1 F_4^2(t)dt = 1/3.$$

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Thus,

$$\int_0^1 P_t(2, 2)dt \le \mathbb{E}(D_{1,2}).$$

Hence, $F_4(t) \in \overline{\mathcal{F}}_2$, and consequently, $F_4(t) \in \widetilde{\mathcal{F}}_2$ (using Proposition 3.1). Furthermore, it can also be seen that $\mathbb{E}\left(D_{1,\,3}\right) = 1/4$, $\mathbb{E}\left(D_{2,\,3}\right) = 1/4$, and

$$\int_0^\infty P_t(3, 3)dt = 1/4.$$

Hence, $F_4(t) \in \overline{\mathcal{F}}_3$, and consequently, $F_4(t) \in \widetilde{\mathcal{F}}_3$. Note that the probability density function $f_4(t)$ corresponding to the distribution function $F_4(t)$ is constant in the interval (0, 1).

(ii) Consider a random variable X having the distribution function $F_5(t, \alpha)$, $\alpha \ge 1$, given by

$$F_5(t, \alpha) = \begin{cases} 0 & \text{if } t \le 0, \\ 2t^{(2^{(1/\alpha)})} - t^{(3^{(1/\alpha)})} & \text{if } 0 \le t \le 1, \\ 1 & \text{if } t \ge 1. \end{cases}$$

Note that $F_5(t, \alpha)$ can be constructed using the distorted distribution $q_7(u)$ (see Table 1 in Section 4) and the Gumbel–Hougaard copula (defined in Section 4), by replacing u with t and θ with α, respectively, in $q_7(u)$. It can be seen that when $\alpha = 2$, $\mathbb{E}\left(D_{1,\,4}\right) = 0.200$, $\mathbb{E}\left(D_{2,\,4}\right) = 0.188$, $\mathbb{E}\left(D_{3,\,4}\right) = 0.183$, and

$$\int_0^\infty P_t(4, 4)dt = \int_0^1 F_5^4(t, 2)dt = 0.181.$$

Thus,

$$\int_0^\infty P_t(4, 4)dt \le \mathbb{E}(D_{3,4}) \le \mathbb{E}(D_{2,4}) \le \mathbb{E}(D_{1,4}).$$

Hence, $F_5(t, 2) \in \overline{\mathcal{F}}_4$, and consequently, $F_5(t, 2) \in \widetilde{\mathcal{F}}_4$ (using Proposition 3.1). Furthermore, it can also be seen that when $\alpha = 2$, $\mathbb{E}\left(D_{1,\,5}\right) = 0.171$, $\mathbb{E}\left(D_{2,\,5}\right) = 0.159$, $\mathbb{E}\left(D_{3,\,5}\right) = 0.154$, $\mathbb{E}\left(D_{4,\,5}\right) = 0.152$, and

$$\int_0^\infty P_t(5, 5)dt = \int_0^1 F_5^5(t, 2)dt = 0.151.$$

Thus,

$$\int_0^\infty P_t(5, 5)dt \le \mathbb{E}(D_{4,5}) \le \mathbb{E}(D_{3,5}) \le \mathbb{E}(D_{2,5}) \le \mathbb{E}(D_{1,5}).$$

Hence, $F_5(t, 2) \in \overline{\mathcal{F}}_5$, and as a consequence, $F_5(t, 2) \in \widetilde{\mathcal{F}}_5$.

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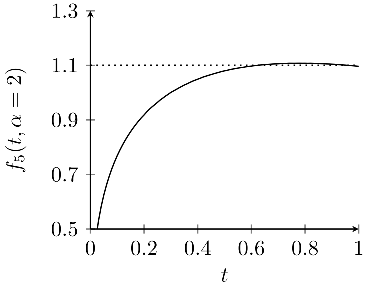

FIGURE 3. Plot of the density function $f_5(t, 2)$.

Figure 3 depicts the density function $f_5(t, 2)$ corresponding to the distribution function $F_5(t, 2)$. It is evident that $f_5(t, 2)$ is not increasing on $(0, 1)$ and is in fact unimodal; i.e., $f_5(t, 2)$ is increasing on $(0, t_0)$ and decreasing on $(t_0, 1)$, where $t_0 \approx 0.78$ (see Basu and Dasgupta [2]).

Thus, the above example reflects that there exist parametric distributions which belong to $\widetilde{\mathcal{F}}_n$, $n \geq 2$, even though their densities are unimodal and not necessarily increasing.

Remark 3.2. Lindqvist *et al.* [15, Proposition 4] established a connection between the decreasing failure rate (DFR) class and $\mathcal{F}_n$, $n \geq 2$. Although we cannot establish a similar connection between the increasing reversed failure rate (IRFR) class and $\overline{\mathcal{F}}_n$, $n \geq 2$, as no distribution with support $(0, \infty)$ can have IRFR (see Block *et al.* [5]), we provide a characterization of $\overline{\mathcal{F}}_n$ using the reverse hazard rate (RH) function. We find that $F \in \overline{\mathcal{F}}_n$, $n \geq 2$, if and only if

$$\mathbb{E}\left(\frac{1}{(j+1)\tilde{r}_F\left(X_{(j+1):n}\right)}\right) \leq \mathbb{E}\left(\frac{1}{j\tilde{r}_F\left(X_{j:n}\right)}\right)$$

for all $j = 1, 2, \ldots, n-1$, where $\tilde{r}_F$ is the RH function of F, given by $\tilde{r}_F(t) = \dfrac{f(t)}{F(t)}$, where $f = F'$, and $X_{j:n}$ is the jth order statistic among a random sample of size n.

4. Comparisons of coherent systems with d.i.d. components

As the classical signature-based mixture representation, given by Samaniego [27] and stated in (1.1), does not necessarily hold when the component lifetimes are d.i.d. (Navarro and Gomis [21]), Navarro *et al.* [18] obtained a representation of the system life distribution F_T as a distorted function of the component life distribution F, i.e.,

$$F_T(t) = q(F(t)), \tag{4.1}$$

where $q : [0, 1] \to [0, 1]$ is an increasing and continuous distortion function such that $q(0) = 0$ and $q(1) = 1$. Similarly, for system reliability $\overline{F}_T = 1 - F_T$, we have

$$\overline{F}_T(t) = \overline{q}(\overline{F}(t)), \tag{4.2}$$

where $\overline{q}(u) = 1 - q(1 - u)$ is called the dual distortion function. Navarro and Gomis [21] used the representation (4.2) to study the MRL comparisons of coherent systems with d.i.d. components.

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To further study coherent systems with identical components, Navarro and Fernández [20] utilized diagonal-dependent copulas. This is a wide class of copulas which includes both exchangeable and some non-exchangeable copulas, and is defined as follows.

Definition 4.1. Let $P \subseteq \{1, \ldots, n\}$, and let $\mathbf{u}_P = (u_1, \ldots, u_n)$ be such that $u_i = u$ if $i \in P$, and $u_i = 1$ if $i \notin P$, for $i = 1, \ldots, n$. An n-dimensional copula C is said to be *diagonal-dependent* (denoted by DD) if

$$C(\mathbf{u}_A) = C(\mathbf{u}_B), \quad \text{for all } A, B \subseteq \{1, \ldots, n\}, \text{ whenever } |A| = |B|.$$

Equivalently, C is DD if, and only if,

$$C(\mathbf{u}_A) = \delta_m(u), \qquad \text{for all } A \subseteq \{1, \ldots, n\}, \quad \text{whenever } |A| = m,$$

for $m = 1, \ldots, n$, where

$$\delta_m(u) := C(\underbrace{u, \ldots, u}_{m \text{ times}}, \underbrace{1, \ldots, 1}_{(n-m)\ times})$$

is the diagonal section for the copula of the marginal distribution of the first m variables. Clearly, $\delta_n(u) = C(\underbrace{u, \ldots, u}_{ntimes})$ and $\delta_1(u) = u$, $u \in [0, 1]$.

The following are some commonly used DD copulas which we will also be using in the sequel (see Nelsen [24]):

- The Clayton–Oakes family of copulas:

$$C(u_1, \ldots, u_n) = \left(\sum_{i=1}^{n} u_i^{1-\theta} - (n-1)\right)^{1/1-\theta}, \qquad n \geq 2, \ \theta > 1.$$

- The Gumbel–Hougaard family of copulas:

$$C(u_1, \ldots, u_n) = \exp\left(-\left[\sum_{i=1}^{n} (-\ln u_i)^{\theta}\right]^{1/\theta}\right), \qquad n \geq 2, \ \theta \geq 1.$$

In this section, we make use of the representation (4.1) and the concept of a DD copula to study MIT comparisons of coherent systems with d.i.d. components. First we present a lemma, involving distorted distributions, which will be helpful in obtaining the main result of this section. We know that the following implications hold:

$$X \leq_{rh} Y \Longrightarrow X \leq_{mit} Y,$$

$$X \leq_{hr} Y \Longrightarrow X \leq_{mrl} Y.$$

However, in certain situations, $\leq_{rh}$ or $\leq_{hr}$ may not hold. To deal with such situations, Belzunce *et al.* [4] provided sufficient conditions under which $\leq_{mit}$ ($\leq_{mrl}$) holds even though $\leq_{rh}$ ($\leq_{hr}$) does not hold (see Theorem 2.3 and Theorem 5.1 in Belzunce *et al.* [4]). To further strengthen these results, Navarro and Gomis [21] obtained sufficient conditions for $\leq_{mrl}$ in terms of dual

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distorted distributions, and Nooghabi *et al.* [25] obtained the following result for $\leq_{mit}$ in terms of distorted distributions. For the sake of clarity, we provide the proof also.

Lemma 4.1. *Consider two non-negative random variables X and Y with distributions F_X and F_Y, respectively, and distorted distribution functions q_X and q_Y based on the same baseline continuous distribution function F. Then the following conditions are equivalent:*

(i) There exists a $t_0 \in (0, \infty)$ such that $F_Y(t)/F_X(t)$ is increasing on $(0, t_0)$ and decreasing on (t_0, ∞).

(ii) There exists a $u_0 \in (0, 1)$ such that $q_Y(u)/q_X(u)$ is increasing on $(0, u_0)$ and decreasing on $(u_0, 1)$.

Proof. Assume (i) is true. Since F is non-decreasing, for $0 < u_1 \leq u_2 \leq u_0 < 1$, there exist $0 < t_1 \leq t_2 \leq t_0 < \infty$ such that $F(t_0) = u_0$, $F(t_1) = u_1$, and $F(t_2) = u_2$. As $F_Y(t)/F_X(t)$ is increasing on $(0, t_0)$, we have

$$\frac{F_Y(t_1)}{F_X(t_1)} = \frac{q_Y(u_1)}{q_X(u_1)} \leq \frac{q_Y(u_2)}{q_X(u_2)} = \frac{F_Y(t_2)}{F_X(t_2)};$$

i.e., $\dfrac{q_Y(u)}{q_X(u)}$ is increasing on $(0, u_0)$. Similarly, for $0 < u_0 \leq u_1 \leq u_2 < 1$, there exist $0 < t_0 \leq t_1 \leq t_2 < \infty$ such that $F(t_0) = u_0$, $F(t_1) = u_1$, and $F(t_2) = u_2$. As $F_Y(t)/F_X(t)$ is decreasing on (t_0, ∞), we have

$$\frac{F_Y(t_1)}{F_X(t_1)} = \frac{q_Y(u_1)}{q_X(u_1)} \geq \frac{q_Y(u_2)}{q_X(u_2)} = \frac{F_Y(t_2)}{F_X(t_2)};$$

i.e., $\dfrac{q_Y(u)}{q_X(u)}$ is decreasing on $(u_0, 1)$. Thus, (i) implies (ii). The proof of the converse is along the same lines and hence is omitted. □

Recall that Belzunce *et al.* [4, Theorem 5.1] showed that if $\mathbb{E}(X) \leq \mathbb{E}(Y)$ and Lemma 4.1(i) holds true, then $X \leq_{mit} Y$. Based on these observations, we have the following result, which provides a sufficient condition for the MIT ordering to hold for coherent systems with d.i.d. components. Note that a similar result for the $\leq_{mrl}$ ordering is given in Navarro and Gomis [21, Theorem 2.3].

Theorem 4.1. *Let S_1 and S_2 be lifetimes of two coherent systems with d.i.d. components such that $\mathbb{E}(S_1) \leq \mathbb{E}(S_2)$. Let $q_1(u)$ and $q_2(u)$ be the corresponding distorted distributions based on the common component life distribution F. If there exists a $u_0 \in (0, 1)$ such that $\dfrac{q_2(u)}{q_1(u)}$ is increasing on $(0, u_0)$ and is decreasing on $(u_0, 1)$, then $S_1 \leq_{mit} S_2$.*

The proof is omitted, as it follows from utilizing Theorem 5.1 in Belzunce *et al.* [4] and replacing q_X (q_Y) by q_1 (q_2) in Lemma 4.1 above. This result is significant because it provides comparisons of coherent systems with d.i.d. components with respect to the $\leq_{mit}$ ordering even when the $\leq_{rh}$ ordering does not hold, by utilizing the distorted distributions. Also note that if the ratio $\dfrac{q_2(u)}{q_1(u)}$ is increasing in $u \in (0, 1)$, then $S_1 \leq_{rh} S_2$ holds, and if the ratio $\dfrac{q_2(u)}{q_1(u)} \leq (\geq) 1$, $u \in (0, 1)$, then $S_1 \leq_{st} (\geq_{st}) S_2$ holds (see Navarro *et al.* [19, Proposition 2.2]). To illustrate the applicability of Theorem 4.1, below we study coherent systems with 1–3 d.i.d. components where the common underlying copula is a DD copula.

TABLE 1. Distorted distributions of coherent systems with 1–3 d.i.d. components.

N	Lifetime	$\Phi(X_1, X_2, X_3)$	$q_N(u)$
1	T_1	$X_{1:1} = X_1$	u
2	T_2	$X_{1:2} = \min\{X_1, X_2\}$	$2u - C(u, u)$
3	T_3	$X_{2:2} = \max\{X_1, X_2\}$	$C(u, u)$
4	T_4	$X_{1:3} = \min\{X_1, X_2, X_3\}$	$3u - 3C(u, u, 1) + C(u, u, u)$
5	T_5	$\min\{X_1, \max\{X_2, X_3\}\}$	$u + C(u, u, 1) - C(u, u, u)$
6	T_6	$X_{2:3}$ (2-out-of-3)	$3C(u, u, 1) - 2C(u, u, u)$
7	T_7	$\max\{X_1, \min\{X_2, X_3\}\}$	$2C(u, u, 1) - C(u, u, u)$
8	T_8	$X_{3:3} = \max\{X_1, X_2, X_3\}$	$C(u, u, u)$

4.1. Comparisons of coherent systems with 1–3 d.i.d. components

First we list the distorted distributions of coherent systems with 1–3 d.i.d. components under a common DD copula (see Table 1).

It is easy to show how $q_N(u)$ for $N = 1, \ldots, 8$ is obtained using a DD copula. For instance, consider $T_6 = X_{2:3}$. The minimal cut sets of the corresponding system are {1, 2}, {1, 3}, and {2, 3}. Let **F** be the joint distribution of (X_1, X_2, X_3), let C be the underlying DD copula, and let F be the common component life distribution. Hence, the system life distribution

$$\begin{aligned} F_{T_6}(t) &= \mathbb{P}\left(\left\{X_{\{1,2\}} < t\right\} \cup \left\{X_{\{1,3\}} < t\right\} \cup \left\{X_{\{2,3\}} < t\right\}\right) \\ &= \mathbb{P}\left(X_{\{1,2\}} < t\right) + \mathbb{P}\left(X_{\{1,3\}} < t\right) + \mathbb{P}\left(X_{\{2,3\}} < t\right) - 2\mathbb{P}\left(X_{\{1,2,3\}} < t\right) \\ &= \mathbf{F}(t, t, \infty) + \mathbf{F}(t, \infty, t) + \mathbf{F}(\infty, t, t) - 2\mathbf{F}(t, t, t) \\ &= 3C(F(t), F(t), 1) - 2C(F(t), F(t), F(t)) = q_6(F(t)), \end{aligned}$$

and $q_6(u) = 3C(u, u, 1) - 2C(u, u, u)$, $u \in (0, 1)$. Under the Gumbel–Hougaard dependence model when $\theta = 2$, it can be seen that the ratio of the distorted distributions of T_6 to T_1, given by

$$\frac{q_6(u)}{q_1(u)} = \frac{3u^{\sqrt{2}} - 2u^{\sqrt{3}}}{u},$$

is increasing on the interval $(0, u_0)$ and is decreasing on the interval $(u_0, 1)$, where $u_0 \approx 0.6$. Hence, on applying Theorem 4.1, it follows that $T_1 \leq_{mit} T_6$ under the Gumbel–Hougaard dependence model when $\theta = 2$, for any distribution F, such that $\mathbb{E}(T_1) \leq \mathbb{E}(T_6)$. In a similar way, we compare all the systems listed in Table 1 with respect to the MIT order, when the common underlying copula is a Clayton–Oakes copula (see Theorem 4.2) or a Gumbel–Hougaard copula (see Theorem 4.3). Note that these comparison results hold true whenever the dependency parameter $\theta \geq 2$.

Theorem 4.2. *Let $T_1, \ldots, T_8$ be the lifetimes of the coherent systems given in Table 1. If the underlying copula is a Clayton–Oakes copula ($\theta \geq 2$), then*

(i) $T_4 \leq_{mit} T_2 \leq_{mit} T_1 \leq_{mit} T_5 \leq_{mit} T_6 \leq_{mit} T_7 \leq_{mit} T_3 \leq_{mit} T_8$, *and*

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(ii) $T_4 \leq_{rh} T_2 \leq_{rh} T_1 \leq_{mit} T_5 \leq_{rh} T_6 \leq_{rh} T_7 \leq_{rh} T_3 \leq_{rh} T_8$,

provided that their means exist and are ordered in the same way.

Proof. On comparing (i) and (ii), it is evident that, except for $T_1 \leq_{mit} T_5$, the rest of the $\leq_{mit}$ orderings in (i) can be strengthened to the $\leq_{rh}$ ordering in (ii). Thus, we first show that $T_1 \leq_{mit} T_5$ holds but $T_1 \nleq_{rh} T_5$. To prove $T_1 \leq_{mit} T_5$, it suffices to show that $\frac{q_5(u)}{q_1(u)}$ is increasing on $(0, u_0)$ and is decreasing on $(u_0, 1)$. Consider

$$\frac{q_5(u)}{q_1(u)} = \frac{u + (2u^{1-\theta} - 1)^{\frac{1}{1-\theta}} - (3u^{1-\theta} - 2)^{\frac{1}{1-\theta}}}{u}, \qquad u \in (0, 1),$$

which on differentiating yields

$$\left(\frac{q_5(u)}{q_1(u)}\right)' = \frac{2(\theta - 1)u^{\theta-2} \cdot \left(3 - 2u^{\theta-1}\right)^{\frac{1}{1-\theta}-1}}{1-\theta} - \frac{(\theta - 1)u^{\theta-2} \cdot \left(2 - u^{\theta-1}\right)^{\frac{1}{1-\theta}-1}}{1-\theta}. \tag{4.3}$$

On equating (4.3) to 0, we obtain

$$u_0 = \left(\frac{3.2^{\frac{1}{\theta}} - 4}{2.2^{\frac{1}{\theta}} - 2}\right)^{\frac{1}{\theta-1}} < 1.$$

Thus, $T_1 \leq_{mit} T_5$, for all $\theta \geq 2$, and $T_1 \nleq_{rh} T_5$. Now, to establish $T_4 \leq_{mit} T_2$, it suffices to show that $T_4 \leq_{rh} T_2$. Consider

$$\frac{q_2(u)}{q_4(u)} = \frac{2u - (2u^{1-\theta} - 1)^{1/1-\theta}}{3u - 3(2u^{1-\theta} - 1)^{1/1-\theta} + (3u^{1-\theta} - 2)^{1/1-\theta}}, \qquad u \in (0, 1),$$

which on differentiating and simplification shows that $\frac{q_2(u)}{q_4(u)}$ is increasing in $u \in (0, 1)$, for all $\theta \geq 2$. In a similar way, it can be proved that $T_2 \leq_{rh} T_1$ and $T_5 \leq_{rh} T_6 \leq_{rh} T_7 \leq_{rh} T_3 \leq_{rh} T_8$, for $\theta \geq 2$. □

It is worth mentioning here that the above result can be stated for the $\leq_{mrl}$ ordering; i.e., under the same assumptions as in Theorem 4.2,

$$T_4 \leq_{mrl} T_2 \leq_{mrl} T_5 \leq_{mrl} T_6 \leq_{mrl} T_7 \leq_{mrl} T_1 \leq_{mrl} T_3 \leq_{mrl} T_8 \tag{4.4}$$

and

$$T_4 \leq_{hr} T_2 \leq_{hr} T_5 \leq_{hr} T_6 \leq_{hr} T_7 \leq_{mrl} T_1 \leq_{hr} T_3 \leq_{hr} T_8, \tag{4.5}$$

provided that their means exist and are ordered in the same way. Note that Navarro and Gomis [21, Theorem 4.1] established (4.4) and (4.5) when $\theta = 2$. However, we strengthen their result by establishing it for a general θ $(\theta \geq 2)$.

Below, we present a result where the dependency among the components is established using a Gumbel–Hougaard copula. The proof is omitted as it similar to that of Theorem 4.2.

Theorem 4.3. *Let* $T_1, \ldots, T_8$ *be the lifetimes of the coherent systems given in Table 1. If the underlying copula is a Gumbel–Hougaard copula* $(\theta \geq 2)$*, then*

(i) $T_4 \leq_{mit} T_2 \leq_{mit} T_1 \leq_{mit} T_5 \leq_{mit} T_6 \leq_{mit} T_7 \leq_{mit} T_3 \leq_{mit} T_8$, *and*

(ii) $T_4 \leq_{rh} T_2 \leq_{rh} T_1 \leq_{mit} T_5 \leq_{rh} T_6 \leq_{rh} T_7 \leq_{rh} T_3 \leq_{rh} T_8$,

provided that their means exist and are ordered in the same way.

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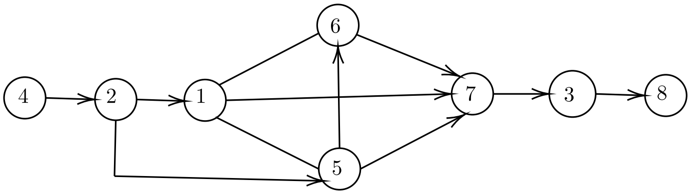

FIGURE 4. MIT (or RH) ordering properties of coherent systems with 1–3 d.i.d. components.

The $\le_{mit}$ (or $\le_{rh}$) ordering relations among coherent systems with 1–3 d.i.d. components when the underlying copula is a Clayton–Oakes copula ($\theta \ge 2$) or a Gumbel–Hougaard copula ($\theta \ge 2$) are given in Figure 4, where the arrows represent the $\le_{rh}$ ordering, and the lines represent the $\le_{mit}$ ordering such that the respective means are ordered.

Now that we have studied the special DD copulas (Theorems 4.2 and 4.3), it is intuitive to ask whether similar results hold true for a general DD copula. To answer this, we present our next result.

Theorem 4.4. *Let C be a DD copula.*

(i) If there exists $u_0 \in (0, 1)$ *such that* $\frac{C(u, u)}{u}$ *is increasing on* $(0, u_0)$ *and is decreasing on* $(u_0, 1)$, *then* $T_2 \le_{mit} T_1 \le_{mit} T_3$.

(ii) If there exists $u_0 \in (0, 1)$ *such that* $\frac{u + C(u, u, 1) - C(u, u, u)}{3u - 3C(u, u, 1) + C(u, u, u)}$ *is increasing on* $(0, u_0)$ *and is decreasing on* $(u_0, 1)$, *then* $T_4 \le_{mit} T_5$.

(iii) If there exists $u_0 \in (0, 1)$ *such that* $\frac{C(u, u, u)}{C(u, u, 1)}$ *is increasing on* $(0, u_0)$ *and is decreasing on* $(u_0, 1)$, *and* $\mathbb{E}(T_6) \le \mathbb{E}(T_7) \le \mathbb{E}(T_8)$, *then* $T_6 \le_{mit} T_7 \le_{mit} T_8$.

Proof.

(i) Consider the ratio of distorted distributions

$$\frac{q_2(u)}{q_1(u)} = 2 - \frac{C(u, u)}{u} \quad \text{and}$$
$$\frac{q_3(u)}{q_1(u)} = \frac{C(u, u)}{u}.$$

Clearly, if $\frac{C(u, u)}{u}$ is increasing on $(0, u_0)$ and is decreasing on $(u_0, 1)$, then both $\frac{q_1(u)}{q_2(u)}$ and $\frac{q_3(u)}{q_1(u)}$ are increasing on $(0, u_0)$ and are decreasing on $(u_0, 1)$. Hence $T_2 \le_{mit} T_1 \le_{mit} T_3$ from Theorem 4.1, since $\mathbb{E}(T_2) \le \mathbb{E}(T_1) \le \mathbb{E}(T_3)$.

(ii) The proof follows from Theorem 4.1 by taking the ratio of distorted distributions of T_5 to T_4 since $\mathbb{E}(T_4) \le \mathbb{E}(T_5)$ for any component life distribution F.

(iii) The proof is omitted as it is along similar lines to that of (i). □

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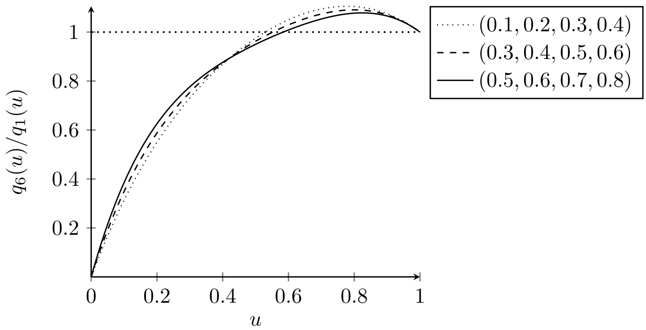

FIGURE 5. Ratio of distorted distributions of T_6 to T_1.

It is important to note that since these stochastic orderings are partial orderings, we cannot combine (i), (ii), and (iii) in Theorem 4.4 to reach conclusions as in (i) and (ii) of Theorems 4.2 and 4.3. However, the conditions of Theorem 4.4 are applicable to all the DD copulas and are easier to check. Furthermore, note that the Clayton–Oakes copula ($\theta \geq 2$) and the Gumbel–Hougaard copula ($\theta \geq 2$) satisfy the conditions mentioned in Theorem 4.4.

Till now, our focus has been on applications of Theorem 4.1 to DD copulas. However, it makes sense to see whether Theorem 4.1 is applicable to non-DD copulas also. To see this, let us consider the Farlie–Gumbel–Morgenstern (FGM) copula as given below.

Example 4.1. Consider the systems whose lifetimes are $T_6 = X_{2:3}$ and $T_1 = X_{1:1}$ composed of d.i.d. component lifetimes with underlying FGM copula given by

$$\begin{aligned} C(u_1, u_2, u_3) = u_1u_2u_3(1 &+ \theta_1(1-u_1)(1-u_2) + \theta_2(1-u_1)(1-u_3) \\ &+ \theta_3(1-u_2)(1-u_3) + \theta_4(1-u_1)(1-u_2)(1-u_3)), \end{aligned}$$

for $|\theta_i| \leq 1,\ i = 1,\ 2,\ 3,\ 4$, and assume that $\theta_i \neq \theta_j$ for $i \neq j$.

The minimal cut sets of the system (with lifetime T_6) are $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$. Hence the system life distribution $F_{T_6}(t)$ is given by

$$C(F(t), F(t), 1) + C(F(t), 1, F(t)) + C(1, F(t), F(t)) - 2C(F(t), F(t), F(t)),$$

where its distorted distribution is

$$q_6(u) = C(u, u, 1) + C(u, 1, u) + C(1, u, u) - 2C(u, u, u), \text{ and } q_1(u) = u.$$

Then, using the FGM copula, we obtain

$$q_6(u) = u^2\left[\sum_{i=1}^{3}\left(1 + \theta_i(1-u)^2\right)\right] - 2u^3\left[1 + \sum_{i=1}^{3}\theta_i(1-u)^2 + \theta_4(1-u)^3\right].$$

The ratio of distorted distributions of T_6 to T_1 for different sets of $(\theta_1, \theta_2, \theta_3, \theta_4)$ are shown in Figure 5.

The curves plotted in Figure 5 have an inverted bathtub shape, and hence $T_1 \leq_{mit} T_6$ whenever $\mathbb{E}(T_1) \leq \mathbb{E}(T_6)$. Furthermore, if the common component life distribution F is uniform (U (0, 1)), then $\mathbb{E}(T_1) = 0.5$ and $\mathbb{E}(T_6) = 0.505$ (for $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\theta_3 = 0.3$,

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$\theta_4 = 0.4$), $\mathbb{E}(T_6) = 0.508$ (for $\theta_1 = 0.3,\ \theta_2 = 0.4,\ \theta_3 = 0.5,\ \theta_4 = 0.6$), and $\mathbb{E}(T_6) = 0.511$ (for $\theta_1 = 0.5,\ \theta_2 = 0.6,\ \theta_3 = 0.7,\ \theta_4 = 0.8$). Therefore, $T_1 \le_{mit} T_6$ for the given sets of θ_i values. However, $T_1 \nleq_{rh} T_6$ and $T_1 \nleq_{st} T_6$, since $q_6(u)/q_1(u)$ is not increasing on $u \in (0, 1)$ and since $q_6(u)/q_1(u)$ takes values bigger than 1 and smaller than 1 as well.

Recall that Navarro and Gomis [21, Example 4.4] considered a non-exchangeable copula and studied the system T_5 for different values of the dependency parameter θ to establish the $\le_{mrl}$ ordering. In a similar way, in Example 4.1 we considered a non-DD copula to establish the $\le_{mit}$ ordering between T_1 and T_6.

5. Discussion and conclusions

In this article, we used the notion of the failure signature (see Samaniego and Navarro [29]) to prove that the MIT order is not preserved in general. However, if the component lifetimes of mixed coherent systems of order n belong to $\overline{\mathcal{F}}_n$ or $\widetilde{\mathcal{F}}_n$, $n \ge 2$, then

$$\mathbf{s}_1 \le_{mit} \mathbf{s}_2 \Longrightarrow T_1 \le_{mit} T_2.$$

This result is significant as it concerns the preservation of the $\le_{mit}$ ordering even when the $\le_{rh}$ ordering does not hold. We believe that similar studies can be conducted for other stochastic orderings which involve distribution functions, such as the increasing concave order, by using the notion of the failure signature. In the article, we also provide various examples from parametric families, as well as the relationship between the proposed results and the concept of order statistics. A natural extension from the i.i.d. case is to consider systems with exchangeable components. It is known from the literature that most of the results in the i.i.d. case can be extended to the case where components lifetimes are exchangeable. To see this, one may refer to Navarro and Rubio [22, Theorem 2.3(i)–(ii)], where it is shown that

$$T_1 \le_{rh} T_2 \quad \text{if } \mathbf{s}_1 \le_{rh} \mathbf{s}_2 \text{ and } X_{1:n} \le_{rh} \ldots \le_{rh} X_{n:n}, \text{ and}$$

$$T_1 \le_{mit} T_2 \quad \text{if } \mathbf{s}_1 \le_{rh} \mathbf{s}_2 \text{ and } X_{1:n} \le_{mit} \ldots \le_{mit} X_{n:n},$$

where $X_1, \ldots, X_n$, are exchangeable component lifetimes, and $X_{i:n}$ is the ith order statistic of $X_1, \ldots, X_n$, for $i = 1, \ldots, n$. Based on these observations, it is intuitive to believe that our results can be extended to components with exchangeable lifetimes.

Moreover, it is known that in the real world, component lifetimes are not always i.i.d. or exchangeable. In fact, they are usually dependent. Thus, to incorporate dependence, we employ DD copulas and distorted functions to obtain stochastic comparisons between the lifetimes T_1 and T_2 with respect to the $\le_{mit}$ ordering. We consider coherent systems with 1–3 d.i.d. components having Clayton–Oakes and Gumbel–Hougaard copula dependency structures, and we show how these systems are ordered with respect to the $\le_{mit}$ ordering for dependency parameter $\theta \ge 2$. It will be interesting to see whether similar results can be established for coherent systems with 1–4 d.i.d. components. To see this, we present a result for the Gumbel–Hougaard copula when the dependency parameter $\theta = 2$.

Theorem 5.1. *Let $T_1, \ldots, T_{28}$ be the lifetimes of coherent systems with 1–4 d.i.d. components (see the second column of Table I in Navarro et al.* [18]*). Then*

$$\begin{aligned} &T_9 \le_{mit} T_4 \le_{mit} T_{10} \le_{mit} T_2 \le_{mit} T_1 \le_{mit} T_{13} \le_{mit} T_5 \le_{mit} T_{12} \le_{mit} T_{11} \le_{mit} T_{14} \\ &\le_{mit} T_{15} \le_{mit} T_{17} =_{st} T_{16} \le_{mit} T_{24} \le_{mit} T_6 =_{st} T_{18} =_{st} T_{19} \le_{mit} T_7 \le_{mit} T_{20} \\ &=_{st} T_{21} \le_{mit} T_3 \le_{mit} T_{25} \le_{mit} T_{22} \le_{mit} T_{23} \le_{mit} T_{26} \le_{mit} T_{27} \le_{mit} T_8 \le_{mit} T_{28}, \end{aligned}$$

provided that their means are ordered in the same way.

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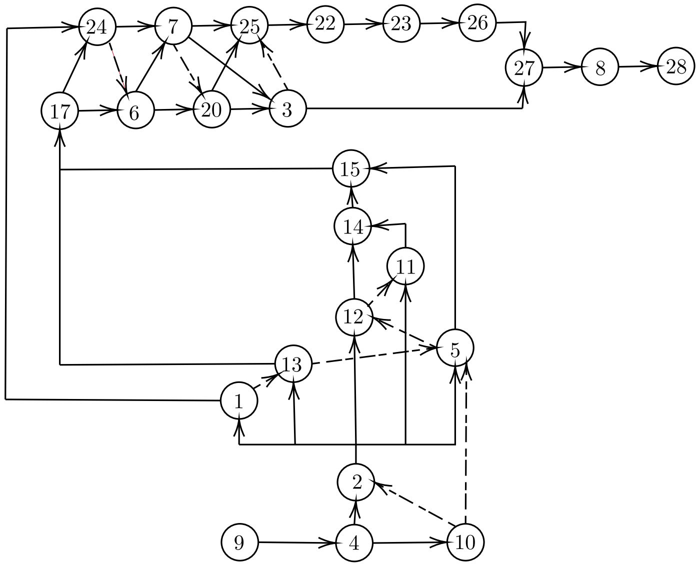

FIGURE 6. MIT (or RH) ordering properties of coherent systems with 1–4 d.i.d. components.

The proof is immediate from Theorem 4.1 upon taking the ratios of the distorted distributions. In a similar way, we have the following orderings:

$$T_9 \le_{rh} T_4 \le_{rh} T_{10} \le_{mit} T_2 \le_{rh} T_1 \le_{mit} T_{13} \le_{mit} T_5 \le_{mit} T_{12} \le_{mit} T_{11} \le_{rh} T_{14}$$
$$\le_{rh} T_{15} \le_{rh} T_{17} =_{st} T_{16} \le_{rh} T_{24} \le_{mit} T_6 =_{st} T_{18} =_{st} T_{19} \le_{rh} T_7 \le_{mit} T_{20}$$
$$=_{st} T_{21} \le_{rh} T_3 \le_{mit} T_{25} \le_{rh} T_{22} \le_{rh} T_{23} \le_{rh} T_{26} \le_{rh} T_{27} \le_{rh} T_8 \le_{rh} T_{28}.$$

The ordering relationships among coherent systems with 1–4 d.i.d. components when the underlying copula is a Gumbel–Hougaard copula ($\theta = 2$) are shown in Figure 6, where

$$i \longrightarrow j \text{ means } T_i \le_{rh} T_j, \text{ and } i \dashrightarrow j \text{ means } T_i \le_{mit} T_j.$$

Although in Theorem 5.1 we have been able to establish the $\le_{mit}$ ordering for a specific dependency parameter θ (= 2), it provides food for thought to consider whether the result can be strengthened to a general θ (as done in Theorems 4.2 and 4.3). Note that not all the $\le_{mit}$ ordering comparisons can be strengthened to the $\le_{rh}$ ordering. Recall that Navarro and Gomis [21, Theorem 3.2] made similar comparisons for coherent systems with 1–4 i.i.d. components, with respect to the $\le_{mrl}$ and $\le_{hr}$ orderings, using the product copula. Thus, in Theorem 5.1, we consider a much more general situation by considering d.i.d. components with the underlying Gumbel–Hougaard copula.

Finally, we would also like to point out that sufficient conditions for the preservation of the $\le_{st}$, $\le_{hr}$, and $\le_{rh}$ orderings when the component lifetimes are non-identical have been addressed in the literature (see Navarro *et al.* [19]). However, the preservation of the $\le_{mit}$ ordering in a similar set-up still remains an open problem.

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