COMPLETELY INDECOMPOSABLE MODULES

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Introduction and summary. The purpose of this paper is to investigate completely indecomposable modules. A completely indecomposable module is an additive abelian group $\mathfrak{V}$ with a ring A as operator domain, where the following four conditions are satisfied.

I-1. A is a commutative ring and has a unit element which is unit operator for $\mathfrak{V}$.

I-2. The submodules of $\mathfrak{V}$ satisfy the ascending chain condition. (Submodule will always mean invariant submodule.)

I-3. The submodules of $\mathfrak{V}$ satisfy the descending chain condition.

I-4. Every submodule of $\mathfrak{V}$ ($\mathfrak{V}$ itself included) is indecomposable.

Properties I-1 and I-2 are equivalent with saying that $\mathfrak{V}$ is a Noetherian module. (See [1] and [2] for Noetherian modules; square brackets refer to the references.) Property I-4 says that if $\mathfrak{W}$ is a submodule of $\mathfrak{V}$, then $\mathfrak{W}$ is not decomposable in a direct sum $\mathfrak{W} = \mathfrak{W}_1 \dot{+} \mathfrak{W}_2$, where $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are submodules of $\mathfrak{W}$. (The symbol $\dot{+}$ will always denote a *direct* sum.) It is easily shown that I-4 is equivalent with saying that $\mathfrak{V}$ has a unique minimal submodule (see remark 1.1).

The notion of a completely indecomposable module arises in a natural way from the study of the classical elementary divisor theory. As is shown in sec. **6**, every indecomposable module which occurs in the classical elementary divisor theory and whose annihilating ideal is not the zero ideal (i.e., for whose submodules the descending chain condition holds) is completely indecomposable.

The main result of this paper is Theorem 5.1. It states that *two completely indecomposable modules are isomorphic* (isomorphic will always mean operator-isomorphic) *if and only if they have the same annihilating ideal.* In the language of ring-representations this means: *two faithful representations of a commutative ring with unit element* 1, *whose respective representation spaces have composition sequences and unique minimal sub-spaces, while* 1 *is unit operator, are equivalent.* (See remark 5.1.) Theorem 5.1 can immediately be extended to Theorem 5.2 on *regular modules* (i.e., modules which are direct sums of completely indecomposable modules; see definition 5.1.) Thus we obtain a generalization to regular modules of that part of the classical elementary divisor theory which is concerned with modules in which the descending chain condition is satisfied (see sec. **6**).

Sec. **1** contains that part of the theory of a completely indecomposable module $\mathfrak{V}$ which follows from the theory of Noetherian modules. Sec. **2** con-

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tains the part of the theory of $\mathfrak{B}$ which is obtained by extending Gröbner's theory of irreducible ideals (see [3]) to completely indecomposable modules. Sec. **3** contains the proof that the operator-endomorphism ring of $\mathfrak{B}$ consists of the multiplications of the elements of $\mathfrak{B}$ with the elements of the operator domain A. This does not imply that $\mathfrak{B}$ is necessarily cyclic (see remark 3.1). Lemma 3.1, which is concerned with the extension of operator-isomorphisms, holds for abelian groups with arbitrary rings, not necessarily commutative, as operator domain. Sec. **4** contains two known lemmas of ring-representation theory which are needed for secs. **5** and **6**. Sec. **5** contains the proof of the main theorem 5.1. In sec. **6**, it is shown how that part of the classical elementary divisor theory which is concerned with modules in which the descending chain condition is satisfied, Steinitz theory of algebraic integers included, can be obtained from the theory of completely indecomposable modules. Here, the main lemma is lemma 6.1, which states that *an indecomposable module which satisfies conditions* I-1, I-2 *and* I-3 *and whose annihilating ideal is intersection-irreducible is completely indecomposable and cyclic.* Sec. **7** contains examples of completely indecomposable modules. The unsolved problems to which this paper gives rise are stated in remarks 6.2 and 7.2.

It is pointed out in remark 7.1 that the theory of completely indecomposable modules gives rise to the question "does every commutative, completely primary ring have a faithful, completely indecomposable representation space?" This question is answered in the affirmative by corollary 9.4 of sec. **9**. In order to prove this corollary it was necessary to introduce in sec. **8** a new composition of operator modules, called *interlacing of modules*, which is a generalization of the direct sum. *Sec.* **8** *is self-contained and demonstrates that interlacing can be used equally well for non-commutative as for commutative operator domains. The author believes that many other uses can be made of the notion of interlacing.* Finally remark 9.2 shows how, as a consequence of corollary 9.4, we can define a "dual vector space" for any commutative ring A with unit element whose ideals satisfy both chain conditions. This dual vector space is a generalization of the dual space defined in [8], p. 558, for the case where A is an algebra of finite rank with respect to a field.

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1. Properties of completely indecomposable modules obtained from the theory of Noetherian modules. Let $\mathfrak{B}$ be a completely indecomposable module with the ring A as right-operator domain. Hence the conditions I-1, I-2, I-3 and I-4 of the introduction are satisfied. The elements of $\mathfrak{B}$ will be denoted by capital Latin letters and the submodules of $\mathfrak{B}$ by capital German letters. The elements of A will be denoted by lower case Greek letters, and the ideals of A by lower case German letters.

Since $\mathfrak{B}$ is a Noetherian module (see [2], sec. 2.1, for the definition of Noetherian module; in [2] the term Noetherian vector space is used) which has a composition series of finite length (this follows from I-2 and I-3), the associated primes of the zero-module 0 of $\mathfrak{V}$ are maximal prime ideals of A. (For the definition of associated primes of a module, see [2], Theorem 2.21. The fact that the associated primes of 0 have to be maximal is stated in Theorem 24 of [1].) Since $\mathfrak{V}$ is itself indecomposable, 0 can have only one associated prime $\mathfrak{p}$, namely the radical of the annihilating ideal $\mathfrak{q}$ of $\mathfrak{V}$. (Since all the associated primes of 0 are maximal, $\mathfrak{V}$ would be decomposable if 0 had more than one associated prime; this follows immediately from Theorem 19 of [1].) Consequently, 0 is a primary submodule of $\mathfrak{V}$ whose fundamental ideal is the annihilating ideal $\mathfrak{q}$ of $\mathfrak{V}$ and whose radical is the maximal ideal $\mathfrak{p}$. (See sec. 2.1 of [2] for the definitions of primary module, fundamental ideal and radical.) This implies that $\mathfrak{q}$ is primary (see Theorem 2.11 of [2]) and that if $V\alpha = 0$, where $V \epsilon \mathfrak{V}$ and $V \neq 0$ and $\alpha \epsilon A$, then $\alpha \epsilon \mathfrak{p}$ (see definition 2.12 of [2]).

Now, let $0 = \mathfrak{V}_0 \subset \mathfrak{V}_1 \subset \ldots \subset \mathfrak{V}_{l-1} \subset \mathfrak{V}_l = \mathfrak{V}$ be a composition series of $\mathfrak{V}$. (The symbol $\subset$ will be used exclusively for *proper* inclusion). Since the fundamental ideal of 0 is primary and has maximal radical, the difference module $\mathfrak{V}_i - \mathfrak{V}_{i-1}$ is operator-isomorphic with $A - \mathfrak{p}$; hence the quotient $\mathfrak{V}_{i-1}: \mathfrak{V}_i = \mathfrak{p}$ for $i = 1, \ldots, l$. (The quotient $\mathfrak{V}_{i-1}: \mathfrak{V}_i$ is defined as the ideal which consists of all $\alpha \epsilon A$, such that $V\alpha \epsilon \mathfrak{V}_{i-1}$ for all $V \epsilon \mathfrak{V}_i$; see sec. 2.1 of [2]. The fact that the difference module $\mathfrak{V}_i - \mathfrak{V}_{i-1}$ is isomorphic with $A - \mathfrak{p}$ is stated in Theorem 2.41 of [2].) In particular, $\mathfrak{V}_1\mathfrak{p} = 0$, which implies that $\mathfrak{V}_1 \subseteq 0: \mathfrak{p}$. (See sec. 2.1 of [2] for the definitions of the product of a module with an ideal and of the quotient of a module by an ideal.) However, if $0 : \mathfrak{p}$ were not a minimal module of $\mathfrak{V}$ (that is, a module which contains 0 as only proper submodule), then $0 : \mathfrak{p}$ would be decomposable in a direct sum of cyclic submodules (see [1], sec. 16). Consequently, property I-4 implies that $\mathfrak{V}_1 = 0 : \mathfrak{p}$, which, since $\mathfrak{V}_1$ is an arbitrary minimal submodule of $\mathfrak{V}$, proves the following lemma.

LEMMA 1.1. *A completely indecomposable module* $\mathfrak{V}$ *has a unique minimal submodule, namely* $0 : \mathfrak{p}$.

REMARK 1.1. If $\mathfrak{V}$ satisfies properties I-1, I-2 and I-3, then I-4 is obviously equivalent with the existence of a unique minimal submodule in $\mathfrak{V}$. In the first place, if $\mathfrak{V}_1$ and $\mathfrak{V}_2$ were two distinct minimal submodules, the submodule $\mathfrak{W} = \mathfrak{V}_1 \dotplus \mathfrak{V}_2$ would be decomposable. Conversely, if $\mathfrak{V}_1$ is a unique minimal submodule of $\mathfrak{V}$, then $\mathfrak{V}_1$ is contained in every submodule of $\mathfrak{V}$, which implies that $\mathfrak{V}$ is completely indecomposable. The main content of lemma 1.1 is the fact that this unique submodule is $0 : \mathfrak{p}$.

REMARK 1.2. It is important for secs. **2**, **5** and **6** to observe that property I-4 is also equivalent with saying that 0 is an intersection-irreducible module (i.e., a module which is not the intersection of two proper divisors). In the first place, if $0 = \mathfrak{W}_1 \cap \mathfrak{W}_2$, where $0 \subset \mathfrak{W}_1$ and $0 \subset \mathfrak{W}_2$, then the sum $(\mathfrak{W}_1, \mathfrak{W}_2)$ of $\mathfrak{W}_1$ and $\mathfrak{W}_2$ is decomposable, $(\mathfrak{W}_1, \mathfrak{W}_2) = \mathfrak{W}_1 \dotplus \mathfrak{W}_2$, and hence $\mathfrak{V}$ is then not completely indecomposable. Conversely, if $\mathfrak{V}$ is not completely indecomposable, $\mathfrak{V}$ contains a decomposable module $\mathfrak{W} = \mathfrak{W}_1 \dot{+} \mathfrak{W}_2$, where $0 \subset \mathfrak{W}_1$ and $0 \subset \mathfrak{W}_2$. Then $0 = \mathfrak{W}_1 \cap \mathfrak{W}_2$ and hence 0 is then not intersection-irreducible.

2. Properties of completely indecomposable modules obtained from Gröbner's theory of irreducible ideals. Gröbner investigated in [3] the theory of an intersection-irreducible ideal $\mathfrak{q}$ of a commutative ring A, where the ideals of A satisfy the ascending chain condition. Since an ideal is intersection-irreducible if it is not the intersection of two proper divisors, the zero module of the difference module $A - \mathfrak{q}$ (difference modules are always considered as modules with A as operator domain) is also intersection-irreducible. Since the zero module of a completely indecomposable module is intersection-irreducible (see remark 1.2), Gröbner's theory, as was shown by Grundy in [1], sec. 16, can be extended to completely indecomposable modules. The following lemmas 2.1 and 2.2, which will be used in secs. **3** and **5**, are proved in [1], sec. 16.

Let $\mathfrak{V}$ again be a completely indecomposable module with A as operator domain and $\mathfrak{q}$ as annihilating ideal; and let $\mathfrak{p}$ again be the radical of 0. Since $\mathfrak{q}$ is a primary ideal of A with maximal radical $\mathfrak{p}$ (see sec. **1**), every divisor of $\mathfrak{q}$, different from A, is a primary ideal with $\mathfrak{p}$ as radical. Furthermore, according to [2], Theorem 2.41, there exists a composition series $\mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_{l'-1} \subset \mathfrak{q}_{l'} = A$ from $\mathfrak{q}$ to A, where the difference module $\mathfrak{q}_i - \mathfrak{q}_{i-1}$ is operator isomorphic with $A - \mathfrak{p}$ and where $\mathfrak{q}_{i-1} : \mathfrak{q}_i = \mathfrak{p}$ for $i = 1, \ldots, l'$.

LEMMA 2.1. *If $\mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_{l'-1} \subset \mathfrak{q}_{l'} = A$ is a composition series from $\mathfrak{q}$ to A, then $0 \subset 0 : \mathfrak{q}_{l'-1} \subset \ldots \subset 0 : \mathfrak{q}_1 \subset \mathfrak{V}$ is a composition series of $\mathfrak{V}$. Conversely, if $0 \subset \mathfrak{V}_1 \subset \ldots \subset \mathfrak{V}_{l-1} \subset \mathfrak{V}$ is a composition series of $\mathfrak{V}$, then $\mathfrak{q} \subset 0 : \mathfrak{V}_{l-1} \subset \ldots \subset 0 : \mathfrak{V}_1 \subset A$ is a composition series from $\mathfrak{q}$ to A. Consequently, the length l' of the difference module $A - \mathfrak{q}$ is equal to the length l of $\mathfrak{V}$.*

LEMMA 2.2. *If $\mathfrak{a}$ is an ideal of A which contains $\mathfrak{q}$, then $0 : (0 : \mathfrak{a}) = \mathfrak{a}$; i.e., the annihilating ideal of the module $0 : \mathfrak{a}$ is $\mathfrak{a}$. In the same way, if $\mathfrak{W}$ is a submodule of $\mathfrak{V}$, then $0 : (0 : \mathfrak{W}) = \mathfrak{W}$.*

It follows from lemma 2.2 that a submodule $\mathfrak{W}$ of $\mathfrak{V}$ is uniquely determined by its annihilating ideal $\mathfrak{a} = 0 : \mathfrak{W}$, since $\mathfrak{W} = 0 : \mathfrak{a}$. Consequently, different submodules of $\mathfrak{V}$ have different annihilating ideals. Furthermore, if $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are two submodules of $\mathfrak{V}$ with the annihilating ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ respectively, then the annihilating ideal of $\mathfrak{W}_3 = \mathfrak{W}_1 \cap \mathfrak{W}_2$ is equal to the sum $(\mathfrak{a}_1, \mathfrak{a}_2)$ of $\mathfrak{a}_1$ and $\mathfrak{a}_2$. In the first place, if $\mathfrak{a}_3$ is the annihilating ideal of $\mathfrak{W}_3$, then $\mathfrak{a}_3 = 0 : \mathfrak{W}_3$, while $0 : (\mathfrak{a}_1, \mathfrak{a}_2) = (0 : \mathfrak{a}_1) \cap (0 : \mathfrak{a}_2) = \mathfrak{W}_1 \cap \mathfrak{W}_2 = \mathfrak{W}_3$, according to lemma 2.2. Hence, $(\mathfrak{a}_1, \mathfrak{a}_2) = 0 : (0 : (\mathfrak{a}_1, \mathfrak{a}_2)) = 0 : \mathfrak{W}_3 = \mathfrak{a}_3$, which proves the following lemma.

LEMMA 2.3. *A submodule $\mathfrak{W}$ of $\mathfrak{V}$ is uniquely determined by its annihilating ideal. If $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are two submodules of $\mathfrak{V}$ with respectively the annihilating ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$, then the annihilating ideal of $\mathfrak{W}_1 \cap \mathfrak{W}_2$ is $(\mathfrak{a}_1, \mathfrak{a}_2)$.*

Let $\mathfrak{a}$ be a divisor of $\mathfrak{q}$. Then there exists a composition series from $\mathfrak{q}$ to A which passes through $\mathfrak{a}$, say $\mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{a} = \mathfrak{q}_h \subset \mathfrak{q}_{h+1} \subset \ldots \subset \mathfrak{q}_l = A$. Hence, the length of the difference module $A - \mathfrak{a}$ is $l - h$. Since, according to lemma 2.1, $0 \subset 0 : \mathfrak{q}_{l-1} \subset \ldots \subset 0 : \mathfrak{a} \subset \ldots \subset 0 : \mathfrak{q}_1 \subset \mathfrak{V}$ is a composition series of $\mathfrak{V}$, the length of the module $0 : \mathfrak{a}$ is $l - h$. This proves lemma 2.4, which will be used in sec. **5** and which is the analogue of Theorem 6 of [3].

LEMMA 2.4. *If $\mathfrak{a}$ is a divisor of $\mathfrak{q}$, the length of the difference module $A - \mathfrak{a}$ is equal to the length of the submodule $0 : \mathfrak{a}$ of $\mathfrak{V}$. In the same way, if $\mathfrak{W}$ is a submodule of $\mathfrak{V}$, the length of $\mathfrak{W}$ is equal to the length of the difference module $A - (0 : \mathfrak{W})$. ($0 : \mathfrak{W}$ is the annihilating ideal of $\mathfrak{W}$.)*

REMARK 2.1. It is obvious that a submodule $\mathfrak{W}$ of a completely indecomposable module is itself completely indecomposable. Since the annihilating ideal of $\mathfrak{W}$ is $0 : \mathfrak{W}$, it follows from lemma 2.1 that the length of $\mathfrak{W}$ is equal to the length of the difference module $A - (0 : \mathfrak{W})$, which gives another proof of lemma 2.4.

3. The endomorphism ring of a completely indecomposable module. The purpose of this section is to prove the following theorem.

THEOREM 3.1. *The operator-endomorphism ring E of a completely indecomposable module $\mathfrak{V}$ with A as operator domain consists of the multiplications of the elements of $\mathfrak{V}$ with the elements of A. Hence, if $\mathfrak{q}$ is the annihilating ideal of $\mathfrak{V}$, then E is ring-isomorphic with the factor ring $A/\mathfrak{q}$.*

Proof. We shall consider E not as a ring, but as a module with A as right-operator domain, according to the following definitions.

$$V(M_1 + M_2) = VM_1 + VM_2, \text{ where } V \in \mathfrak{V} \text{ and } M_1, M_2 \in E;$$

$$V(M\alpha) = (VM)\alpha, \text{ where } V \in \mathfrak{V}, M \in E \text{ and } \alpha \in A.$$

To any $\alpha \in A$, there corresponds the endomorphism $M(\alpha)$, defined by $VM(\alpha) = V\alpha$ for all $V \in \mathfrak{V}$. (These endomorphisms $M(\alpha)$ are the multiplications mentioned in Theorem 3.1. Endomorphism and homomorphism will always mean operator-endomorphism and operator-homomorphism.) The correspondence $\alpha \rightarrow M(\alpha)$ is clearly an operator-homomorphism from A into E, where both A and E are considered as A-modules. Since the kernel of this homomorphism is $\mathfrak{q}$, E contains a submodule E' which is operator isomorphic with the difference module $A - \mathfrak{q}$. All we have to show is that $E' = E$. However, E' is an A-module of finite length; namely, the length of E' is equal to the length of $A - \mathfrak{q}$ and hence, according to lemma 2.1, is equal to the length l of $\mathfrak{V}$. Consequently, all we have to prove is that the A-module E has a composition series of length l. Hereto, let $0 = \mathfrak{V}_0 \subset \mathfrak{V}_1 \subset \ldots \subset \mathfrak{V}_{l-1} \subset \mathfrak{V}_l = \mathfrak{V}$ be a composition series of $\mathfrak{V}$. Let E_i, for $i = 0, \ldots, l$, be the submodule of E which consists of those endomorphisms which annihilate $\mathfrak{V}_i$; i.e., $VM = 0$ for all $V \in \mathfrak{V}_i$ and $M \in E_i$. Then, obviously $E_0 = E$ and E_l is the zero-endomorphism Ω, while $E = E_0 \supseteq E_1 \supseteq \ldots \supseteq E_{l-1} \supseteq E_l = \Omega$. We will prove that this series is a composition series of E. It will be enough to show that, for any fixed integer i, where $1 \leq i \leq l$, the difference module $E_{i-1} - E_i$ is operator-isomorphic with the difference module $A - \mathfrak{p}$, where $\mathfrak{p}$ is the radical of 0 ($\mathfrak{p}$ is a maximal ideal of A according to sec. 1). Let $M \in E_{i-1}$, then $\mathfrak{V}_i M$ is a submodule of $\mathfrak{V}$ which is homomorphic with $\mathfrak{V}_i$. Since $\mathfrak{V}_{i-1}M = 0$, the length of $\mathfrak{V}_i M$ is at most 1. Consequently, since $\mathfrak{V}_1$ is the only submodule of $\mathfrak{V}$ of length 1, $\mathfrak{V}_i M \subseteq \mathfrak{V}_1$. Now, let V_1 be a fixed non-zero element of $\mathfrak{V}_1$ and let V_i be a fixed element of $\mathfrak{V}_i$ where V_i not $\in \mathfrak{V}_{i-1}$. Then, $\mathfrak{V}_1$ is a cyclic module with V_1 as generator and $\mathfrak{V}_i = (\mathfrak{V}_{i-1}, V_i)$. (This means that $\mathfrak{V}_i$ is generated by $\mathfrak{V}_{i-1}$ and V_i; i.e., an element of $\mathfrak{V}_i$ can always be written as $W_{i-1} + V_i\alpha$ where $W_{i-1} \in \mathfrak{V}_{i-1}$ and $\alpha \in A$.) It follows that there exists an $\alpha \in A$ such that $V_i M = V_1\alpha$. Let $\bar{\alpha}$ denote the coset of α modulo $\mathfrak{p}$. Then, since $\mathfrak{p}$ is the annihilating ideal of V_1, the correspondence $M \rightarrow \bar{\alpha}$ is an operator-homomorphism H from E_{i-1} into $A - \mathfrak{p}$. The kernel of H is clearly E_i and hence the difference module $E_{i-1} - E_i$ is operator-isomorphic with a submodule of $A - \mathfrak{p}$. Since the only submodules of $A - \mathfrak{p}$ are the zero-module and $A - \mathfrak{p}$ itself, all that remains to be shown is that $E_{i-1} - E_i$ is not the zero-module, i.e., that $E_{i-1} \supset E_i$. However, it follows from lemma 2.1 that $0 : \mathfrak{V}_i \subset 0 : \mathfrak{V}_{i-1}$ and consequently, there exists an element α of A such that $\alpha \in 0 : \mathfrak{V}_{i-1}$ and α not $\in 0 : \mathfrak{V}_i$. The endomorphism $VM(\alpha) = V\alpha$, for all $V \in \mathfrak{V}$, is clearly an element of E_{i-1} which is not contained in E_i; hence Theorem 3.1 is proved.

REMARK 3.1. Any cyclic module $\mathfrak{V}$ which has a ring A as operator domain and which satisfies property I-1 is operator-isomorphic with the difference module $A - \mathfrak{a}$, where $\mathfrak{a}$ is the annihilating ideal of $\mathfrak{V}$. As is well known, the operator-endomorphisms of $A - \mathfrak{a}$ are just the multiplications of the elements of $A - \mathfrak{a}$ with the elements of A (see, for instance, [4], sec. 120) and hence the operator-endomorphism ring of $A - \mathfrak{a}$ is ring-isomorphic with the factor ring $A/\mathfrak{a}$. In other words, Theorem 3.1 always holds for a cyclic module. As is shown by example 7.3, a completely indecomposable module is not necessarily cyclic.

It follows immediately from Theorem 3.1 that if $\mathfrak{V}$ is a completely indecomsable module with A as operator domain, 0 as zero-module and $\mathfrak{p}$ as radical of 0, then the automorphism-group of $\mathfrak{V}$ consists of the multiplications of elements of $\mathfrak{V}$ with elements $\alpha \in A$, where α not $\in \mathfrak{p}$. The elements of $\mathfrak{p}$ give rise to the endomorphisms of $\mathfrak{V}$ with non-zero kernel.

COROLLARY 3.1. *Let $\mathfrak{V}$ be a completely indecomposable module and let $\mathfrak{W}$ be a submodule of $\mathfrak{V}$. Then, every operator-endomorphism M^* of $\mathfrak{W}$ can be extended to an operator-endomorphism M of $\mathfrak{V}$. If M^* is an automorphism of $\mathfrak{W} \neq 0$, M is an automorphism of $\mathfrak{V}$.*

Proof. Let A be the operator domain of $\mathfrak{V}$. Then $\mathfrak{W}$, as a submodule of a completely indecomposable module, is itself completely indecomposable and has A as operator domain. Hence, it follows from Theorem 3.1 that if M^* is an endomorphism of $\mathfrak{W}$, there exists an $\alpha \in A$, such that $VM^* = V\alpha$ for all

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$V \epsilon \mathfrak{W}$. The endomorphism $VM = V\alpha$ for all $V \epsilon \mathfrak{V}$ is clearly an extension of M^* to an endomorphism M of $\mathfrak{V}$. If M^* is an automorphism of $\mathfrak{W} \neq 0$, α not $\epsilon\, \mathfrak{p}^*$, where $\mathfrak{p}^*$ is the radical of the zero-module 0 of $\mathfrak{W}$. Hence, $\mathfrak{p}^*$ is the radical of the annihilating ideal $\mathfrak{q}^*$ of $\mathfrak{W}$ (see sec. **1**). If $\mathfrak{q}$ is the annihilating ideal of $\mathfrak{V}$ and $\mathfrak{p}$ is the radical of $\mathfrak{q}$, then $\mathfrak{q} \subset \mathfrak{q}^* \subset A$ and hence $\mathfrak{p} \subseteq \mathfrak{p}^* \subset A$. Since $\mathfrak{p}$ and $\mathfrak{p}^*$ are maximal prime ideals of A, $\mathfrak{p} = \mathfrak{p}^*$ and hence α not $\epsilon\, \mathfrak{p}$, which implies that M is an automorphism of $\mathfrak{V}$.

Corollary 3.1 can be used to characterize completely indecomposable modules. Precisely, *an indecomposable module* $\mathfrak{V}$ *with a ring* A *as operator domain, where conditions* I-1, I-2, *and* I-3 *are satisfied, is completely indecomposable if and only if every operator-endomorphism of any submodule* $\mathfrak{W}$ *of* $\mathfrak{V}$ *can be extended to an operator-endomorphism of* $\mathfrak{V}$. The "only if part" is proved by corollary 3.1. Now suppose that $\mathfrak{V}$ is indecomposable, that conditions I-1, I-2, I-3 are satisfied and that, if $\mathfrak{W}$ is any submodule of $\mathfrak{V}$, every operator-endomorphism of $\mathfrak{W}$ can be extended to an operator-endomorphism of $\mathfrak{V}$. Then, if $\mathfrak{W}$ is decomposable, say $\mathfrak{W} = \mathfrak{W}_1 \dot{+} \mathfrak{W}_2$, the projection P^* of $\mathfrak{W}$ on $\mathfrak{W}_1$ is an operator-endomorphism of $\mathfrak{W}$. (If $W \epsilon \mathfrak{W}$, then $W = W_1 + W_2$, and $WP^* = W_1$.) This P^* could not be extended to an operator-endomorphism P of $\mathfrak{V}$, since P would then be a not-nilpotent endomorphism of $\mathfrak{V}$ with non-zero kernel which, according to Fitting's lemma (see [5], p. 11) is not possible since $\mathfrak{V}$ is indecomposable. Hence every submodule of $\mathfrak{V}$ is indecomposable, which means that $\mathfrak{V}$ is completely indecomposable.

Corollary 3.1 and the following lemma on arbitrary operator groups will be used in the proof of Theorem 5.1.

LEMMA 3.1. *Let* $\mathfrak{V}$ *and* $\mathfrak{W}$ *be two additive abelian groups with the same ring* A *as operator domain.* (*A is not necessarily commutative and none of the properties* I-1 *through* I-4 *are assumed.*) *Let* $\mathfrak{V}_1$ *and* $\mathfrak{V}_2$ *be two submodules of* $\mathfrak{V}$ *and let* $\mathfrak{W}_1$ *and* $\mathfrak{W}_2$ *be two submodules of* $\mathfrak{W}$. *Let* I_1 *be an operator-isomorphism between* $\mathfrak{V}_1$ *and* $\mathfrak{W}_1$ *and let* I_2 *be an operator-isomorphism between* $\mathfrak{V}_2$ *and* $\mathfrak{W}_2$. ($\mathfrak{V}_1$, $\mathfrak{V}_2$, $\mathfrak{W}_1$ *and* $\mathfrak{W}_2$ *all have* A *as operator domain.*) *Let* I_1 *map* $\mathfrak{V}_3 = \mathfrak{V}_1 \cap \mathfrak{V}_2$ *isomorphically onto* $\mathfrak{W}_3 = \mathfrak{W}_1 \cap \mathfrak{W}_2$; *i.e.,* I_1 *induces an operator-isomorphism* I^*_1 *between* $\mathfrak{V}_3$ *and* $\mathfrak{W}_3$. *In the same way, let* I_2 *map* $\mathfrak{V}_3$ *isomorphically onto* $\mathfrak{W}_3$; *i.e.,* I_2 *induces an operator-isomorphism* I^*_2 *between* $\mathfrak{V}_3$ *and* $\mathfrak{W}_3$. *Finally, let* $I^*_1 = I^*_2 = I_3$. *Then, there exists an operator-isomorphism* I *between* ($\mathfrak{V}_1$, $\mathfrak{V}_2$) *and* ($\mathfrak{W}_1$, $\mathfrak{W}_2$) *which is simultaneously an extension of* I_1, I_2 *and* I_3.

Proof. Let $V \epsilon (\mathfrak{V}_1, \mathfrak{V}_2)$; hence $V = V_1 + V_2$ where $V_1 \epsilon \mathfrak{V}_1$ and $V_2 \epsilon \mathfrak{V}_2$. (($\mathfrak{V}_1$, $\mathfrak{V}_2$) denotes the submodule generated by $\mathfrak{V}_1$ and $\mathfrak{V}_2$.) We define $VI = V_1I_1 + V_2I_2$ and assert that I is an operator-isomorphism between ($\mathfrak{V}_1$, $\mathfrak{V}_2$) and ($\mathfrak{W}_1$, $\mathfrak{W}_2$). In the first place, we show that VI does not depend on the choice of V_1 and V_2. Let $V = V'_1 + V'_2$ where $V'_1 \epsilon \mathfrak{V}_1$ and $V'_2 \epsilon \mathfrak{V}_2$; then $V_1 - V'_1 = V'_2 - V_2$ is an element of $\mathfrak{V}_3$ and hence $(V_1 - V'_1)I_1 = (V'_2 - V_2)\, I_2$, which implies that $V_1I_1 - V'_1I_1 = V'_2I_2 - V_2I_2$ and hence that $VI = V_1I_1 + V_2I_2 = V'_1I_1 + V'_2I_2$. In the second place, $VI \epsilon (\mathfrak{W}_1, \mathfrak{W}_2)$ since $V_1I_1 \epsilon \mathfrak{W}_1$ and $V_2I_2 \epsilon$

$\mathfrak{W}_2$. One can prove immediately that $(V + V')I = VI + V'I$ and $(V\alpha)I = (VI)\alpha$, where V and V' are elements of $(\mathfrak{V}_1, \mathfrak{V}_2)$ and where $\alpha \epsilon A$, and also that I maps $(\mathfrak{V}_1, \mathfrak{V}_2)$ *onto* $(\mathfrak{W}_1, \mathfrak{W}_2)$. We now show that the kernel of I is the zero-element 0 of $\mathfrak{V}$. Hereto, let $VI = 0'$, where $V \epsilon (\mathfrak{V}_1, \mathfrak{V}_2)$ and where $0'$ is the zero element of $\mathfrak{W}$. Then $V = V_1 + V_2$, where $V_1 \epsilon \mathfrak{V}_1$ and $V_2 \epsilon \mathfrak{V}_2$, and $VI = V_1I_1 + V_2I_2 = 0'$ and hence $V_1I_1 = -V_2I_2$. Since $V_1I_1 \epsilon \mathfrak{W}_1$ and $-V_2I_2 \epsilon \mathfrak{W}_2$, V_1I_1 and $-V_2I_2$ are both elements of $\mathfrak{W}_3$. Since I_1 maps $\mathfrak{V}_1$ isomorphically onto $\mathfrak{W}_1$ and, at the same time, $\mathfrak{V}_3$ isomorphically onto $\mathfrak{W}_3$, $V_1 \epsilon \mathfrak{V}_3$ and, in the same way, $-V_2 \epsilon \mathfrak{V}_3$. Consequently, $V_1I_1 = V_1I_3$ and $-V_2I_2 = -V_2I_3$ and hence $V_1I_1 + V_2I_2 = V_1I_3 + V_2I_3 = (V_1 + V_2)I_3 = 0'$ which implies, since I_3 is an isomorphism, that $0 = V_1 + V_2 = V$. Finally, I is an extension of I_1, since if $V \epsilon \mathfrak{V}_1$, then $VI = VI_1 + 0I_2 = VI_1$ and, in the same way, I is an extension of I_2 and I_3.

Let $\mathfrak{V}, \mathfrak{V}_1, \mathfrak{V}_2, \mathfrak{V}_3, \mathfrak{W}, \mathfrak{W}_1, \mathfrak{W}_2, \mathfrak{W}_3$ have the same meaning as in lemma 3.1. Another way of formulating lemma 3.1 is as follows.

If there exist an operator-isomorphism I_3 between $\mathfrak{V}_3$ and $\mathfrak{W}_3$, an operator-isomorphism I_1 between $\mathfrak{V}_1$ and $\mathfrak{W}_1$, and an operator-isomorphism I_2 between $\mathfrak{V}_2$ and $\mathfrak{W}_2$, where I_1 and I_2 are extensions of I_3, then I_1, I_2 and I_3 can be extended simultaneously to an operator-isomorphism I between $(\mathfrak{V}_1, \mathfrak{V}_2)$ and $(\mathfrak{W}_1, \mathfrak{W}_2)$.

4. Two lemmas of ring-representation theory. The following lemmas 4.1 and 4.2, which will be used in secs. 5 and 6, are immediate corollaries of two lemmas proved by Professor R. Brauer in a course on group theory, given during the summer of 1947 at the University of Michigan. The proofs of lemmas 4.1 and 4.2, which are added here for the sake of completeness, can be derived immediately from Professor Brauer's proofs of the more general lemmas.

LEMMA 4.1. *Let $\mathfrak{V}$ be an additive abelian group with a ring A as operator domain where property* I-1 *is satisfied and where $\mathfrak{V}$ has a unique maximal submodule $\mathfrak{W}$. Let $\mathfrak{a}$ be the annihilating ideal of $\mathfrak{V}$. Then, $\mathfrak{V}$ is operator isomorphic with $A - \mathfrak{a}$.*

Proof. Since $\mathfrak{W}$ is a unique maximal submodule of $\mathfrak{V}$, $\mathfrak{W} \subset \mathfrak{V}$ and $\mathfrak{W}$ contains every proper submodule of $\mathfrak{V}$. Let $V \epsilon \mathfrak{V}$, where V not $\epsilon\, \mathfrak{W}$. The submodule VA of $\mathfrak{V}$ is not contained in $\mathfrak{W}$ since $V.1 = V$ not $\epsilon\, \mathfrak{W}$. Hence, $VA = \mathfrak{V}$ which implies that $\mathfrak{V}$ is cyclic and hence operator isomorphic with $A - \mathfrak{a}$, where $\mathfrak{a}$ is the annihilating ideal of $\mathfrak{V}$.

LEMMA 4.2. *Let $\mathfrak{V}$ be an additive abelian group with a ring A as operator domain where property* I-1 *is satisfied. Let $\mathfrak{a}$ be the annihilating ideal of $\mathfrak{V}$ and suppose that $\mathfrak{V}$ has a submodule $\mathfrak{W}$ such that $\mathfrak{V} - \mathfrak{W}$ is operator isomorphic with $A - \mathfrak{a}$. Then, $\mathfrak{V}$ is decomposable in a direct sum, $\mathfrak{V} = \mathfrak{W} \dotplus \mathfrak{V}_1$ where $\mathfrak{V}_1$ is operator-isomorphic with $A - \mathfrak{a}$.*

Proof. Let T denote the natural homomorphism from $\mathfrak{V}$ onto $\mathfrak{V} - \mathfrak{W}$; i.e., for any $V \epsilon \mathfrak{V}$, VT is the coset of V modulo $\mathfrak{W}$. Let Q denote the isomorphism

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between $\mathfrak{V} - \mathfrak{W}$ and $A - \mathfrak{a}$, whose existence is stated in lemma 4.2. Then $P = TQ$ (first T, then Q) is an operator-homomorphism from $\mathfrak{V}$ *onto* $A - \mathfrak{a}$ with kernel $\mathfrak{W}$. Let V_1 be an element of $\mathfrak{V}$ such that $V_1P = \bar{1}$, where $\bar{1}$ is the coset of the unit element 1 of A modulo $\mathfrak{a}$. We claim that we can take V_1A for the $\mathfrak{V}_1$ of lemma 4.2. (V_1A is the submodule of $\mathfrak{V}$ which consists of the elements $V_1\alpha$, where $\alpha \in A$.) In the first place, P maps V_1A isomorphically onto $A - \mathfrak{a}$, since $(V_1\alpha)P = (V_1P)\alpha = \bar{1}\alpha = \bar{\alpha}$ for any $\alpha \in A$ (bars denote cosets modulo $\mathfrak{a}$). This proves already that V_1A is operator-isomorphic with $A - \mathfrak{a}$ and that $\mathfrak{W} \cap V_1A = 0$, where 0 is the zero-element of $\mathfrak{V}$. In the second place, for any $V \in \mathfrak{V}$, we can find an $\alpha \in A$ such that $VP = (V_1\alpha)P$ and hence such that $V - V_1\alpha = W$, where W lies in the kernel $\mathfrak{W}$ of P. Consequently, $V = W + V_1\alpha$ which proves that $\mathfrak{V} = \mathfrak{W} \dot{+} \mathfrak{V}_1$.

5. Criterion of isomorphism of completely indecomposable modules. We can now prove the following criterion that two completely indecomposable modules be isomorphic.

THEOREM 5.1. *Two completely indecomposable modules with the same operator domain are operator-isomorphic if and only if they have the same annihilating ideals.*

Proof. Let $\mathfrak{V}$ and $\mathfrak{W}$ be two completely indecomposable modules with the same operator domain A. Hence, properties I-1, I-2, I-3 and I-4 are satisfied. It is obvious that if $\mathfrak{V}$ and $\mathfrak{W}$ are operator-isomorphic, they have the same annihilating ideal. Hence we assume that $\mathfrak{V}$ and $\mathfrak{W}$ have the same annihilating ideal $\mathfrak{q}$ and prove that then $\mathfrak{V}$ and $\mathfrak{W}$ are operator-isomorphic.

Since the lengths of $\mathfrak{V}$ and of $\mathfrak{W}$ are both equal to the length of the difference module $A - \mathfrak{q}$ (see lemma 2.1), let l be the common length of $\mathfrak{V}$, $\mathfrak{W}$ and $A - \mathfrak{q}$. If $l = 1$, both $\mathfrak{V}$ and $\mathfrak{W}$ are cyclic and hence operator-isomorphic with $A - \mathfrak{q}$ and hence isomorphic with each other. Consequently, we assume that $l > 1$ and we make the induction hypothesis that Theorem 5.1 has been proved for $l = 1, 2, \ldots, l - 1$. There are two cases to be considered.

Case 1. *One of the two modules, say* $\mathfrak{V}$, *has only one maximal submodule.* Since $\mathfrak{V}$ has a unique maximal submodule, it follows from lemma 4.1 that $\mathfrak{V}$ is operator-isomorphic with $A - \mathfrak{q}$. This implies in particular that $A - \mathfrak{q}$ is completely indecomposable and hence that $A - \mathfrak{q}$ has a unique minimal submodule. Consequently, there exists only one ideal $\mathfrak{q}_1$ such that $\mathfrak{q} \subset \mathfrak{q}_1$ and where the length of the difference module $A - \mathfrak{q}_1$ is $l - 1$. If $0'$ is the zero-module of $\mathfrak{W}$, the length of $\mathfrak{W}_{l-1} = 0' : \mathfrak{q}_1$ is $l - 1$ (see lemma 2.4) and hence $\mathfrak{W}_{l-1}$ is a maximal submodule of $\mathfrak{W}$. We assert that $\mathfrak{W}_{l-1}$ is the only maximal submodule of $\mathfrak{W}$. Let $\mathfrak{W}'_{l-1}$ be any maximal submodule of $\mathfrak{W}$ where $\mathfrak{q}'_1$ is the annihilating ideal of $\mathfrak{W}'_{l-1}$. Then the length of $\mathfrak{W}'_{l-1}$ is $l - 1$ and hence the length of $A - \mathfrak{q}'_1$ is also $l - 1$ (see lemma 2.4); consequently, $\mathfrak{q}'_1 = \mathfrak{q}_1$ and, according to lemma 2.2, $\mathfrak{W}'_{l-1} = 0' : \mathfrak{q}'_1 = 0' : \mathfrak{q}_1 = \mathfrak{W}_{l-1}$, which proves the assertion. Lemma 4.1 then implies that $\mathfrak{W}$ is also operator-isomorphic with $A - \mathfrak{q}$ and hence that $\mathfrak{V}$ and $\mathfrak{W}$ are isomorphic.

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Case 2. *Both modules* $\mathfrak{V}$ *and* $\mathfrak{W}$ *have at least two different maximal submodules.* Let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be two different maximal submodules of $\mathfrak{V}$, which implies that $\mathfrak{V} = (\mathfrak{V}_1, \mathfrak{V}_2)$. The length of both $\mathfrak{V}_1$ and $\mathfrak{V}_2$ is $l - 1$. If $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are the annihilating ideals of respectively $\mathfrak{V}_1$ and $\mathfrak{V}_2$, then $\mathfrak{q}_1 = 0 : \mathfrak{V}_1$ and $\mathfrak{q}_2 = 0 : \mathfrak{V}_2$, where 0 is the zero-module of $\mathfrak{V}$. Furthermore, $\mathfrak{q}_1 \neq \mathfrak{q}_2$ according to lemma 2.3, and the length of both the difference modules $A - \mathfrak{q}_1$ and $A - \mathfrak{q}_2$ is $l - 1$ (see lemma 2.4). Consider the submodules $\mathfrak{W}_1 = 0' : \mathfrak{q}_1$ and $\mathfrak{W}_2 = 0' : \mathfrak{q}_2$ of $\mathfrak{W}$, where $0'$ is the zero-module of $\mathfrak{W}$. According to lemma 2.3, $\mathfrak{W}_1 \neq \mathfrak{W}_2$, and according to lemma 2.4 the length of both $\mathfrak{W}_1$ and $\mathfrak{W}_2$ is $l - 1$; i.e., $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are two distinct, maximal submodules of $\mathfrak{W}$, which implies that $\mathfrak{W} = (\mathfrak{W}_1, \mathfrak{W}_2)$. Since $\mathfrak{V}_1$ and $\mathfrak{W}_1$ are completely indecomposable modules (a submodule of a completely indecomposable module is obviously completely indecomposable) and have the same annihilating ideal $\mathfrak{q}_1$, it follows from the induction hypothesis that $\mathfrak{V}_1$ and $\mathfrak{W}_1$ are operator-isomorphic. In the same way, $\mathfrak{V}_2$ and $\mathfrak{W}_2$ are operator-isomorphic. Hence, let I_1 be an operator-isomorphism which maps $\mathfrak{V}_1$ onto $\mathfrak{W}_1$ and let I_2 be an operator-isomorphism which maps $\mathfrak{V}_2$ onto $\mathfrak{W}_2$. We claim that I_1 maps $\mathfrak{V}_3 = \mathfrak{V}_1 \cap \mathfrak{V}_2$ isomorphically onto $\mathfrak{W}_3 = \mathfrak{W}_1 \cap \mathfrak{W}_2$ and that I_2 maps $\mathfrak{V}_3$ isomorphically onto $\mathfrak{W}_3$. In the first place, the annihilating ideal of $\mathfrak{V}_3$ and $\mathfrak{W}_3$ is $(\mathfrak{q}_1, \mathfrak{q}_2)$ according to lemma 2.3. In the second place, both $\mathfrak{V}_3 I_1$ and $\mathfrak{V}_3 I_2$ are submodules of $\mathfrak{W}$ which are operator-isomorphic with $\mathfrak{V}_3$. Hence these modules have $(\mathfrak{q}_1, \mathfrak{q}_2)$ as annihilating ideal, which implies, according to lemma 2.3, that $\mathfrak{V}_3 I_1 = \mathfrak{V}_3 I_2 = \mathfrak{W}_3$. Let I^*_1 and I^*_2 be the operator-isomorphisms which map $\mathfrak{V}_3$ onto $\mathfrak{W}_3$ and which are induced respectively by I_1 and I_2; i.e., if $V \in \mathfrak{V}_3$, then $VI^*_1 = VI_1 \in \mathfrak{W}_3$ and $VI^*_2 = VI_2 \in \mathfrak{W}_3$. We cannot as yet apply lemma 3.1 since it may be that $I^*_1 \neq I^*_2$. However, we shall change I_1 into a new operator-isomorphism J_1 such that $J^*_1 = I^*_2$. Hereto, let $(I^*_1)^{-1}$ be the inverse of I^*_1; hence, $(I^*_1)^{-1}$ maps $\mathfrak{W}_3$ operator-isomorphically onto $\mathfrak{V}_3$. Then, $(I^*_1)^{-1} I^*_2$ is clearly an operator-automorphism, say H^*, of $\mathfrak{W}_3$ onto itself. Since $\mathfrak{W}_3 \subseteq \mathfrak{W}_1$, it follows from corollary 3.1 that there exists an operator-automorphism H of $\mathfrak{W}_1$ onto itself, where H is an extension of H^*; i.e., if $W \in \mathfrak{W}_3$, then $WH = WH^*$. For J_1 we then take the operator-isomorphism $J_1 = I_1 H$ which clearly maps $\mathfrak{V}_1$ onto $\mathfrak{W}_1$. Furthermore, if J^*_1 is the operator-isomorphism between $\mathfrak{V}_3$ and $\mathfrak{W}_3$, induced by J_1, then $J^*_1 = I^*_2$ because, if $V \in \mathfrak{V}_3$, then $VJ^*_1 = VI_1 H = VI^*_1 H$ and, since $VI^*_1 \in \mathfrak{W}_3$, $VI^*_1 H = VI^*_1 H^* = VI^*_1 (I^*_1)^{-1} I^*_2 = VI^*_2 = VI_2$. We can then conclude from lemma 3.1 that there exists an operator-isomorphism between $(\mathfrak{V}_1, \mathfrak{V}_2) = \mathfrak{V}$ and $(\mathfrak{W}_1, \mathfrak{W}_2) = \mathfrak{W}$ which proves Theorem 5.1.

Remark 5.1. The module $\mathfrak{V}$ in Theorem 5.1 can be considered as a representation space of the representation $\alpha \rightarrow M(\alpha)$ of A, where $\alpha \in A$ and where $M(\alpha)$ is the operator-endomorphism $VM(\alpha) = V\alpha$ of $\mathfrak{V}$. This representation of A is faithful if and only if the annihilating ideal $\mathfrak{q}$ of $\mathfrak{V}$ is the zero-ideal. Furthermore, according to Theorem 5.1, two completely indecomposable modules are operator-isomorphic if they have the same operator domain A and each has the zero-ideal as annihilating ideal. Consequently, as stated in the introduction, two faithful representations of A, both of which give rise to completely indecomposable representation spaces, are equivalent.

The following corollary of Theorem 5.1 will be used in sec. **6**.

COROLLARY 5.1. *A completely indecomposable module is cyclic if and only if its annihilating ideal is intersection-irreducible.*

Proof. Let $\mathfrak{B}$ be a completely indecomposable module with A as operator domain and $\mathfrak{q}$ as annihilating ideal. If $\mathfrak{B}$ is cyclic, $\mathfrak{B}$ is operator-isomorphic with the difference module $A - \mathfrak{q}$, which implies that then $A - \mathfrak{q}$ is completely indecomposable and hence $\mathfrak{q}$ is intersection-irreducible (see remark 1.2). Conversely, if $\mathfrak{q}$ is intersection-irreducible, $A - \mathfrak{q}$ is completely indecomposable. Consequently, $\mathfrak{B}$ and $A - \mathfrak{q}$ are then two completely indecomposable modules with the same annihilating ideal $\mathfrak{q}$. Theorem 5.1 then implies that $\mathfrak{B}$ and $A - \mathfrak{q}$ are operator-isomorphic and hence that $\mathfrak{B}$ is then cyclic.

DEFINITION 5.1. *An additive abelian group $\mathfrak{B}$ with a ring A as operator domain is called a regular module if property* I-1 *is satisfied and if $\mathfrak{B}$ is the direct sum of a finite number of completely indecomposable submodules.*

REMARK 5.2. The term "regular" is taken from [3], where regular rings are defined. A regular ring, considered as a module with itself as operator domain, is a regular module in the sense of definition 5.1.

Let $\mathfrak{B}$ be any additive abelian group with a ring A as operator domain, where conditions I-1, I-2 and I-3 are satisfied. Then, $\mathfrak{B}$ is a direct sum of indecomposable submodules $\mathfrak{B} = \mathfrak{B}_1 \dot{+} \ldots \dot{+} \mathfrak{B}_h$, where $\mathfrak{B}_i$ is an indecomposable module for $i = 1, \ldots, h$. If $\mathfrak{B} = \mathfrak{B}'_1 \dot{+} \ldots \dot{+} \mathfrak{B}'_{h'}$ is another decomposition of $\mathfrak{B}$ into a direct sum of indecomposable submodules, $h = h'$ and, after a suitable ordering, $\mathfrak{B}_i$ is operator-isomorphic with $\mathfrak{B}'_i$ for $i = 1, \ldots, h$, according to the Krull-Schmidt theorem (see [5], p. 12). Consequently, the annihilating ideals of $\mathfrak{B}_1, \ldots, \mathfrak{B}_h$ are completely determined by $\mathfrak{B}$ and do not depend on the particular decomposition of $\mathfrak{B}$ which is chosen.

DEFINITION 5.2. *Let $\mathfrak{B}$ be an additive abelian group with a ring A as operator domain where conditions* I-1, I-2 *and* I-3 *are satisfied. Let $\mathfrak{B} = \mathfrak{B}_1 \dot{+} \ldots \dot{+} \mathfrak{B}_h$ be a decomposition of $\mathfrak{B}$ into a direct sum of indecomposable submodules. Then, the annihilating ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ respectively of $\mathfrak{B}_1, \ldots, \mathfrak{B}_h$ are called the elementary ideals of $\mathfrak{B}$.*

REMARK 5.3. Since $\mathfrak{B}_1, \ldots, \mathfrak{B}_h$ are indecomposable modules with composition sequences of finite length, the elementary ideals are always primary ideals of A with maximal associated primes (see sec. **1**). The reason for calling $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ the elementary ideals of $\mathfrak{B}$, rather than the elementary divisors of $\mathfrak{B}$, is that $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ correspond to the primary factors of the classical elementary divisors and not to the elementary divisors themselves (see remark 6.1).

Since conditions I-1, I-2 and I-3 are clearly satisfied in a regular module, the elementary ideals of a regular module are defined by definition 5.2. The following theorem is an immediate corollary of Theorem 5.1 and constitutes an

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extension of the classical theory of elementary divisors to regular modules (see sec. 6).

THEOREM 5.2. *Two regular modules with the same operator domain are operator-isomorphic if and only if they have the same elementary ideals.*

The following corollary is an immediate consequence of corollary 5.1.

COROLLARY 5.2. *A regular module* $\mathfrak{B}$ *with* A *as operator domain and whose elementary ideals* $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ *are intersection-irreducible ideals of* A *is a direct sum of cyclic submodules; namely,* $\mathfrak{B}$ *is operator-isomorphic with the direct sum of the* h *cyclic modules* $A - \mathfrak{q}_1, \ldots, A - \mathfrak{q}_h$.

Proof. $\mathfrak{B}$ is a direct sum $\mathfrak{B} = \mathfrak{B}_1 \dot{+} \ldots \dot{+} \mathfrak{B}_h$, where $\mathfrak{B}_i$ is a completely indecomposable submodule of $\mathfrak{B}$ with $\mathfrak{q}_i$ as annihilating ideal for $i = 1, \ldots, h$. Consequently, according to corollary 5.1, $\mathfrak{B}_i$ is cyclic and operator-isomorphic with $A - \mathfrak{q}_i$, which proves corollary 5.2.

6. The classical elementary divisors. We first derive the following lemma whose proof is modelled after Professor R. Brauer's lecture "The Normal Form of a Matrix," given during the summer of 1947 at the University of Michigan.

LEMMA 6.1. *Let* $\mathfrak{B}$ *be an additive abelian group with a ring* A *as operator domain where conditions* I-1, I-2 *and* I-3 *are satisfied. Furthermore, let* $\mathfrak{B}$ *be an indecomposable module and let the annihilating ideal* $\mathfrak{q}$ *of* $\mathfrak{B}$ *be an intersection-irreducible ideal of* A. *Then,* $\mathfrak{B}$ *is a cyclic module and is consequently operator-isomorphic with* $A - \mathfrak{q}$, *which implies that* $\mathfrak{B}$ *is completely indecomposable.*

Proof. Since $\mathfrak{q}$ is intersection-irreducible, the difference module $A - \mathfrak{q}$ is completely indecomposable (see remark 1.2); hence all we have to show is that $\mathfrak{B}$ is cyclic. Hereto we show first that, independent of whether $\mathfrak{B}$ is indecomposable or not, $\mathfrak{B}$ contains a submodule $\mathfrak{W}$ such that $\mathfrak{B} - \mathfrak{W}$ is operator-isomorphic with $A - \mathfrak{q}$. Let S be the collection of submodules of $\mathfrak{B}$ whose fundamental ideals are equal to $\mathfrak{q}$; i.e., $\mathfrak{W}' \in S$ if and only if $\mathfrak{W}' : \mathfrak{B} = \mathfrak{q}$. Since S contains the zero-module of $\mathfrak{B}$, S is not empty and hence I-2 assures that S contains a maximal element $\mathfrak{W}$. We assert that $\mathfrak{W}$ has the required property. We first show that $\mathfrak{W}$ is intersection-irreducible. Suppose that $\mathfrak{W} = \mathfrak{W}_1 \cap \mathfrak{W}_2$, where $\mathfrak{W} \subset \mathfrak{W}_1 \subset \mathfrak{B}$ and $\mathfrak{W} \subset \mathfrak{W}_2 \subset \mathfrak{B}$. Then, if $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are the fundamental ideals respectively of $\mathfrak{W}_1$ and $\mathfrak{W}_2$, $\mathfrak{q} = \mathfrak{q}_1 \cap \mathfrak{q}_2$ and, since $\mathfrak{W}$ is maximal in S, $\mathfrak{q} \subset \mathfrak{q}_1$ and $\mathfrak{q} \subset \mathfrak{q}_2$. This, however, is impossible since $\mathfrak{q}$ is intersection-irreducible, which shows that $\mathfrak{W}$ is intersection-irreducible. Hence the zero-module of the difference module $\mathfrak{B} - \mathfrak{W}$ is intersection-irreducible. Since conditions I-1, I-2 and I-3 are clearly satisfied by $\mathfrak{B} - \mathfrak{W}$, it follows from remark 1.2 that $\mathfrak{B} - \mathfrak{W}$ is a completely indecomposable module. Since the annihilating ideal of $\mathfrak{B} - \mathfrak{W}$ is $\mathfrak{W} : \mathfrak{B} = \mathfrak{q}$ and $\mathfrak{q}$ is intersection-irreducible, it follows from corollary 5.1 that $\mathfrak{B} - \mathfrak{W}$ is cyclic and consequently operator-isomorphic with $A - \mathfrak{q}$. We can now conclude from lemma 4.2 that $\mathfrak{B}$ is decomposable in a direct sum $\mathfrak{B} = \mathfrak{B}_1 \dot{+} \mathfrak{B}_2$, where $\mathfrak{B}_1$ is operator-isomorphic with $A - \mathfrak{q}$. Hence, since $\mathfrak{B}$ is assumed to be indecomposable, $\mathfrak{B} = \mathfrak{B}_1$ and $\mathfrak{B}$ is operator-isomorphic with $A - \mathfrak{q}$ which shows that $\mathfrak{B}$ is cyclic.

The following corollary extends corollary 5.2 from regular modules to arbitrary modules which satisfy conditions I-1, I-2, I-3.

COROLLARY 6.1. *Let $\mathfrak{B}$ be an additive abelian group which has a ring A as operator domain and which satisfies conditions* I-1, I-2 *and* I-3. *Suppose that the elementary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ of $\mathfrak{B}$ are intersection-irreducible. Then $\mathfrak{B}$ is a direct sum of cyclic sub-modules. Namely, $\mathfrak{B}$ is operator-isomorphic with the direct sum of the h cyclic modules $A - \mathfrak{q}_1, \ldots, A - \mathfrak{q}_h$.*

Proof. $\mathfrak{B}$ is a direct sum $\mathfrak{B} = \mathfrak{B}_1 \dot{+} \ldots \dot{+} \mathfrak{B}_h$, where $\mathfrak{B}_i$ is an indecomposable module with $\mathfrak{q}_i$ as annihilating ideal for $i = 1, \ldots, h$. Consequently, according to lemma 6.1, $\mathfrak{B}_i$ is cyclic and operator-isomorphic with $A - \mathfrak{q}_i$, which proves corollary 6.1.

In the classical elementary divisor theory one deals with additive abelian groups which satisfy conditions I-1 and I-2 and have a ring A as operator domain, where A is either a Euclidean domain (see [4], sec. 108), a principal ideal ring, or the ring of integers of an algebraic number field (see [6]). All these rings have the property that every non-zero primary ideal, i.e., every primary ideal whose associated prime is maximal, is intersection-irreducible. Consequently, the following corollary, which follows immediately from corollary 6.1, contains that part of the classical elementary divisor theory which deals with modules in which the descending chain condition is satisfied; that is with modules whose annihilating ideal is not the zero-ideal.

COROLLARY 6.2. *Let $\mathfrak{B}$ be an additive abelian group with a ring A as operator domain where conditions* I-1, I-2 *and* I-3 *are satisfied. Suppose that every primary ideal of A whose associated prime ideal is maximal, is intersection-irreducible. Then $\mathfrak{B}$ is a direct sum of cyclic modules. Namely, $\mathfrak{B}$ is operator-isomorphic with the direct sum of the h cyclic modules $A - \mathfrak{q}_1, \ldots, A - \mathfrak{q}_h$, where $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ are the elementary ideals of $\mathfrak{B}$.*

REMARK 6.1. In the rings which occur in the classical elementary divisor theory, every primary ideal is a power of its associated prime ideal. Let A be such a ring. Then, if all the elementary ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ of corollary 6.2 have the same associated prime ideal, $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \ldots \subseteq \mathfrak{q}_h$ after a suitable ordering. In other words, $\mathfrak{B}$ is then the direct sum of cyclic submodules whose annihilating ideals divide each other, and the elementary ideals are then exactly the classical elementary divisors of $\mathfrak{B}$. If among the associated prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ of $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ different ones occur, then $(\mathfrak{p}_i, \mathfrak{p}_j) = A$ whenever $\mathfrak{p}_i \neq \mathfrak{p}_j$. As a result one can easily show how the cyclic modules of corollary 6.2 can then be combined to give again the classical statement that $\mathfrak{B}$ is the direct sum of cyclic modules whose annihilating ideals $\mathfrak{a}_1 \subseteq \ldots \subseteq \mathfrak{a}_k$ divide each other. These $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ are then the classical elementary divisors and the $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ are the primary factors of the elementary divisors. This implies in particular that the $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ and the $\mathfrak{q}_1, \ldots, \mathfrak{q}_h$ determine each other completely, from which it follows that two modules are operator-isomorphic if and only if they have the same elementary divisors.

REMARK 6.2. Let $\mathfrak{B}$ be a regular module with a ring A as operator domain and $\mathfrak{q}$ as annihilating ideal. If, as in the case of the classical elementary divisor theory, $\mathfrak{B}$ is the direct sum of cyclic submodules whose annihilating ideals divide each other, then the centre C of the operator-endomorphism ring of $\mathfrak{B}$ consists of the multiplications of the elements of $\mathfrak{B}$ with the elements of A, and consequently C is ring-isomorphic with $A/\mathfrak{q}$. (This is the actual content of [7]. If A is the ring which consists of the scalar polynomials of a square matrix M, this statement is equivalent with saying that the matrices which commute with all matrices which commute with M, are the scalar polynomials of M.) The same statement about the centre C of the operator-endomorphism ring of $\mathfrak{B}$ can obviously be made, according to Theorem 3.1, if $\mathfrak{B}$ is a completely indecomposable module, that is a regular module which is indecomposable. Whether the same statement is also correct for the centre of the operator-endomorphism ring of an arbitrary regular module is unsolved.

7. Examples of completely indecomposable and regular modules.

EXAMPLE 7.1. *The classical case.* Let A be a commutative principal ideal ring with unit element and without divisors of zero or let A be the ring of integers of an algebraic number field of finite degree. Let $\mathfrak{A}$ be the m-dimensional column vector space which consists of the columns of length m whose components are elements of A. Let $\mathfrak{B}$ be a submodule of $\mathfrak{A}$ of rank m. Then, the difference module $\mathfrak{V} = \mathfrak{A} - \mathfrak{B}$ is a regular module with A as operator domain. In the first place, it is well known that conditions I-1 and I-2 are satisfied in $\mathfrak{V}$. In the second place, the annihilating ideal of $\mathfrak{V}$ is not the zero-ideal since the rank of $\mathfrak{B}$ is m. This implies, since the non-zero associated prime ideals of A are maximal ideals, that condition I-3 is also satisfied in $\mathfrak{V}$. We can then conclude from corollary 6.2 that $\mathfrak{V}$ is regular. In this classical case, according to the same corollary, $\mathfrak{V}$ is indecomposable if and only if $\mathfrak{V}$ is cyclic and hence operator-isomorphic with $A - \mathfrak{q}$ where $\mathfrak{q}$ is the annihilator of $\mathfrak{V}$. Consequently, the completely indecomposable modules which occur in this case are all cyclic modules.

EXAMPLE 7.2. *Irreducible ideals.* Let A be a commutative ring with unit element whose ideals satisfy the ascending chain condition. Let $\mathfrak{q}$ be an intersection-irreducible ideal of A whose associated prime $\mathfrak{p}$ is a maximal ideal (an intersection-irreducible ideal is necessarily primary). Then conditions I-1, I-2 and I-3 are satisfied in the difference module $\mathfrak{V} = A - \mathfrak{q}$ and, according to remark 1.2, $\mathfrak{V}$ is a completely indecomposable module with A as operator domain. If $\mathfrak{p}$ is not maximal, a completely indecomposable module can be obtained by constructing the ring of quotients of the ring $A/\mathfrak{q}$, where the non-divisors of zero of $A/\mathfrak{q}$ are admitted as denominators (see [3], p. 215).

Regular modules can of course be constructed at will by forming direct sums of the modules of examples 7.1 and 7.2. All the completely indecomposable modules mentioned in the previous examples are cyclic modules. The author is indebted to Professor R. M. Thrall for the following example which enables us to construct non-cyclic completely indecomposable modules at will.

EXAMPLE 7.3. *Duals of vector spaces.* Let $A = K[x_1, \ldots, x_n]$ be a polynomial domain in n variables $x_1, \ldots, x_n$ where K is a field. Let $\mathfrak{q}$ be a primary ideal of A whose associated prime ideal $\mathfrak{p}$ is maximal (i.e., $\mathfrak{p}$ is a zero-dimensional prime ideal of A). Then, the difference module $\mathfrak{A} = A - \mathfrak{q}$ is a (K, A)-module which, with respect to K, is a vector space of finite rank, say m. The dual vector space $\mathfrak{V}$ of $\mathfrak{A}$ consists, according to [8], p. 558, of the linear functionals from $\mathfrak{A}$ into K where, if $V \in \mathfrak{V}$ and $\alpha \in A$, the linear functional $V\alpha$ is defined by $(V\alpha)(x) = V(x\alpha)$ for all $x \in \mathfrak{A}$. It follows that $\mathfrak{V}$ is again a (K, A)-module which, with respect to K, has finite rank m, while the annihilating ideal of $\mathfrak{V}$ is $\mathfrak{q}$. Since $\mathfrak{V}$ is an A-module with finite K-rank, conditions I-1, I-2 and I-3 are satisfied in $\mathfrak{V}$. Furthermore, to every submodule $\mathfrak{B}$ of $\mathfrak{A}$, one can associate the submodule $\mathfrak{W}(\mathfrak{B})$ of $\mathfrak{V}$ which consists of the functionals which vanish on $\mathfrak{B}$. According to [8], this establishes a one to one correspondence between the submodules of $\mathfrak{A}$ and the submodules of $\mathfrak{V}$, such that $\mathfrak{B}$ consists of all the elements of $\mathfrak{A}$ which are annihilated by the functionals of $\mathfrak{W}(\mathfrak{B})$ and such that $\mathfrak{B}_1 \subset \mathfrak{B}_2$ implies $\mathfrak{W}(\mathfrak{B}_1) \supset \mathfrak{W}(\mathfrak{B}_2)$. It follows easily that, since the A-module $\mathfrak{A}$ has the unique maximal submodule $\mathfrak{B}_0 = \mathfrak{p} - \mathfrak{q}$, the A-module $\mathfrak{V}$ has the unique minimal submodule $\mathfrak{W}_0 = \mathfrak{W}(\mathfrak{B}_0)$ and hence that $\mathfrak{V}$ is a completely indecomposable A-module (see remark 1.1). Since $\mathfrak{V}$ has $\mathfrak{q}$ as annihilating ideal, $\mathfrak{V}$ cannot be cyclic if we choose for $\mathfrak{q}$ an ideal which is not intersection-irreducible, according to corollary 5.1. For instance, if $\mathfrak{q}$ is the ideal (x^2, xy, y^2) of the polynomial domain $A = K\ [x, y]$, then the dual vector space of the vector space $A - \mathfrak{q}$ is a non-cyclic, completely indecomposable, A-module.

REMARK 7.1. Let $\mathfrak{q}$ be a primary ideal with maximal associated prime of the polynomial ring $A = K[x_1, \ldots, x_n]$. The factor ring $A/\mathfrak{q}$ is then a "completely primary" commutative ring. (See [9], p. 96, for the notion of a completely primary ring.) At the same time, $A/\mathfrak{q}$ is an algebra with finite rank with respect to the field K. Conversely, every commutative completely primary ring which is at the same time an algebra with finite rank with respect to a field K, can be obtained in this way. Furthermore, we have seen in example 7.3, that the dual vector space $\mathfrak{V}$ of the vector space $A - \mathfrak{q}$ is a completely indecomposable A-module with $\mathfrak{q}$ as annihilating ideal. Hence, if $\mathfrak{V}$ is considered as a module with $A/\mathfrak{q}$ as operator domain, then $\mathfrak{V}$ is a faithful, completely indecomposable representation space of $A/\mathfrak{q}$. Consequently we see that *every commutative, completely primary ring which is, at the same time, an algebra of finite rank with respect to a field K has a faithful, completely indecomposable representation space.* Conversely, it follows easily from sec. **1** that, if a ring A has a faithful, completely indecomposable representation space, then A is a commutative, completely primary ring. It will be shown in sec. **9** that every commutative, completely primary ring A has a faithful, completely indecomposable representation space, even if A is not an algebra of finite rank with respect to a field.

REMARK 7.2. If in condition I-1 we allow A to be a non-commutative ring, then the conditions I-1, I-2, I-3 and I-4 define the notion of a completely indecomposable module with a not necessarily commutative ring A as operator domain. If A is a (not necessarily commutative) completely primary ring, which is at the same time an algebra with finite rank m with respect to a field K, Theorem 5.1 still holds; that is, two faithful, completely indecomposable representation spaces $\mathfrak{V}_1$ and $\mathfrak{V}_2$ of A are A-isomorphic. This can be shown easily by means of the construction of dual vector spaces, as was pointed out by Professor Thrall. In the first place, $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are easily seen to be vector spaces of finite rank m with respect to K; hence we can construct the dual vector spaces $\mathfrak{V}^*_1$ and $\mathfrak{V}^*_2$ of $\mathfrak{V}_1$ and $\mathfrak{V}_2$ respectively. Since $\mathfrak{V}_1$ and $\mathfrak{V}_2$ have unique minimal submodules, $\mathfrak{V}^*_1$ and $\mathfrak{V}^*_2$ have unique maximal submodules; therefore $\mathfrak{V}^*_1$ and $\mathfrak{V}^*_2$ are both A-isomorphic with A, considered as an A-module (lemma 4.1 holds also for non-commutative rings). Hence, $\mathfrak{V}^*_1$ and $\mathfrak{V}^*_2$ are A-isomorphic, which implies that $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are A-isomorphic. It is not known whether Theorem 5.1 remains valid for non-commutative operator domains which are not algebras of finite rank with respect to a field.

8. Interlacing of modules. This section is self-contained and describes a new composition of modules, called interlacing of modules, which will be used in the proof of Theorem 9.1. In order to stress the generality of this composition, which is a generalization of the direct sum of modules, we describe it also for non-commutative operator domains.

Let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be two A-modules, that is any two additive abelian groups with the ring A as right operator domain. We make no assumptions whatsoever about chain conditions or about A; in particular, we do not assume that A is commutative or has a unit element. Let $\mathfrak{W}_1$ be a submodule of $\mathfrak{V}_1$ and $\mathfrak{W}_2$ a submodule of $\mathfrak{V}_2$, where $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are isomorphic; let J be an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. (*Module, of course, always means A-module and isomorphism always means A-isomorphism.*) We want to construct a new module $\mathfrak{V}$ which, intuitively speaking, can be thought of as having been obtained by interlacing $\mathfrak{V}_1$ and $\mathfrak{V}_2$, where this interlacing is to be carried out by identifying the elements of $\mathfrak{W}_1$ and $\mathfrak{W}_2$ which correspond under J. Algebraically, this means that we want $\mathfrak{V}$ to have the following three properties.

8.1 *There exists an isomorphism I_1 from $\mathfrak{V}_1$ onto a submodule $\mathfrak{V}'_1$ of $\mathfrak{V}$ and an isomorphism I_2 from $\mathfrak{V}_2$ onto a submodule $\mathfrak{V}'_2$ of $\mathfrak{V}$.*

8.2 $\mathfrak{V} = (\mathfrak{V}'_1, \mathfrak{V}'_2)$.

8.3 *If J_1 is the contraction of I_1 on $\mathfrak{W}_1$ and J_2 the contraction of I_2 on $\mathfrak{W}_2$, then $\mathfrak{W}_1 J_1 = \mathfrak{W}_2 J_2 = \mathfrak{V}'_1 \cap \mathfrak{V}'_2$ and $J_1 J_2^{-1} = J$.*

($J_1 J_2^{-1}$ means first J_1 and then J_2^{-1}. The term "contraction" is used in the usual sense; hence, if $W \in \mathfrak{W}_1$, then $WJ_1 = WI_1$ etc.) Observe that 8.3 says that $W_1J_1 = W_2J_2$, where $W_1 \in \mathfrak{W}_1$ and $W_2 \in \mathfrak{W}_2$, if and only if $W_1J = W_2$, i.e., if and only if W_1 and W_2 correspond under J. Hence 8.3 states indeed that the interlacing of $\mathfrak{V}_1$ and $\mathfrak{V}_2$ has been carried out by identifying the elements of $\mathfrak{W}_1$ and $\mathfrak{W}_2$ which correspond under J.

DEFINITION 8.1. *Let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be two modules with respectively $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as submodules and let J be an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. Then, a module $\mathfrak{V}$ which satisfies conditions* 8.1, 8.2 *and* 8.3 *is said to have been obtained by interlacing $\mathfrak{V}_1$ and $\mathfrak{V}_2$. We call $\mathfrak{W}_1$ and $\mathfrak{W}_2$ the "laces" and J the "lacing isomorphism" and say that $\mathfrak{V}_1$ and $\mathfrak{V}_2$ were interlaced by lacing the laces $\mathfrak{W}_1$ and $\mathfrak{W}_2$ together according to the lacing isomorphism J.*

We now prove the following theorem.

THEOREM 8.1. *Let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be two modules with respectively $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as submodules and let J be an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. Then there always exists a module $\mathfrak{V}$ which is obtained by interlacing $\mathfrak{V}_1$ and $\mathfrak{V}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism. This $\mathfrak{V}$ is unique except for isomorphisms. Conversely, if $\mathfrak{V}$ is a module and $\mathfrak{V}'_1$ and $\mathfrak{V}'_2$ are two submodules of $\mathfrak{V}$ where $(\mathfrak{V}'_1, \mathfrak{V}'_2) = \mathfrak{V}$, then $\mathfrak{V}$ can always be obtained by interlacing any two modules $\mathfrak{V}_1$ and $\mathfrak{V}_2$ which are isomorphic with respectively $\mathfrak{V}'_1$ and $\mathfrak{V}'_2$.*

Proof. We first prove the uniqueness of $\mathfrak{V}$. Hereto, let $\mathfrak{V}_1$, $\mathfrak{W}_1$, $\mathfrak{V}_2$, $\mathfrak{W}_2$, J and $\mathfrak{V}$ have the same meaning as in Theorem 8.1 and let I_1, $\mathfrak{V}'_1$, I_2, $\mathfrak{V}'_2$, J_1 and J_2 have the same meaning as in conditions 8.1, 8.2 and 8.3. Then, since $\mathfrak{V} = (\mathfrak{V}'_1, \mathfrak{V}'_2)$, every element V of $\mathfrak{V}$ can be written as $V = V_1I_1 + V_2I_2$, where $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$. This representation of V is not unique, as the following statement indicates.

STATEMENT 8.1. *The equality $V_1I_1 + V_2I_2 = V'_1I_1 + V'_2I_2$, where $V_1, V'_1 \in \mathfrak{V}_1$ and $V_2, V'_2 \in \mathfrak{V}_2$, is equivalent to the following two conditions:*

8.4 $$V_1 - V'_1 \in \mathfrak{W}_1 \text{ and } V'_2 - V_2 \in \mathfrak{W}_2;$$

8.5 $$(V_1 - V'_1)J = V'_2 - V_2.$$

We prove statement 8.1 by observing that $V_1I_1 + V_2I_2 = V'_1I_1 + V'_2I_2$ is the same as $(V_1 - V'_1)I_1 = (V'_2 - V_2)I_2$. Since I_1 maps $\mathfrak{V}_1$ onto $\mathfrak{V}'_1$ and I_2 maps $\mathfrak{V}_2$ onto $\mathfrak{V}'_2$, we conclude that $(V_1 - V'_1)I_1 \in \mathfrak{V}'_1 \cap \mathfrak{V}'_2$ and $(V'_2 - V_2)I_2 \in \mathfrak{V}'_1 \cap \mathfrak{V}'_2$. Consequently, it follows from 8.3, that $V_1 - V'_1 \in \mathfrak{W}_1$ and $V'_2 - V_2 \in \mathfrak{W}_2$ which proves 8.4. We can then conclude that $(V_1 - V'_1)I_1 = (V_1 - V'_1)J_1$ and that $(V'_2 - V_2)I_2 = (V'_2 - V_2)J_2$, which implies that $(V_1 - V'_1)J_1 = (V'_2 - V_2)J_2$ and hence that $(V_1 - V'_1)J_1J_2^{-1} = V'_2 - V_2$. It then follows from condition 8.3 that $(V_1 - V'_1)J = V'_2 - V_2$ which proves 8.5. Conversely, suppose that 8.4 and 8.5 are satisfied. We can then write for 8.5, $(V_1 - V'_1)\cdot J_1J_2^{-1} = V'_2 - V_2$, hence $(V_1 - V'_1)J_1 = (V'_2 - V_2)J_2$, hence $(V_1 - V'_1)I_1 = (V'_2 - V_2)I_2$, which shows that $V_1I_1 + V_2I_2 = V'_1I_1 + V'_2I_2$; consequently, statement 8.1 is proved.

The uniqueness of $\mathfrak{V}$ now follows easily. Namely, suppose that $\mathfrak{V}^*$ is also obtained by interlacing $\mathfrak{V}_1$ and $\mathfrak{V}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism. Let I^*_1 and I^*_2 have the same meaning for $\mathfrak{V}^*$ as I_1 and I_2 have for $\mathfrak{V}$. We associate to the element $V_1I_1 + V_2I_2$ of $\mathfrak{V}$, where $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$, the element $V_1I^*_1 + V_2I^*_2$ of $\mathfrak{V}^*$ and we claim that this correspondence is an isomorphism from $\mathfrak{V}$ onto $\mathfrak{V}^*$. In the first place, it is an immediate consequence of statement 8.1 that $V_1I_1 + V_2I_2 = V'_1I_1 + V'_2I_2$, where $V_1, V'_1 \in \mathfrak{V}_1$ and $V_2, V'_2 \in \mathfrak{V}_2$, if and only if $V_1I^*_1 + V_2I^*_2 = V'_1I^*_1 + V'_2I^*_2$; consequently, the correspondence is well defined and one-one. The fact that this correspondence is an isomorphism (A-isomorphism (!)) then follows trivially and hence the uniqueness of $\mathfrak{V}$ is proved.

We now prove the existence of $\mathfrak{V}$. Hereto, let $\mathfrak{V}_1$, $\mathfrak{W}_1$, $\mathfrak{V}_2$, $\mathfrak{W}_2$ and J have the same meaning as in Theorem 8.1. We consider the pairs (V_1, V_2), where $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$, and define the following equivalence relation for them.

DEFINITION 8.2. *Two pairs* (V_1, V_2) *and* (V'_1, V'_2), *where* $V_1, V'_1 \in \mathfrak{V}_1$ *and* $V_2, V'_2 \in \mathfrak{V}_2$, *are equivalent if conditions* 8.4 *and* 8.5 *are satisfied:*

8.4 $$V_1 - V'_1 \in \mathfrak{W}_1 \text{ and } V'_2 - V_2 \in \mathfrak{W}_2.$$

8.5 $$(V_1 - V'_1)J = V'_2 - V_2.$$

(This definition of equivalence is indeed the natural one, as follows from statement 8.1 and the fact that the pair (V_1, V_2) corresponds to the element $V_1I_1 + V_2I_2$ of the module $\mathfrak{V}$ which we are trying to construct.) It can be proved without the slightest difficulty that the equivalence relation of definition 8.2 satisfies the conditions of reflexivity, symmetry and transitivity. The class of pairs which is equivalent to (V_1, V_2), where $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$, is denoted by $[V_1, V_2]$; hence $[V_1, V_2] = [V'_1, V'_2]$ if and only if conditions 8.4 and 8.5 are satisfied. *The elements of* $\mathfrak{V}$ *are defined as these classes* $[V_1, V_2]$ *of equivalent pairs. Addition of the elements* $[V_1, V_2]$ *and* $[V'_1, V'_2]$ of $\mathfrak{V}$ *and multiplication by elements* $\alpha \in A$ *are defined by:*

8.6 $$[V_1, V_2] + [V'_1, V'_2] = [V_1 + V'_1, V_2 + V'_2].$$

8.7 $$[V_1, V_2]\,\alpha = [V_1\alpha, V_2\alpha].$$

The proof that definitions 8.6 and 8.7 do not depend on the representatives which are used for the classes $[V_1, V_2]$ and $[V'_1, V'_2]$ and that $\mathfrak{V}$ is thus made into an A-module gives no difficulty whatsoever. We proceed to show that $\mathfrak{V}$ satisfies conditions 8.1, 8.2 and 8.3. Condition 8.1 is proved by observing that the mapping $V_1I_1 = [V_1, 0_2]$, where $V_1 \in \mathfrak{V}_1$ and 0_2 is the zero element of $\mathfrak{V}_2$, is clearly an isomorphism from $\mathfrak{V}_1$ onto a submodule $\mathfrak{V}'_1$ of $\mathfrak{V}$; in the same way $V_2I_2 = [0_1, V_2]$, where $V_2 \in \mathfrak{V}_2$ and 0_1 is the zero element of $\mathfrak{V}_1$, is an isomorphism from $\mathfrak{V}_2$ onto a submodule $\mathfrak{V}'_2$ of $\mathfrak{V}$. Condition 8.2 then follows from the fact that the arbitrary element $[V_1, V_2]$ of $\mathfrak{V}$ is the sum of V_1I_1 and V_2I_2:

$$[V_1, V_2] = [V_1, 0_2] + [0_1, V_2] = V_1I_1 + V_2I_2.$$

In order to prove 8.3, we observe that $\mathfrak{W}_1J_1$ consists of the elements $[W_1, 0_2]$, where $W_1 \in \mathfrak{W}_1$, and $\mathfrak{W}_2J_2$ of the elements $[0_1, W_2]$ where $W_2 \in \mathfrak{W}_2$. Since $[W_1, 0_2] = [0_1, W_1J]$ we conclude that $\mathfrak{W}_1J_1 \subseteq \mathfrak{W}_2J_2$, while $[0_1, W_2] = [W_2J^{-1}, 0_2]$ shows that $\mathfrak{W}_2J_2 \subseteq \mathfrak{W}_1J_1$; hence $\mathfrak{W}_1J_1 = \mathfrak{W}_2J_2$. An element V of $\mathfrak{V}$ lies in

$\mathfrak{V}'_1 \cap \mathfrak{V}'_2$ if and only if it can be written as $V = [V_1, 0_2] = [0_1, V_2]$, where $V_1 \epsilon \mathfrak{V}_1$ and $V_2 \epsilon \mathfrak{V}_2$. This implies that $V_1 \epsilon \mathfrak{W}_1$ and hence that $V \epsilon \mathfrak{W}_1 J_1$; consequently, $\mathfrak{V}'_1 \cap \mathfrak{V}'_2 \subseteq \mathfrak{W}_1 J_1$. Conversely, if $V \epsilon \mathfrak{W}_1 J_1$, then $V = [W_1, 0_2]$ where $W_1 \epsilon \mathfrak{W}_1$. This already proves that $V \epsilon \mathfrak{V}'_1$ and hence that $\mathfrak{W}_1 J_1 \subseteq \mathfrak{V}'_1$. Since, furthermore, $[W_1, 0_2] = [0_1, W_1 J] \epsilon \mathfrak{V}'_2$, we conclude that $\mathfrak{W}_1 J_1 = \mathfrak{V}'_1 \cap \mathfrak{V}'_2$ and hence that $\mathfrak{W}_1 J_1 = \mathfrak{W}_2 J_2 = \mathfrak{V}'_1 \cap \mathfrak{V}'_2$. Finally, if $W_1 \epsilon \mathfrak{W}_1$, then $W_1 J_1 J_2^{-1} = [W_1, 0_2]\, J_2^{-1} = [0_1, W_1 J]\, J_2^{-1} = W_1 J$ which proves that $J_1 J_2^{-1} = J$; hence 8.3 is fully proved. In order to prove the last part of Theorem 8.1, let $\mathfrak{V}$ be a module which has two submodules $\mathfrak{V}'_1$ and $\mathfrak{V}'_2$ such that $\mathfrak{V} = (\mathfrak{V}'_1, \mathfrak{V}'_2)$. Let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be two modules which are isomorphic with respectively $\mathfrak{V}'_1$ and $\mathfrak{V}'_2$. Let I_1 be an isomorphism from $\mathfrak{V}_1$ onto $\mathfrak{V}'_1$ and I_2 an isomorphism from $\mathfrak{V}_2$ onto $\mathfrak{V}'_2$. Let $\mathfrak{W}_1$ be the submodule $(\mathfrak{V}'_1 \cap \mathfrak{V}'_2) I_1^{-1}$ of $\mathfrak{V}_1$ and let $\mathfrak{W}_2$ be the submodule $(\mathfrak{V}'_1 \cap \mathfrak{V}'_2) I_2^{-1}$ of $\mathfrak{V}_2$. Let J_1 be the contraction of I_1 on $\mathfrak{W}_1$ and J_2 the contraction of I_2 on $\mathfrak{W}_2$. Since J_1 and J_2 map respectively $\mathfrak{W}_1$ and $\mathfrak{W}_2$ onto $\mathfrak{V}'_1 \cap \mathfrak{V}'_2$, the isomorphism $J = J_1 J_2^{-1}$ is a well-defined isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. We claim that if we interlace $\mathfrak{V}_1$ and $\mathfrak{V}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism, we obtain a module $\mathfrak{V}^*$ which is isomorphic with $\mathfrak{V}$. We can consider the elements of $\mathfrak{V}^*$, as above, as the classes $[V_1, V_2]$ of equivalent pairs, where $V_1 \epsilon \mathfrak{V}_1$ and $V_2 \epsilon \mathfrak{V}_2$. If then we associate to the element $[V_1, V_2]$ of $\mathfrak{V}^*$ the element $V_1 I_1 + V_2 I_2$ of $\mathfrak{V}$, we clearly obtain an isomorphism (A-isomorphism!) from $\mathfrak{V}^*$ onto $\mathfrak{V}$. This completes the proof of Theorem 8.1.

REMARK 8.1. If we interlace two modules $\mathfrak{V}_1$ and $\mathfrak{V}_2$, using the zero modules of $\mathfrak{V}_1$ and $\mathfrak{V}_2$ as laces, we obtain the usual direct sum of $\mathfrak{V}_1$ and $\mathfrak{V}_2$. Hence the interlacing of modules can be considered as a generalization of the direct sum of modules.

The following statement and lemma will be used in the next section.

STATEMENT 8.2. *Let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be two A-modules with respectively $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as submodules. Let $\mathfrak{W}_1$ and $\mathfrak{W}_2$ be A-isomorphic and let J be an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. Let $\mathfrak{V}$ be obtained by interlacing $\mathfrak{V}_1$ and $\mathfrak{V}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism. Then, if I_1, I_2, $\mathfrak{V}'_1$ and $\mathfrak{V}'_2$ have the same meaning as in conditions* 8.1, 8.2 *and* 8.3 *and if V_1 and V_2 denote elements of respectively $\mathfrak{V}_1$ and $\mathfrak{V}_2$:*

8.8 $\quad V_1 I_1 + V_2 I_2 \epsilon \mathfrak{V}'_1$ *if and only if* $V_2 \epsilon \mathfrak{W}_2$;
$\quad V_1 I_1 + V_2 I_2 \epsilon \mathfrak{V}'_2$ *if and only if* $V_1 \epsilon \mathfrak{W}_1$.

8.9 $\quad V_1 I_1 + V_2 I_2 \epsilon \mathfrak{V}'_1 \cap \mathfrak{V}'_2$ *if and only if*
$\quad V_1 \epsilon \mathfrak{W}_1$ *and* $V_2 \epsilon \mathfrak{W}_2$.

Proof. If $V_1 I_1 + V_2 I_2 \epsilon \mathfrak{V}'_1$, then $V_1 I_1 + V_2 I_2 = V'_1 I_1 + 0_2 I_2$, where $V'_1 \epsilon \mathfrak{V}_1$ and 0_2 is the zero element of $\mathfrak{V}_2$. Statement 8.1 then implies that $V_2 \epsilon \mathfrak{W}_2$. Conversely, if $V_2 \epsilon \mathfrak{W}_2$, that same statement implies that $V_1 I_1 + V_2 I_2 = (V_1 + V_2 J^{-1}) I_1 + 0_2 I_2 \epsilon \mathfrak{V}'_1$ and hence the first part of 8.8 is proved. The second part of 8.8 is proved in the same way while 8.9 is an immediate consequence of 8.8.

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Lemma 8.1. *Let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be two A-modules with respectively $\mathfrak{a}_1$ and $\mathfrak{a}_2$ as annihilators. Let $\mathfrak{W}_1$ be a submodule of $\mathfrak{V}_1$ and $\mathfrak{W}_2$ a submodule of $\mathfrak{V}_2$, where $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are A-isomorphic. Let J be an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. The module $\mathfrak{V}$, which is obtained by interlacing $\mathfrak{V}_1$ and $\mathfrak{V}_2$ using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism, has $\mathfrak{a}_1 \cap \mathfrak{a}_2$ as annihilating ideal. If furthermore $\mathfrak{V}_1$ and $\mathfrak{V}_2$ have respective finite lengths l_1 and l_2 and hence $\mathfrak{W}_1$ and $\mathfrak{W}_2$ have the common finite length, say λ, then $\mathfrak{V}$ has finite length $l = l_1 + l_2 - \lambda$.*

Proof. (A module is said to have finite length if it has a composition series of finite length.) We know that $\mathfrak{V} = (\mathfrak{V}'_1, \mathfrak{V}'_2)$ and hence that the annihilating ideal $\mathfrak{a}$ of $\mathfrak{V}$ is the intersection of the annihilating ideal $\mathfrak{a}'_1$ of $\mathfrak{V}'_1$ and $\mathfrak{a}'_2$ of $\mathfrak{V}'_2$. The fact that $\mathfrak{V}'_1$ is isomorphic with $\mathfrak{V}_1$ and $\mathfrak{V}'_2$ with $\mathfrak{V}_2$ implies that $\mathfrak{a}'_1 = \mathfrak{a}_1$ and $\mathfrak{a}'_2 = \mathfrak{a}_2$ and hence that $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$. Now suppose that $\mathfrak{V}_1$ and $\mathfrak{V}_2$ have respective finite lengths l_1 and l_2 and that the common length of $\mathfrak{W}_1$ and $\mathfrak{W}_2$ is λ. Then $\mathfrak{V}'_1$ and $\mathfrak{V}'_2$ have respective finite lengths l_1 and l_2 and, since property 8.3 implies that $\mathfrak{V}'_1 \cap \mathfrak{V}'_2$ is isomorphic with $\mathfrak{W}_1$ and $\mathfrak{W}_2$, the module $\mathfrak{V}'_1 \cap \mathfrak{V}'_2$ has finite length λ. The difference module $\mathfrak{V} - \mathfrak{V}'_1$ satisfies the relation $\mathfrak{V} - \mathfrak{V}'_1 = (\mathfrak{V}'_1, \mathfrak{V}'_2) - \mathfrak{V}'_1 = \mathfrak{V}'_2 - \mathfrak{V}'_1 \cap \mathfrak{V}'_2$, which implies that $\mathfrak{V}$ has finite length l, where $l - l_1 = l_2 - \lambda$ and hence $l = l_1 + l_2 - \lambda$. This completes the proof of lemma 8.1.

Remark 8.2. It is interesting to note that lemma 8.1 implies that the annihilating ideal of the module $\mathfrak{V}$, obtained by interlacing $\mathfrak{V}_1$ and $\mathfrak{V}_2$, does not depend on the laces and lacing isomorphism used; it depends only on the annihilating ideals of $\mathfrak{V}_1$ and $\mathfrak{V}_2$. In particular, lacing always produces a faithful representation from two faithful representations. The length of $\mathfrak{V}$ depends on the length of the laces, but not on the lacing isomorphism.

In the next section we investigate only those properties of interlacing which are needed for the proof of Theorem 9.1. The author believes, however, that a systematic investigation of interlacing would be worth while.

9. The faithful, completely indecomposable representation space of a completely primary ring. We now return to the study of modules with commutative operator domains. All notations and conventions will be the same as in the first seven sections of this paper. Hence the term module will be used only for A-module where A is always a commutative ring with unit element and where condition I-1 of the introduction is assumed to hold. Isomorphism always means A-isomorphism and, as before, the radical of the annihilating ideal of a module is called the radical of the module. The following corollary is an immediate consequence of lemma 8.1 and the fact that the radical of the intersection of two ideals is the intersection of the radicals of these ideals.

Corollary 9.1. *Let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be two A-modules with respectively $\mathfrak{r}_1$ and $\mathfrak{r}_2$ as radicals. Then, if $\mathfrak{V}$ is obtained by interlacing $\mathfrak{V}_1$ and $\mathfrak{V}_2$, no matter what laces and lacing isomorphism are used, the radical of $\mathfrak{V}$ is $\mathfrak{r}_1 \cap \mathfrak{r}_2$.*

Since the proof of Theorem 9.1 depends on interlacing by means of lacing isomorphisms which are not "extendable," we discuss the notion of extendable isomorphisms in the following definition and lemma. We remind the reader of the convention that the symbol $\subset$ is used exclusively for *proper* inclusion.

DEFINITION 9.1. *Let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be two A-modules with respectively $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as submodules. Let $\mathfrak{W}_1$ and $\mathfrak{W}_2$ be A-isomorphic and J an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. Then J is called extendable if the following two conditions are satisfied.*

9.1 *There exist submodules $\mathfrak{W}'_1$ and $\mathfrak{W}'_2$ of respectively $\mathfrak{V}_1$ and $\mathfrak{V}_2$, where $\mathfrak{W}_1 \subset \mathfrak{W}'_1$ and $\mathfrak{W}_2 \subset \mathfrak{W}'_2$,*

9.2 *J can be extended to an isomorphism J′ from $\mathfrak{W}'_1$ onto $\mathfrak{W}'_2$.*

It is clear that, if $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are not-isomorphic A-modules with $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as respective *maximal* submodules, then any isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$ is not extendable.

LEMMA 9.1. *Let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be two A-modules with respectively $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as submodules. Let $\mathfrak{W}_1$ and $\mathfrak{W}_2$ be A-isomorphic and J an isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. Then J is extendable if and only if there exist elements $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$, where:*

9.3 V_1 not $\in \mathfrak{W}_1$ *and* V_2 not $\in \mathfrak{W}_2$.

9.4 $\mathfrak{W}_1 : V_1 = \mathfrak{W}_2 : V_2$; *we denote this ideal by* $\mathfrak{c}$.

9.5 $(V_1\gamma)J = V_2\gamma$ *for all* $\gamma \in \mathfrak{c}$.

Proof. (The quotient of modules is defined as before. The quotient $\mathfrak{W}_1 : V_1$ of a module $\mathfrak{W}_1$ by an element V_1 is defined as the quotient of $\mathfrak{W}_1$ by the cyclic module generated by V_1. Consequently, $V_1\gamma \in \mathfrak{W}_1$ and $V_2\gamma \in \mathfrak{W}_2$ for all $\gamma \in \mathfrak{c}$, and hence condition 9.5 makes sense.) Let $\mathfrak{V}_1$, $\mathfrak{W}_1$, $\mathfrak{V}_2$, $\mathfrak{W}_2$ and J have the same meaning as in lemma 9.1 where J is extendable. Let $\mathfrak{W}'_1$, $\mathfrak{W}'_2$ and J' have the same meaning as in definition 9.1. Then, because of condition 9.1, we can choose an element $V_1 \in \mathfrak{W}'_1$ where V_1 not $\in \mathfrak{W}_1$. We denote V_1J' by V_2 and claim that V_1 and V_2 satisfy conditions 9.3, 9.4 and 9.5. In the first place J' is an extension of J and $\mathfrak{W}_1J = \mathfrak{W}_2$; hence V_1 not $\in \mathfrak{W}_1$ implies V_2 not $\in \mathfrak{W}_2$ and 9.3 is satisfied. Secondly the module $(\mathfrak{W}_1, V_1)$, generated by $\mathfrak{W}_1$ and V_1, is mapped isomorphically onto $(\mathfrak{W}_2, V_2)$ by J' while $\mathfrak{W}_1J' = \mathfrak{W}_1J = \mathfrak{W}_2$. This implies, since we are dealing with operator isomorphisms, that $\mathfrak{W}_1 : (\mathfrak{W}_1, V_1) = \mathfrak{W}_2 : (\mathfrak{W}_2, V_2)$. This, however, is exactly condition 9.4 since $\mathfrak{W}_1 : (\mathfrak{W}_1, V_1) = \mathfrak{W}_1 : \mathfrak{W}_1 \cap \mathfrak{W}_1 : V_1 = \mathfrak{V}_1 \cap \mathfrak{W}_1 : V_1 = \mathfrak{W}_1 : V_1$ and in the same way $\mathfrak{W}_2 : (\mathfrak{W}_2, V_2) = \mathfrak{W}_2 : V_2$. Finally, since J' is an operator isomorphism, $(V_1\gamma)J' = (V_1J')\gamma = V_2\gamma$. If furthermore $\gamma \in \mathfrak{c} = \mathfrak{W}_1 : V_1$, then $V_1\gamma \in \mathfrak{W}_1$ and hence $(V_1\gamma)J' = (V_1\gamma)J$, from which we conclude that $(V_1\gamma)J = V_2\gamma$, which proves 9.5. Conversely, let $\mathfrak{V}_1$, $\mathfrak{W}_1$, $\mathfrak{V}_2$, $\mathfrak{W}_2$ again have the same meaning as above where now J is any A-isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$. We assume that there exist elements $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$ which satisfy conditions 9.3, 9.4 and 9.5 and prove that then J is extendable. We choose $\mathfrak{W}'_1 = (\mathfrak{W}_1, V_1)$ and $\mathfrak{W}'_2 = (\mathfrak{W}_2, V_2)$ and con-

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sequently, as follows from 9.3, condition 9.1 is satisfied. Furthermore, J can be extended to an isomorphism J' from $\mathfrak{W}'_1$ onto $\mathfrak{W}'_2$ by defining:

$$(W_1 + V_1\alpha)J' = W_1J + V_2\alpha, \text{ for } W_1 \in \mathfrak{W}_1 \text{ and } \alpha \in A.$$

In order to show that this mapping J' is well defined, let $W_1 + V_1\alpha = W'_1 + V_1\alpha'$, where W_1, $W'_1 \in \mathfrak{W}_1$ and α, $\alpha' \in A$. Then $W_1 - W'_1 = V_1(\alpha' - \alpha)$ and consequently $\alpha' - \alpha \in \mathfrak{W}_1 : V_1 = \mathfrak{c}$. Condition 9.4 then gives that $V_2(\alpha' - \alpha) \in \mathfrak{W}_2$ and 9.5 that $(V_1(\alpha' - \alpha))J = V_2(\alpha' - \alpha)$. This proves that $W_1J - W'_1J = (W_1 - W'_1)J = V_2\alpha' - V_2\alpha$ and hence that $W_1J + V_2\alpha = W'_1J + V_2\alpha'$; consequently, J' is well defined. It is trivial to show that J' is a homomorphism from $\mathfrak{W}'_1$ onto $\mathfrak{W}'_2$. We now prove that the zero element 0_1 of $\mathfrak{W}'_1$ is the only element of $\mathfrak{W}'_1$ which is mapped by J' on the zero element 0_2 of $\mathfrak{W}'_2$. Hereto, let $(W_1 + V_1\alpha)J' = 0_2$, where $W_1 \in \mathfrak{W}_1$ and $\alpha \in A$. Then $W_1J + V_2\alpha = 0_2$ and hence $W_1J = -V_2\alpha$ which shows that $\alpha \in \mathfrak{W}_2 : V_2 = \mathfrak{c}$. It then follows from conditions 9.4 and 9.5 that $V_1\alpha \in \mathfrak{W}_1$ and $(V_1\alpha)J = V_2\alpha$. We conclude that $W_1J + (V_1\alpha)J = (W_1 + V_1\alpha)J = 0_2$ and hence, since J is an isomorphism, that $W_1 + V_1\alpha = 0_1$. This completes the proof of lemma 9.1.

The usefulness of the notion of extendable isomorphism for interlacing is apparent from the following lemma and its corollaries.

LEMMA 9.2. *Let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be two A-modules with the same radical $\mathfrak{p}$, where $\mathfrak{p}$ is a maximal prime ideal of A. Let 0_1 and 0_2 be the zero elements of $\mathfrak{V}_1$ and $\mathfrak{V}_2$ and let $\mathfrak{W}_1$ be a submodule of $\mathfrak{V}_1$ and $\mathfrak{W}_2$ a submodule of $\mathfrak{V}_2$, where $0_1 : \mathfrak{p} \subseteq \mathfrak{W}_1$ and $0_2 : \mathfrak{p} \subseteq \mathfrak{W}_2$. Let J be a not-extendable isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$ and let $\mathfrak{V}$ be obtained by interlacing $\mathfrak{V}_1$ and $\mathfrak{V}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism. Then, if 0 is the zero element of $\mathfrak{V}$ and if $\mathfrak{V}'_1$ and $\mathfrak{V}'_2$ have the customary meaning of definitions* 8.1, 8.2 *and* 8.3:

9.6 *The radical of $\mathfrak{V}$ is $\mathfrak{p}$.*

9.7 $0 : \mathfrak{p} \subseteq \mathfrak{V}'_1 \cap \mathfrak{V}'_2$.

9.8 *The three modules $0 : \mathfrak{p}$, $0_1 : \mathfrak{p}$ and $0_2 : \mathfrak{p}$ are isomorphic.*

Proof. It is *always* true, according to corollary 9.1, that, if two modules with the same radical $\mathfrak{p}$ are interlaced, the resulting module has as radical $\mathfrak{p} \cap \mathfrak{p} = \mathfrak{p}$; hence 9.6 is proved. Furthermore, since J is an operator isomorphism and since $0_1 : \mathfrak{p} \subseteq \mathfrak{W}_1$ and $0_2 : \mathfrak{p} \subseteq \mathfrak{W}_2$, the module $0_1 : \mathfrak{p}$ is mapped by J onto $0_2 : \mathfrak{p}$ and hence $0_1 : \mathfrak{p}$ is isomorphic with $0_2 : \mathfrak{p}$. In order to prove 9.7 and to prove that $0_1 : \mathfrak{p}$ is isomorphic with $0 : \mathfrak{p}$, let I_1, I_2, J_1 and J_2 have the meaning of definitions 8.1, 8.2 and 8.3. Hence $\mathfrak{V}_1I_1 = \mathfrak{V}'_1$ and $\mathfrak{V}_2I_2 = \mathfrak{V}'_2$ and the elements of $\mathfrak{V}$ can be written as $V_1I_1 + V_2I_2$, where $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$. We now prove the following statement.

STATEMENT 9.1. *The element $V_1I_1 + V_2I_2$ of $\mathfrak{V}$, where $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$, belongs to $0 : \mathfrak{p}$ if and only if the following two conditions are satisfied:*

9.9 $V_1 \in \mathfrak{W}_1$ *and* $V_2 \in \mathfrak{W}_2$.

9.10 $V_1J + V_2 \in 0_2 : \mathfrak{p}$.

Suppose that conditions 9.9 and 9.10 are satisfied. Then, for any $\pi \in \mathfrak{p}$, certainly $V_1\pi \in \mathfrak{W}_1$ and $-V_2\pi \in \mathfrak{W}_2$ and $(V_1\pi)J = -V_2\pi$. This is, according to statement 8.1, equivalent with $(V_1I_1+ V_2I_2)\pi = 0_1I_1+ 0_2I_2= 0$ and hence $V_1I_1+ V_2I_2 \in 0 : \mathfrak{p}$. Conversely, let $V_1I_1+ V_2I_2 \in 0 : \mathfrak{p}$; i.e., for any $\pi \in \mathfrak{p}$, $(V_1\pi)I_1+(V_2\pi)I_2= 0_1I_1+ 0_2I_2$. Again, this is equivalent with $V_1\pi \in \mathfrak{W}_1$ and $-V_2\pi \in \mathfrak{W}_2$ and $(V_1\pi)J = -V_2\pi$ for all $\pi \in \mathfrak{p}$. Hence we have the following statement.

STATEMENT 9.2. *If* $V_1I_1+ V_2I_2 \in 0 : \mathfrak{p}$, *then*

9.11 $$V_1 \in \mathfrak{W}_1 : \mathfrak{p} \textit{ and } V_2 \in \mathfrak{W}_2 : \mathfrak{p};$$

9.12 $$(V_1\pi)J = -V_2\pi \textit{ for all } \pi \in \mathfrak{p}.$$

We first conclude that either condition 9.9 is satisfied or *simultaneously* V_1 not $\in$ $\mathfrak{W}_1$ and V_2 not $\in \mathfrak{W}_2$. Hereto, let $V_1 \in \mathfrak{W}_1$ which implies that V_1J is defined and hence that $(V_1\pi)J = (V_1J)\pi$. Condition 9.12 then states that $(V_1J + V_2)\pi = 0_2$ for all $\pi \in \mathfrak{p}$, i.e., that $V_1J + V_2 \in 0_2 : \mathfrak{p}$. Consequently, since $0_2 : \mathfrak{p} \subseteq \mathfrak{W}_2$ and $V_1J \in \mathfrak{W}_2$, we see that $V_2 \in \mathfrak{W}_2$. We show, in the same way, that if $V_2 \in \mathfrak{W}_2$ then $V_1 \in \mathfrak{W}_1$ and hence the conclusion is proved. Secondly, we show that V_1 not $\in \mathfrak{W}_1$ and V_2 not $\in \mathfrak{W}_2$ contradicts the hypothesis that J is not extendable. If V_1 not $\in \mathfrak{W}_1$, then $\mathfrak{W}_1 : V_1 \neq A$, while 9.11 implies that $\mathfrak{p} \subseteq \mathfrak{W}_1 : V_1$; hence, since $\mathfrak{p}$ is maximal, $\mathfrak{p} = \mathfrak{W}_1 : V_1$. In the same way, V_2 not $\in \mathfrak{W}_2$ implies that $\mathfrak{p} = \mathfrak{W}_2 : V_2$. Consequently, if V_1 not $\in \mathfrak{W}_1$ and V_2 not $\in \mathfrak{W}_2$, conditions 9.3, 9.4 and 9.5 are satisfied for $\mathfrak{W}_1$, $\mathfrak{W}_2$, V_1, $-V_2$ and J; and J is extendable. This proves that 9.9 is satisfied while, as was pointed out above, 9.12 becomes 9.10 when 9.9 is satisfied. Hence statement 9.1 is completely proved. It then immediately follows from conditions 9.9 and 8.9 that $0 : \mathfrak{p} \subseteq \mathfrak{V}'_1 \cap \mathfrak{V}'_2$. Hence 9.7 is proved and all there remains to be shown is that $0_1 : \mathfrak{p}$ is isomorphic with $0 : \mathfrak{p}$. We prove this fact by showing that I_1 maps $0_1 : \mathfrak{p}$ onto $0 : \mathfrak{p}$; i.e., that $(0_1 : \mathfrak{p})I_1= 0 : \mathfrak{p}$. If $V \in 0 : \mathfrak{p}$, then $V = V_1I_1+ V_2I_2$ where 9.9 and 9.10 are satisfied. Then V_2J^{-1} is defined and hence $V_1I_1+ V_2I_2=(V_1+ V_2J^{-1})I_1+ 0_2I_2$. Condition 9.10 states that $V_1+ V_2J^{-1} \in 0_1 : \mathfrak{p}$ and hence $V \in (0_1 : \mathfrak{p})I_1$ which proves that $0 : \mathfrak{p} \subseteq (0_1 : \mathfrak{p})I_1$. Conversely, if $V = V_1I_1+ V_2I_2 \in (0_1 : \mathfrak{p})I_1$, where $V_1 \in \mathfrak{V}_1$ and $V_2 \in \mathfrak{V}_2$, then $V_1I_1+ V_2I_2= V'_1I_1+ 0_2I_2$, where $V'_1 \in 0_1 : \mathfrak{p}$. This implies that $V_1- V'_1 \in \mathfrak{W}_1$ and $-V_2 \in \mathfrak{W}_2$ and $(V_1- V'_1)J = -V_2$. Since $V'_1 \in 0_1 : \mathfrak{p} \subseteq \mathfrak{W}_1$, we see that $V_1 \in \mathfrak{W}_1$ and 9.9 is proved. Furthermore, $(V_1- V'_1)J = -V_2$ implies that $V_1J + V_2= V'_1J$, while $V'_1 \in 0_1 : \mathfrak{p}$ implies that $V'_1J \in 0_2 : \mathfrak{p}$. Hence 9.10 is satisfied, which shows that $(0_1 : \mathfrak{p})I_1 \subseteq 0 : \mathfrak{p}$ and hence that $(0_1 : \mathfrak{p})I_1= 0 : \mathfrak{p}$. This completes the proof of lemma 9.2.

If $\mathfrak{V}$ is an A-module of finite length with radical $\mathfrak{p}$ and zero element 0, the length of $0 : \mathfrak{p}$ is usually referred to as the first Loewy invariant of $\mathfrak{V}$. The importance of lemma 9.2 lies in the fact that it states a condition for the invariance of this Loewy invariant under interlacing. The following immediate corollary of lemma 9.2 formulates this invariance explicitly.

COROLLARY 9.2. *Let* $\mathfrak{V}_1$ *and* $\mathfrak{V}_2$ *be two A-modules of finite length. Let* $\mathfrak{V}$ *be obtained by interlacing* $\mathfrak{V}_1$ *and* $\mathfrak{V}_2$ *where the radicals of* $\mathfrak{V}_1$ *and* $\mathfrak{V}_2$, *the laces,*

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and the lacing isomorphism satisfy the conditions of lemma 9.2. *Then,* $\mathfrak{B}_1$, $\mathfrak{B}_2$ *and* $\mathfrak{B}$ *have the same first Loewy invariant and the same radical.*

We know that a module $\mathfrak{B}$ of finite length is completely indecomposable if and only if its first Loewy invariant is equal to one. Furthermore, the radical $\mathfrak{p}$ of a completely indecomposable module $\mathfrak{B}$ is always a maximal ideal and every non-zero submodule of $\mathfrak{B}$ always contains $0 : \mathfrak{p}$. This proves the following simple and important corollary on interlacing of completely indecomposable modules.

COROLLARY 9.3. *Let* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *be two completely indecomposable* A*-modules with the same radical* $\mathfrak{p}$. *Let* $\mathfrak{B}$ *be obtained by interlacing* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$, *using non-zero laces and a not-extendable lacing isomorphism. Then* $\mathfrak{B}$ *is completely indecomposable and has radical* $\mathfrak{p}$.

REMARK 9.1. We can see from the following that we can not omit the condition that the lacing isomorphism is not extendable from corollary 9.3 (and hence not from corollary 9.2 and lemma 9.2). Choose for the $\mathfrak{B}_1$ and $\mathfrak{B}_2$ of corollary 9.3 one and the same completely indecomposable A-module $\mathfrak{B}_1$ of length $l_1 > 1$ and with $\mathfrak{q}_1$ as annihilating ideal; hence $\mathfrak{B}_1 = \mathfrak{B}_2$ and, according to lemma 2.1, the length of $\mathfrak{q}_1$ is also l_1. Let $\mathfrak{W}_1$ be a maximal submodule of $\mathfrak{B}_1$ and let J be an A-automorphism from $\mathfrak{W}_1$ onto itself; then $\mathfrak{W}_1$ has length $l_1 - 1 > 0$ and hence is not the zero module of $\mathfrak{B}_1$ and, according to corollary 3.1, J is extendable to an automorphism from $\mathfrak{B}_1$ onto $\mathfrak{B}_1$. Let $\mathfrak{B}$ be obtained by interlacing $\mathfrak{B}_1$ with itself, using $\mathfrak{W}_1$ as both laces and J as lacing isomorphism. Then, all the conditions of corollary 9.3 are satisfied except that J is extendable, while we can see as follows that $\mathfrak{B}$ is not completely indecomposable. According to lemma 8.1, the length of $\mathfrak{B}$ is $l_1 + l_1 - (l_1 - 1) = l_1 + 1$ and the annihilating ideal of $\mathfrak{B}$ is $\mathfrak{q}_1 \cap \mathfrak{q}_1 = \mathfrak{q}_1$. Consequently, since $\mathfrak{q}_1$ has length l_1, it follows from lemma 2.1 that $\mathfrak{B}$ is not completely indecomposable.

We have now developed the theory of interlacing far enough to prove Theorem 9.1. Before doing this, however, we give one example of interlacing in order to demonstrate the power of this composition for the construction of counterexamples.

EXAMPLE 9.1. We know that the length of any completely indecomposable module and of any cyclic module is always equal to the length of its annihilator. We counterexample the converse of this statement by constructing an A-module $\mathfrak{B}$ which has the following properties.

9.13 $\mathfrak{B}$ satisfies the properties I-1, I-2 and I-3 of the introduction.

9.14 The zero element of $\mathfrak{B}$ is a primary module of $\mathfrak{B}$ and the radical of $\mathfrak{B}$ is a maximal ideal of A.

9.15 $\mathfrak{B}$ is indecomposable.

9.16 The length of $\mathfrak{B}$ is equal to the length of its annihilator.

9.17 $\mathfrak{B}$ is neither completely indecomposable nor cyclic.

We do this by choosing a Noetherian ring A with unit element (i.e., a commutative ring with unit element whose ideals satisfy the ascending chain condition) and two ideals $\mathfrak{q}_1$ and $\mathfrak{q}_2$ in A which have the following properties.

9.18 $\mathfrak{q}_1 \neq \mathfrak{q}_2$ and $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are primary ideals with the same maximal associated prime $\mathfrak{p}$.

9.19 The difference modules $\mathfrak{V}_1 = A - \mathfrak{q}_1$ and $\mathfrak{V}_2 = A - \mathfrak{q}_2$ both have finite length equal to 3.

9.20 $\mathfrak{q}_1 : \mathfrak{p} = \mathfrak{q}_2 : \mathfrak{p} = \mathfrak{p}$.

9.21 $\mathfrak{q}_1 \cap \mathfrak{q}_2 = \mathfrak{p}^2$ and the length of the difference module $A - \mathfrak{p}^2$ is equal to 4.

We then have in $\mathfrak{V}_1$ and $\mathfrak{V}_2$ two A-modules of finite length 3 with the same maximal radical $\mathfrak{p}$, while $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are the annihilating ideals of respectively $\mathfrak{V}_1$ and $\mathfrak{V}_2$. Consequently, since $\mathfrak{q}_1 \neq \mathfrak{q}_2$, $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are not isomorphic. Furthermore, if 0_1 and 0_2 denote the zero elements of respectively $\mathfrak{V}_1$ and $\mathfrak{V}_2$, property 9.20 implies that both modules $\mathfrak{W}_1 = 0_1 : \mathfrak{p}$ and $\mathfrak{W}_2 = 0_2 : \mathfrak{p}$ have length 2; hence the first Loewy invariant of both $\mathfrak{V}_1$ and $\mathfrak{V}_2$ is 2 and $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are maximal submodules of respectively $\mathfrak{V}_1$ and $\mathfrak{V}_2$. Furthermore, $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are isomorphic since each of these modules is isomorphic with the direct sum $A - \mathfrak{p} \dotplus A - \mathfrak{p}$. Let J be any fixed isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$; then, according to the sentence following definition 9.1, J is not extendable. We claim that the module $\mathfrak{V}$ which is obtained by interlacing $\mathfrak{V}_1$ and $\mathfrak{V}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism, has the required properties. In the first place, it follows from lemma 8.1, that $\mathfrak{V}$ has length $3 + 3 - 2 = 4$ and that the annihilating ideal of $\mathfrak{V}$ is $\mathfrak{q}_1 \cap \mathfrak{q}_2 = \mathfrak{p}^2$ which, according to 9.21, also has length 4; hence 9.13 and 9.16 are satisfied. Furthermore, according to corollary 9.1, the radical of $\mathfrak{V}$ is equal to $\mathfrak{p} \cap \mathfrak{p} = \mathfrak{p}$ which, since $\mathfrak{p}$ is maximal, implies that the zero module of $\mathfrak{V}$ is primary and hence 9.14 is proved. In the second place, since the conditions of corollary 9.2 are satisfied, the first Loewy invariant of $\mathfrak{V}$ is 2 and hence $\mathfrak{V}$ is not completely indecomposable. If $\mathfrak{V}$ were cyclic, $\mathfrak{V}$ would be A-isomorphic with the difference module $A - \mathfrak{p}^2$. This is not possible since $\mathfrak{V}$ contains two distinct maximal submodules $\mathfrak{V}'_1$ and $\mathfrak{V}'_2$ which are isomorphic with respectively $A - \mathfrak{q}_1$ and $A - \mathfrak{q}_2$, while $A - \mathfrak{p}^2$ has the unique maximal submodule $\mathfrak{p} - \mathfrak{p}^2$; hence 9.17 is proved. In order to prove 9.15, suppose that $\mathfrak{V} = \mathfrak{A}_1 \dotplus \mathfrak{A}_2$, where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are non-zero submodules of $\mathfrak{V}$. Then, either $\mathfrak{A}_1$ has length 1 and $\mathfrak{A}_2$ has length 3, or both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have length 2. We can easily see that in both cases, since $\mathfrak{A}_1 \cap \mathfrak{A}_2 = 0$, the modules $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have to be completely indecomposable. Hence, if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ denote the annihilating ideals of respectively $\mathfrak{A}_1$ and $\mathfrak{A}_2$, the lengths of $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are equal to those of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. This, together with the facts that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are primary ideals with $\mathfrak{p}$ as associated prime and that $\mathfrak{p}^2 = \mathfrak{a}_1 \cap \mathfrak{a}_2$, leads easily to a contradiction; hence 9.15 is proved. We can choose for A the ring $A = K[x, y, z]$ which consists of the polynomials in three variables x, y and z with coefficients in the field K; and for $\mathfrak{q}_1$ and $\mathfrak{q}_2$ we can choose respectively the primary ideals $\mathfrak{q}_1 = (x, y^2, yz, z^2)$ and $\mathfrak{q}_2 = (x^2, y, xz, z^2)$ of A. Then,

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x not ϵ $\mathfrak{q}_2$ and hence $\mathfrak{q}_1 \neq \mathfrak{q}_2$, while the common maximal associated prime of $\mathfrak{q}_1$ and $\mathfrak{q}_2$ is $\mathfrak{p} = (x, y, z)$; hence 9.18 is satisfied. We easily prove that $\mathfrak{q}_1 \cap \mathfrak{q}_2 = (x^2, y^2, z^2, xy, xz, yz) = \mathfrak{p}^2$ and that the following two sequences are both composition series from $\mathfrak{p}^2$ to A:

$$\mathfrak{p}^2 \subset \mathfrak{q}_1 \subset (x, y, z^2) \subset \mathfrak{p} \subset A,$$
$$\mathfrak{p}^2 \subset \mathfrak{q}_2 \subset (x, y, z^2) \subset \mathfrak{p} \subset A.$$

This implies both 9.21 and 9.19. Finally, since $\mathfrak{p}^2 \subset \mathfrak{q}_1$, we have $\mathfrak{p} \subseteq \mathfrak{q}_1 : \mathfrak{p}$, while $\mathfrak{q}_1 : \mathfrak{p} \subseteq \mathfrak{p}$ follows from the fact that $\mathfrak{q}_1 : \mathfrak{p}$ is primary and has $\mathfrak{p}$ as associated prime. Hence $\mathfrak{p} = \mathfrak{q}_1 : \mathfrak{p}$ and, in the same way, $\mathfrak{p} = \mathfrak{q}_2 : \mathfrak{p}$, which proves 9.20 and completes example 9.1.

The proof of Theorem 9.1 uses the following simple fact of Noetherian rings A with unit element. (If $\mathfrak{q}$ is an ideal of A, a proper divisor $\mathfrak{q}_1$ of $\mathfrak{q}$ is of course called a minimal divisor of $\mathfrak{q}$, if there exists no ideal $\mathfrak{a}$ such that $\mathfrak{q} \subset \mathfrak{a} \subset \mathfrak{q}_1$.)

LEMMA 9.3. *Let A be a Noetherian ring with unit element and let $\mathfrak{q}$ be a primary ideal of A with maximal associated prime ideal $\mathfrak{p}$. Then, either $\mathfrak{q}$ is intersection-irreducible, or $\mathfrak{q}$ has two distinct minimal divisors $\mathfrak{q}_1$ and $\mathfrak{q}_2$. In the latter case, $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are always primary ideals with $\mathfrak{p}$ as associated prime, while $\mathfrak{q} = \mathfrak{q}_1 \cap \mathfrak{q}_2$; furthermore, $(\mathfrak{q}_1, \mathfrak{q}_2)$ is then a common minimal divisor of $\mathfrak{q}_1$ and $\mathfrak{q}_2$ and is also primary with $\mathfrak{p}$ as associated prime.*

Proof. We know that, since $\mathfrak{p}$ is maximal, every divisor of $\mathfrak{q}$, except A, is primary and has $\mathfrak{p}$ as associated prime and that $\mathfrak{V} = A - \mathfrak{q}$ has finite length. If $\mathfrak{q}$ is not intersection-irreducible, $\mathfrak{V}$ is not completely indecomposable and hence $\mathfrak{V}$ has then at least two distinct minimal submodules $\mathfrak{W}_1$ and $\mathfrak{W}_2$. Consequently, $\mathfrak{W}_1 \cap \mathfrak{W}_2 = 0$ and $(\mathfrak{W}_1, \mathfrak{W}_2)$ is a common minimal divisor of $\mathfrak{W}_1$ and $\mathfrak{W}_2$. The lemma then follows immediately from considering the ideals $\mathfrak{q}_1$, $\mathfrak{q}_2$ and $(\mathfrak{q}_1, \mathfrak{q}_2)$ which are mapped onto respectively $\mathfrak{W}_1$, $\mathfrak{W}_2$ and $(\mathfrak{W}_1, \mathfrak{W}_2)$ by the natural operator-homomorphism from A onto $A - \mathfrak{q}$.

THEOREM 9.1. *Let A be a Noetherian ring with unit element. Let $\mathfrak{q}$ be a primary ideal of A with maximal associated prime ideal $\mathfrak{p}$. Then there exists a completely indecomposable A-module $\mathfrak{V}$ with $\mathfrak{q}$ as annihilator.*

Proof. If the length of $\mathfrak{q}$ is 1, then $\mathfrak{q} = \mathfrak{p}$ and $A - \mathfrak{q}$ is a completely indecomposable A-module with $\mathfrak{q}$ as annihilator. Hence we can make the induction hypothesis that Theorem 9.1 has been proved when $\mathfrak{q}$ has length $1, 2, \ldots, l-1$. (The length of $\mathfrak{q}$ is the length of $A - \mathfrak{q}$.) Suppose that $\mathfrak{q}$ has length l. If $\mathfrak{q}$ is intersection-irreducible, $A - \mathfrak{q}$ is a completely indecomposable A-module (see example 7.2) with $\mathfrak{q}$ as annihilator. If $\mathfrak{q}$ is not intersection-irreducible then, according to lemma 9.3, $\mathfrak{q} = \mathfrak{q}_1 \cap \mathfrak{q}_2$ where $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are primary ideals with $\mathfrak{p}$ as associated prime and $l - 1$ as length. The induction hypothesis then guarantees the existence of two completely indecomposable A-modules $\mathfrak{V}_1$ and $\mathfrak{V}_2$ with $\mathfrak{q}_1$ and $\mathfrak{q}_2$ as respective annihilators. The modules $\mathfrak{V}_1$ and $\mathfrak{V}_2$ have the same radical $\mathfrak{p}$ since $\mathfrak{p}$ is the common associated prime of $\mathfrak{q}_1$ and $\mathfrak{q}_2$, while $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are not isomorphic since $\mathfrak{q}_1 \neq \mathfrak{q}_2$. Furthermore, it follows from sec. **2** and the fact that $(\mathfrak{q}_1, \mathfrak{q}_2)$ is a common minimal divisor of $\mathfrak{q}_1$ and $\mathfrak{q}_2$, that $\mathfrak{V}_1$ and

$\mathfrak{B}_2$ contain non-zero maximal submodules $\mathfrak{W}_1 \subset \mathfrak{B}_1$ and $\mathfrak{W}_2 \subset \mathfrak{B}_2$ which both have $(\mathfrak{q}_1, \mathfrak{q}_2)$ as annihilator. Hence $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are completely indecomposable A-modules with the same annihilator, which implies, according to Theorem 5.1, that $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are isomorphic. Let J be any fixed isomorphism from $\mathfrak{W}_1$ onto $\mathfrak{W}_2$ and let $\mathfrak{B}$ be the A-module which is obtained by interlacing $\mathfrak{B}_1$ and $\mathfrak{B}_2$, using $\mathfrak{W}_1$ and $\mathfrak{W}_2$ as laces and J as lacing isomorphism. We claim that $\mathfrak{B}$ has the required properties. In the first place, it follows from lemma 8.1 that the annihilating ideal of $\mathfrak{B}$ is $\mathfrak{q}_1 \cap \mathfrak{q}_2 = \mathfrak{q}$. In the second place, since $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are maximal submodules of the not-isomorphic modules $\mathfrak{B}_1$ and $\mathfrak{B}_2$, J is not extendable. Hence all the conditions of corollary 9.3 are satisfied which proves that $\mathfrak{B}$ is completely indecomposable. This completes the proof of Theorem 9.1.

COROLLARY 9.4. *Every completely primary ring A with unit element whose ideals satisfy both chain conditions has a faithful, completely indecomposable representation space.*

Proof. The zero ideal of A is primary and has a maximal associated prime ideal. Hence this zero ideal can be used as the $\mathfrak{q}$ of Theorem 9.1 which proves corollary 9.4.

REMARK 9.2. Let A be a completely primary ring with unit element which is at the same time an algebra of finite rank with respect to a field. Then the dual vector space of the regular representation of A defined in [8], p. 558, coincides with the faithful, completely indecomposable representation space of corollary 9.4. (See example 7.3.) There is no doubt that, if A is not an algebra, the faithful, completely indecomposable representation space of corollary 9.4 must be considered as the correct generalization of the dual vector space of [8]. Now, let A be any commutative ring with unit element whose ideals satisfy both chain conditions. Let $\mathfrak{o}$ be the zero ideal of A and let $\mathfrak{o} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s$ be the Noether decomposition of $\mathfrak{o}$. The regular representation space of A is the direct sum of the difference modules $A - \mathfrak{q}_1 \dotplus \ldots \dotplus A - \mathfrak{q}_s$; and the ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ satisfy the requirements of Theorem 9.1. Then, if $\mathfrak{B}_1, \ldots, \mathfrak{B}_s$ denote the completely indecomposable A-modules with respectively $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ as annihilators, we define the direct sum $\mathfrak{B}_1 \dotplus \ldots \dotplus \mathfrak{B}_s$ as the dual vector space of A. It follows from lemma II-B of [8], p. 559, that, if A is an algebra of finite rank with respect to a field, this dual vector space coincides with the dual vector space of the regular representation of A as defined in [8]. Again there is no doubt that, if A is not an algebra, the above definition of dual vector space is the correct generalization of the dual vector space of [8].

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