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THE CURVATURE AND TOPOLOGICAL PROPERTIES OF HYPERSURFACES WITH CONSTANT SCALAR CURVATURE

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In this paper, we consider n ($n \geqslant 3$)-dimensional compact oriented connected hypersurfaces with constant scalar curvature $n(n-1)r$ in the unit sphere $S^{n+1}(1)$. We prove that, if $r \geqslant (n-2)/(n-1)$ and $S \leqslant (n-1)\bigl(n(r-1)+2\bigr)/(n-2) + (n-2)/\bigl(n(r-1)+2\bigr)$, then either M is diffeomorphic to a spherical space form if $n = 3$; or M is homeomorphic to a sphere if $n \geqslant 4$; or M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $c^2 = (n-2)/(nr)$ and S is the squared norm of the second fundamental form of M.

1. INTRODUCTION

Let M be an n-dimensional hypersurface in the unit sphere $S^{n+1}(1)$ of dimension $n+1$. Suppose the scalar curvature $n(n-1)r$ of M is constant and $r \geqslant 1$. Cheng and Yau [**3**] and Li [**7**] obtained some characterisation theorems in terms of the sectional curvature or the squared norm of the second fundamental form of M respectively. We should notice that the condition $r \geqslant 1$ plays an essential role in the proofs of their theorems. On the other hand, for any $0 < c < 1$, by considering the standard immersions $S^{n-1}(c) \subset R^n, S^1(\sqrt{1-c^2}) \subset R^2$ and taking the Riemannian product immersion $S^1(\sqrt{1-c^2}) \times S^{n-1}(c) \hookrightarrow R^2 \times R^n$, we obtain a hypersurface $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$, where $r = (n-2)/(nc^2) > 1 - (2/n)$. Hence, not all Riemannian products $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ are covered by the results of [**3, 7**]; since the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ has only two distinct principal curvatures and its scalar curvature $n(n-1)r$ is constant and satisfies $r > 1 - (2/n)$. Hence, Cheng [**4**] asked the following interesting problem:

PROBLEM 1. ([**4**]). Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If $r > 1 - (2/n)$ and $S \leqslant (n-1)\bigl(n(r-1)+2\bigr)/(n-2) + (n-2)/\bigl(n(r-1)+2\bigr)$, then is M isometric to either a totally umbilical hypersurface or the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$?

Cheng [**4**] said that when $r = (n-2)/(n-1)$, he answered the Problem 1 affirmatively. For the general case, Problem 1 is still open.

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In this paper, we try to solve Problem 1. We shall give a topological answer, which relies on the Lawson-Simons formula ([8]) for the nonexistence of stable k-currents, which enables us to eliminate the homology groups and to show M is a homology sphere. We prove the following

THEOREM. *Let M be an $n(n \geqslant 3)$-dimensional compact oriented connected hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If $r \geqslant (n-2)/(n-1)$ and $S \leqslant (n-1)\big(n(r-1)+2\big)/(n-2) + (n-2)/\big(n(r-1)+2\big)$, then either M is diffeomorphic to a spherical space form if $n=3$; or M is homeomorphic to a sphere if $n \geqslant 4$; or M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $c^2 = (n-2)/(nr)$.*

2. Preliminaries

Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$. We take a local orthonormal frame field $e_1, \cdots, e_{n+1}$ in $S^{n+1}(1)$, restricted to $M, e_1, \cdots, e_n$ are tangent to M. Let $\omega_1, \cdots, \omega_{n+1}$ be the dual coframe fields in $S^{n+1}(1)$. We use the following convention on the ranges of indices: $1 \leqslant A, B, C, \cdots, \leqslant n+1; 1 \leqslant i, j, k, \cdots, \leqslant n$. The struture equations of $S^{n+1}(1)$ are given by

$$d\omega_A = -\sum_{B=1}^{n+1} \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_{C=1}^{n+1} \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{C,D=1}^{n+1} R_{ABCD}\omega_C \wedge \omega_D, \tag{2.2}$$

$$R_{ABCD} = (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \tag{2.3}$$

where R_{ABCD} denotes the components of the curvature tensor of $S^{n+1}(1)$. Then, in M

$$\omega_{n+1} = 0. \tag{2.4}$$

It follows from Cartan's Lemma that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, h_{ij} = h_{ji}. \tag{2.5}$$

The second fundamental form B and the mean curvature of M are defined by $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$ and $nH = \sum_i h_{ii}$, respectively. The structure equations of M are given by

$$d\omega_i = -\sum_{k=1}^{n} \omega_{ik} \wedge \omega_k, \omega_{ij} + \omega_{ji} = 0, \tag{2.6}$$

$$d\omega_{ij} = -\sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l=1}^{n} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.7}$$

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.8}$$

where R_{ijkl} denotes the components of the Riemannian curvature tensor of M. From the above equation, we have

$$n(n-1)r = n(n-1) + n^2H^2 - S, \tag{2.9}$$

where $n(n-1)r$ is the scalar curvature of M and $S = \sum_{i,j=1}^{n} h_{ij}^2$ is the squared norm of the second fundamental form of M.

The Codazzi equation and the Ricci identities are

$$h_{ijk} = h_{ikj}, \tag{2.10}$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im}R_{mjkl} + \sum_m h_{jm}R_{mikl}, \tag{2.11}$$

where the first and the second covariant derivatives of h_{ij} are defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{ik}\omega_{kj} - \sum_k h_{jk}\omega_{ki}, \tag{2.12}$$

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l h_{ijl}\omega_{lk} - \sum_l h_{ilk}\omega_{lj} - \sum_l h_{ljk}\omega_{li}. \tag{2.13}$$

We need the following Lemmas.

Lemma 1. ([5] or [9].) *Let $A = (a_{ij}), i,j = 1, \cdots, n$ be a symmetric $(n \times n)$ matrix, $n \geqslant 2$. Assume that $A_1 = \mathrm{tr}A$, $A_2 = \sum_{i,j}(a_{ij})^2$, then*

$$\sum_i (a_{in})^2 - A_1a_{nn} \leqslant \frac{1}{n^2}\{n(n-1)A_2 + (n-2)\sqrt{n-1}|A_1|\sqrt{nA_2-(A_1)^2} - 2(n-1)(A_1)^2\}. \tag{2.14}$$

We prove the following algebraic Lemma by a simple and direct method.

Lemma 2. *Let $A = (a_{ij}), i,j = 1, \cdots, n$ be a symmetric $(n \times n)$ matrix, $p + q = n$, $p, q \geqslant 2$ are positive integers. Assume that $\sum_{s=1}^{p} a_{ss} + \sum_{t=p+1}^{n} a_{tt} = A_1$, $\sum_{i=1}^{n}(a_{ii})^2 = \widetilde{A}_2$. Then*

$$\left(\sum_{s=1}^{p} a_{ss}\right)^2 - A_1\left(\sum_{s=1}^{p} a_{ss}\right) \leqslant \frac{1}{n^2}\left\{pqn\widetilde{A}_2 - 2pq(A_1)^2 + |p-q|\sqrt{pq}|A_1|\sqrt{n\widetilde{A}_2 - (A_1)^2}\right\}. \tag{2.15}$$

Proof: By Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\widetilde{A}_2 &= \sum_{s=1}^{p}(a_{ss})^2 + \sum_{t=p+1}^{n}(a_{tt})^2 \geqslant \frac{1}{p}\left(\sum_{s=1}^{p} a_{ss}\right)^2 + \frac{1}{q}\left(\sum_{t=p+1}^{n} a_{tt}\right)^2 \\ &= \frac{n}{pq}\left(\sum_{s=1}^{p} a_{ss}\right)^2 - \frac{2}{q}A_1\left(\sum_{s=1}^{p} a_{ss}\right) + \frac{1}{q}(A_1)^2.\end{aligned} \tag{2.16}$$

Hence

$$\left(\sum_{s=1}^{p} a_{ss}\right)^2 - \frac{2p}{n}A_1\left(\sum_{s=1}^{p} a_{ss}\right) + \frac{p}{n}(A_1)^2 - \frac{pq}{n}\widetilde{A}_2 \leqslant 0. \tag{2.17}$$

From (2.17) we have

$$\frac{pA_1}{n} - \frac{\sqrt{pq}}{n}\sqrt{n\widetilde{A}_2 - (A_1)^2} \leqslant \sum_{s=1}^{p} a_{ss} \leqslant \frac{pA_1}{n} + \frac{\sqrt{pq}}{n}\sqrt{n\widetilde{A}_2 - (A_1)^2}. \tag{2.18}$$

From (2.17) we also have

$$\left(\sum_{s=1}^{p} a_{ss}\right)^2 - A_1\left(\sum_{s=1}^{p} a_{ss}\right) \leqslant \frac{pq}{n}\widetilde{A}_2 - \frac{p}{n}(A_1)^2 + \frac{p-q}{n}A_1\left(\sum_{s=1}^{p} a_{ss}\right). \tag{2.19}$$

By (2.18) we have

$$\left(\sum_{s=1}^{p} a_{ss}\right)^2 - A_1\left(\sum_{s=1}^{p} a_{ss}\right) \leqslant \frac{pq}{n}\widetilde{A}_2 - \frac{p}{n}(A_1)^2 + \frac{(p-q)p}{n^2}(A_1)^2 + \left|\frac{p-q}{n}A_1\right|\frac{\sqrt{pq}}{n}\sqrt{n\widetilde{A}_2 - (A_1)^2}.$$

Hence (2.15) holds. Lemma 2 is proved. ∎

From [8] we have the following result.

Lemma 3. ([8].) *Let M be a compact n-dimensional submanifold of the unit sphere $S^{n+m}(1)$ with second fundamental form B, and let p, q be positive integers such that $1 < p, q < n-1, p+q = n$. If the inequality*

$$\sum_{s=1}^{p}\sum_{t=p+1}^{n}(2 \mid B(e_s, e_t) \mid^2 - \langle B(e_s, e_s), B(e_t, e_t)\rangle) < pq, \tag{2.20}$$

holds for any point of M and any local orthonormal frame field $\{e_s, e_t\}$ on M, then $H_p(M, Z) = H_q(M, Z) = 0$, where $H_s(M, Z)$ denotes the s-th homology group of M whth integer coefficients.

Remark. Lemma 3 is ture for general submanifold with any codimension m of $S^{n+m}(c)$, of course is true for hypersurface of $S^{n+1}(1)$.

Lemma 4. ([11] or [1].) *Let $\mu_i, i = 1, \cdots, n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2, \beta = constant \geqslant 0$, then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leqslant \sum_i \mu_i^3 \leqslant \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{2.21}$$

and the equality holds in (2.21) if and only if at least $(n-1)$ of the μ_i are equal.

From Aubin [2, see p. 344], we have.

Lemma 5. ([2].) *If the Ricci curvature of a compact Riemannian manifold is non-negative and positive at somewhere, then the manifold carries a metric with positive Ricci curvature.*

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3. Proof of Theorem

Proof: For a given point $P \in M$, we choose an orthonormal frame field $e_1, \cdots, e_n$, such that $h_{ij} = \lambda_i \delta_{ij}$. From (2.10) and (2.11) by a standard calculation we have

$$\frac{1}{2}\Delta S = \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{3.1}$$

Let $\mu_i = \lambda_i - H$ and $f^2 = \sum_i \mu_i^2$, we have

$$\sum_i \mu_i = 0, \; f^2 = S - nH^2, \tag{3.2}$$

$$\sum_i \lambda_i^3 = \sum_i \mu_i^3 + 3Hf^2 + nH^3. \tag{3.3}$$

From (2.8) we get $R_{ijij} = 1 + \lambda_i \lambda_j$, putting this into (3.1), by (3.2),(3.3) we get

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + \frac{1}{2}\sum_{i,j}(1 + \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2 \\ &= \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + nS - n^2H^2 - S^2 + nH\sum_i \lambda_i^3 \\ &= \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + nf^2 + nH^2f^2 - f^4 + nH\sum_i \mu_i^3. \end{aligned} \tag{3.4}$$

By Lemma 4, we get

$$\frac{1}{2}\Delta S \geqslant \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + f^2\left\{n + nH^2 - f^2 - n|H|\frac{n-2}{\sqrt{n(n-1)}}f\right\}. \tag{3.5}$$

We denote

$$P_H(f) = n + nH^2 - f^2 - n|H|\frac{n-2}{\sqrt{n(n-1)}}f. \tag{3.6}$$

From (2.9) we know $f^2 = S - nH^2 = (n-1)/n\big[S - n(r-1)\big]$, then by (2.9) we write $P_H(f)$ as

$$\begin{aligned} P_r(S) = n + n(r-1) &- \frac{n-2}{n}\big[S - n(r-1)\big] \\ &- \frac{n-2}{n}\sqrt{\big[n(n-1)(r-1) + S\big]\big[S - n(r-1)\big]}. \end{aligned} \tag{3.7}$$

Hence(3.5) can be written as

$$\frac{1}{2}\Delta S \geqslant \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + \frac{n-1}{n}\big[S - n(r-1)\big]P_r(S). \tag{3.8}$$

On the other hand, for any point and any unit vector $v \in T_P M$, we choose a local orthonormal frame field $e_1, \cdots, e_n$ such that $e_n = v$, we have from Gauss equation (2.8) that the Ricci curvature $\mathrm{Ric}(v,v)$ of M with respect to v is expressed as

$$\mathrm{Ric}(v,v) = (n-1) + nHh_{nn} - \sum_{i=1}^{n} h_{in}^2. \tag{3.9}$$

By Lemma 1,(3.6) and (3.7) we get

$$\mathrm{Ric}(v,v) \geqslant \frac{n-1}{n}\left[n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|f - f^2\right] = \frac{n-1}{n}P_r(S). \tag{3.10}$$

When $S \leqslant (n-1)\big(n(r-1)+2\big)/(n-2) + (n-2)/\big(n(r-1)+2\big)$, we know this is equivalent to

$$\left\{n+n(r-1)-\frac{n-2}{n}[S-n(r-1)]\right\}^2 \geqslant \frac{(n-2)^2}{n^2}\{n(n-1)(r-1)+S\}\{S-n(r-1)\}. \tag{3.11}$$

Since $r \geqslant (n-2)/(n-1)$, then we get $r-1 \geqslant -1/(n-1)$ and $n(r-1)+2 \geqslant (n-2)/(n-1)$, hence

$$\begin{aligned}
&n + n(r-1) - \frac{n-2}{n}[S - n(r-1)] \\
&\geqslant n + 2(n-1)(r-1) - \frac{n-2}{n}\left[(n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}\right] \\
&= n + 2(n-1)(r-1) - \frac{n-1}{n}[n(r-1)+2] - \frac{(n-2)^2}{n}\frac{1}{n(r-1)+2} \\
&= \frac{n^2-2(n-1)}{n} + (n-1)(r-1) - \frac{(n-2)^2}{n}\frac{1}{n(r-1)+2} \\
&\geqslant \frac{n^2-2(n-1)}{n} - 1 - \frac{(n-2)^2}{n}\frac{n-1}{n-2} = 0.
\end{aligned}$$

Obviously, by (2.9) and $f^2 = (n-1)/n[S - n(r-1)]$, we have $n(n-1)(r-1) + S \geqslant 0, S - n(r-1) \geqslant 0$. Hence from (3.11) we have

$$n+n(r-1)-\frac{n-2}{n}[S-n(r-1)] \geqslant \frac{n-2}{n}\sqrt{[n(n-1)(r-1)+S]\,[S-n(r-1)]}, \tag{3.12}$$

that is

$$P_r(S) \geqslant 0. \tag{3.13}$$

From (3.10),(3.13) we have $\mathrm{Ric}(v,v) \geqslant 0$ at all points of M.

CASE (i). When $S < (n-1)\big(n(r-1)+2\big)/(n-2) + (n-2)/\big(n(r-1)+2\big)$ holds at all points of M, or it holds at somewhere of M, then we all have the fundamental group of M is finite.

In fact, when $S < (n-1)\big(n(r-1)+2\big)/(n-2) + (n-2)/\big(n(r-1)+2\big)$ holds at all points of M, from the assertions above, we have $\mathrm{Ric}(v,v) > 0$ at all points of M. Hence by the classical Myers Theorem, we know that the fundamental group of M is finite.

When $S < (n-1)\big(n(r-1)+2\big)/(n-2) + (n-2)/\big(n(r-1)+2\big)$ holds at some points of M, from the assertions above, we know that $\mathrm{Ric}(v,v) > 0$ holds at such points of M. From Lemma 5, we know that there exists a metric on M such that the Ricci curvature is positive on M. Hence, we also know that the fundamental group of M is finite.

Therefore, the proof of Theorem in the case where $n = 3$ following directly from the Hamilton Theorem (see [**6**]) which states that a compact and connected oriented Riemannian 3-manifold with positive Ricci curvature is diffeomorphic to a spherical space form.

Now, we consider the case $n \geqslant 4$. Taking any positive integers p, q such that $p + q = n$, $1 < p, q < n-1$. Then $pq = p(n-p) = n + (p-1)n - p^2 \geqslant n + (p-1)(p+2) - p^2 = n + (p-2) \geqslant n$. Let $T = \mathrm{tr}(h_{ij}) = \sum_{s=1}^{p} h_{ss} + \sum_{t=p+1}^{n} h_{tt}$, $\widetilde{S} = \sum_i (h_{ii})^2$, $S = \sum_{i,j} (h_{ij})^2$, then we have

$$2\sum_{s=1}^{p}\sum_{t=p+1}^{n}(h_{st})^2 + \frac{pq}{n}\widetilde{S} \leqslant \frac{pq}{n}\left[2\sum_{s=1}^{p}\sum_{t=p+1}^{n}(h_{st})^2 + \widetilde{S}\right] \leqslant \frac{pq}{n}S. \tag{3.14}$$

When $p \geqslant q$, $|p-q| = p-q = n-2q < n-2$, when $p < q$, $|p-q| = q-p = n-2p < n-2$, therefore, $|p-q| < n-2$ for all p, q and $\sqrt{pq} \geqslant \sqrt{n} > \sqrt{n-1}$.

By Lemma 2,(3.14) and $\widetilde{S} \leqslant S$, we make use of the same calculation for general submanifold in [**12**], we get for hypersurface that

$$\begin{aligned}
&\sum_{s=1}^{p}\sum_{t=p+1}^{n}\Big(2\mid B(e_s,e_t)\mid^2 - \big\langle B(e_s,e_s), B(e_t,e_t)\big\rangle\Big)\\
&\quad = 2\sum_{s=1}^{p}\sum_{t=p+1}^{n}(h_{st})^2 - \sum_{s=1}^{p}\sum_{t=p+1}^{n} h_{ss}h_{tt}\\
&\quad = 2\sum_{s=1}^{p}\sum_{t=p+1}^{n}(h_{st})^2 + \Big(\sum_{s=1}^{p} h_{ss}\Big)^2 - T\Big(\sum_{s=1}^{p} h_{ss}\Big)\\
&\quad \leqslant 2\sum_{s=1}^{p}\sum_{t=p+1}^{n}(h_{st})^2 + \frac{pq}{n}\widetilde{S} - \frac{2pq}{n^2}T^2 + \frac{|p-q|}{n^2}\sqrt{pq}|T|\sqrt{n\widetilde{S} - T^2}\\
&\quad \leqslant \frac{pq}{n}S - \frac{2pq}{n^2}T^2 + \frac{|p-q|}{n^2}\sqrt{pq}|T|\sqrt{n\widetilde{S} - T^2}\\
&\quad \leqslant \frac{pq}{n}\left[S - 2nH^2 + \frac{|p-q|}{\sqrt{pq}}|H|\sqrt{nS - n^2H^2}\right]\\
&\quad < \frac{pq}{n}\left[S - 2nH^2 + \frac{\sqrt{n}(n-2)}{\sqrt{n-1}}|H|\sqrt{S - n^2H^2}\right]
\end{aligned}$$

$$= -\frac{pq}{n}\left[n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|f - f^2\right] + pq.$$

Therefore, from (3.6) or (3.7) and (3.13) we have

$$\sum_{s=1}^{p}\sum_{t=p+1}^{n}\Big(2|B(e_s,e_t)|^2 - \langle B(e_s,e_s), B(e_t,e_t)\rangle\Big) < -\frac{pq}{n}P_r(S) + pq < pq. \tag{3.15}$$

Hence from Lemma 3 $H_p(M,Z) = H_q(M,Z) = 0$, for all $1 < p, q < n-1, p+q = n$. Since $H_{n-2}(M,Z) = 0$, taking the same discussion in [**10**], by the universal coefficient theorem $H^{n-1}(M,Z)$ has no torsion and consequently $H_1(M,Z)$ has no torsion by Poincare duality. By our assumption, since the fundamental group $\pi_1(M)$ of M is finite, hence $H_1(M,Z) = 0$, so M is a homology sphere. The above arguments can be applied to the universal covering $\widetilde{M}$ of M. Since $\widetilde{M}$ is a homology sphere which is simple connected, that is $\pi_1(\widetilde{M}) = 0$, it is also a homotopy sphere. By the generalised Poincare conjecture (Smale $n \geqslant 5$, Freedman $n = 4$) we have $\widetilde{M}$ is homeomorphic to a sphere and hence we have a homotopy sphere M which is covered by a sphere $\widetilde{M}$, so by a result of Sjerve [**13**] we have $\pi_1(M) = 0$, and hence M is homeomorphic to a sphere.

Case (ii). $S \equiv (n-1)(n(r-1)+2)/(n-2) + (n-2)/(n(r-1)+2)$ on M, from the discussion above this is equivalent to $P_r(S) = 0$. Since the scalar curvature $n(n-1)r$ is constant, thus S is constant, and by (2.9) H is also constant. Hence the equalities in (3.8),(3.5) and (2.21) in Lemma 4 hold. If $r \geqslant (n-2)/(n-1)$, since $S = (n-1)(n(r-1)+2)/(n-2) + (n-2)/(n(r-1)+2) > (n-1)(n(r-1)+2)/(n-2) > n(r-1)$, then $f^2 = (n-1)/n[S - n(r-1)] \neq 0$, that is M is not umbilical. When the equality in (2.21) holds, by Lemma 4 M is of only two distinct principal curvatures, one with multiplicity 1 and the other with multiplicity $n-1$. After renumberation if necessary, we can assume that $\lambda = \lambda_1 = \cdots = \lambda_{n-1}, \mu = \lambda_n$. When the equalities in (3.8) or(3.5) hold. We have

$$h_{ijk} = 0. \tag{3.16}$$

Choose a local frame of orthonormal vector fields such that $h_{ij} = \lambda_i\delta_{ij}$, from (2.6) $\omega_{ii} = 0$. Let $i = j$ in (2.12), from (3.16) and (2.12) we have $0 = d\lambda_i - 2\sum_k h_{ik}\omega_{ki} = d\lambda_i$, hence λ_i is constant, again from (2.12) we have

$$0 = \lambda_i\omega_{ij} + \lambda_j\omega_{ji} = (\lambda_i - \lambda_j)\omega_{ij}, \tag{3.17}$$

then for $\lambda_i \neq \lambda_j$

$$\omega_{ij} = 0. \tag{3.18}$$

From (2.7) and (3.18), if $\lambda_i \neq \lambda_j$, then

$$0 = d\omega_{ij} = -\sum_k \omega_{ik}\wedge\omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l. \tag{3.19}$$

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If for some k such that $\omega_{ik} \neq 0$ and $\omega_{kj} \neq 0$,then by (3.17) we have $\lambda_i = \lambda_k = \lambda_j$, this contradicts to $\lambda_i \neq \lambda_j$, so $\sum\limits_{k,l} R_{ijkl}\omega_k \wedge \omega_l = 0$, thus, if $\lambda_i \neq \lambda_j$ we have

$$R_{ijkl} = 0. \tag{3.20}$$

From(3.20) and the Guass equation (2.8) we have $1 + \lambda_i\lambda_j = 0$ for $\lambda_i \neq \lambda_j$, that is

$$1 + \lambda\mu = 0. \tag{3.21}$$

From (2.9) we have

$$\mu = \frac{n(r-1)}{2\lambda} - \frac{n-2}{2}\lambda. \tag{3.22}$$

Hence from (3.21),(3.22) we get $\lambda^2 = \big(n(r-1)+2\big)/(n-2)$ and $\mu^2 = (n-2)/\big(n(r-1)+2\big)$. Thus we get that M is isoparametric. Therefore, M is isometric to the Riemannian prodnct $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $c^2 = (n-2)/(nr)$. The Theorem is proved. ∎

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