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GLOBAL HYPOELLIPTICITY OF A CLASS OF SECOND ORDER OPERATORS

ADALBERTO P. BERGAMASCO AND SÉRGIO LUÍS ZANI

ABSTRACT. We show that almost all perturbations $P - \lambda$, $\lambda \in \mathbf{C}$, of an arbitrary constant coefficient partial differential operator P are globally hypoelliptic on the torus. We also give a characterization of the values $\lambda \in \mathbf{C}$ for which the operator $D_t^2 - 2D_x^2 - \lambda$ is globally hypoelliptic; in particular, we show that the addition of a term of order zero may destroy the property of global hypoellipticity of operators of principal type, contrary to that happens with the usual (local) hypoellipticity.

1. **Introduction.** The following property (called Weyl's Lemma) is valid for the Laplacian $\Delta = D_1^2 + D_2^2$: if u is a weak solution of $\Delta u = 0$ then u is smooth (*i.e.* C^∞).

The same result is true for solutions of the non-homogeneous equation: if $\Delta u = f \in \mathrm{C}^\infty$ then $u \in \mathrm{C}^\infty$. This property (of the regularity of all solutions) is called *hypoellipticity*.

Another important property concerns perturbation by lower-order terms: if Q is an arbitrary operator of order ≤ 1 (possibly with variable smooth coefficients) then the operator $\Delta + Q$ is also hypoelliptic.

We stress the fact that the concept of hypoellipticity as well as the results mentioned above have a *local* character; for instance, if V is a neighborhood of a point x_0 and if $\Delta u \in \mathrm{C}^\infty(V)$ then $u \in \mathrm{C}^\infty(V)$.

In this work we are interested in the *global hypoellipticity* of certain (constant coefficient) operators, which means: if f is smooth and 2π-periodic in each variable and if u is a 2π-periodic solution of $Pu = f$ in $\mathbf{R}^n$ then u itself is smooth. In view of the periodicity all objects may be thought of as being defined on the torus $\mathbf{T}^n$, hence the name *global* hypoellipticity.

We abbreviate the statement "P is *globally hypoelliptic*" by writing "P is GH".

We call attention to the fact that if P is hypoelliptic near each point then P is GH. That the converse is not true can be seen through the examples involving wave equations. Indeed the function $u(t,x) = |x+ct|^5$ is C^2 but not C^∞ and it satisfies $u_{tt} - c^2 u_{xx} = 0 \in \mathrm{C}^\infty$, hence the operator $P = D_t^2 - c^2 D_x^2$ is not hypoelliptic; on the other hand P is GH if c is a non-Liouville irrational (see Section 2).

Our main goal is to study the influence of lower-order terms on the property of global hypoellipticity; as we shall see, the addition of terms of order zero may turn a non-GH operator into a GH one (for instance, if P is an arbitrary operator of order two then $P - \lambda$

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is GH for almost all $\lambda \in \mathbf{R}$ (or $\mathbf{C}$)). We recall that for operators of principal type, the addition of lower-order terms cannot destroy the property of (local) hypoellipticity [8].

The following is an interesting example: the operator $P = D_t^2 - 2D_x^2$ is GH but the operator $P_1 = D_t^2 - 2D_x^2 - 1$ is not GH.

We also give a characterization of the values of $\lambda \in \mathbf{C}$ for which the operator $P_\lambda = D_t^2 - 2D_x^2 - \lambda$ is GH.

A real number α is called a *Liouville number* if for every $j \in \mathbf{N}$ there exist $C > 0$ and infinitely many numbers m/n such that

$$\left|\alpha - \frac{m}{n}\right| < C|n|^{-j}.$$

We recall two results from [1]:

THEOREM ([1]). *Let P be a linear partial differential operator on $\mathbf{T}^n$ with constant coefficients. Then P is* GH *if and only if the following condition holds: there exist real numbers $C > 0$ and K such that*

$$|P(m)| \geq C|m|^K, \tag{1}$$

for all $m \in \mathbf{Z}^n$ with $|m| \geq C^{-1}$.

COROLLARY ([1]). *Let $P = D_t - \alpha D_x$, $\alpha \in \mathbf{C}$ acting on $\mathbf{T}^2$. Then P is not* GH *if and only if α is either a rational number or a Liouville irrational.*

We point out that conditions such as (1) above and Liouville numbers occur in the so-called small denominators problem of Celestial Mechanics; for a recent survey we refer to [6], where regularity results for nonlinear operators are proved.

We note that the work of Herz [3] seems to have been the first to point out the role of Liouville numbers in the study of a global property (namely, closedness of the range of a vector field) in the C^∞ framework.

2. **Results.** The theorem below is a generalization (and an easy consequence) of the corollary above to operators of higher order.

THEOREM 1. *Let $P = \sum_{j=0}^m a_j D_t^{m-j} D_x^j$ be a homogeneous partial differential operator on $\mathbf{T}^2$ with constant complex coefficients. Then the following holds:*

(i) If $a_0 = a_m = 0$, then P is not GH.

(ii) If $a_0 \neq 0$ (or $a_m \neq 0$) then there are complex constants $\alpha_1, \ldots, \alpha_m$ such that

$$P = a_0(D_t - \alpha_1 D_x) \cdots (D_t - \alpha_m D_x) \quad or$$
$$P = a_m(D_x - \alpha_1 D_t) \cdots (D_x - \alpha_m D_t)$$

and P is not GH *if and only if there exists j such that α_j is either a rational number or a Liouville irrational.*

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In particular, the operator $P = D_t^2 - c^2 D_x^2$ is GH on $\mathbf{T}^2$ if and only if c is neither rational nor a Liouville irrational.

Thus, for example, $P = D_t^2 - 2D_x^2$ is GH since $\sqrt{2}$ is algebraic of degree 2 and as such it satisfies $|\sqrt{2} - m/n| \geq Cn^{-2}$, for some $C > 0$ and for all $m \in \mathbf{Z}$, $n \in \mathbf{N}$.

We now pose the following question: if we add a term of order zero to a GH operator, does it remain GH? In other words, if P is GH and $\lambda \in \mathbf{C}$, is $P - \lambda$ a GH operator?

The answer is not always yes; indeed if $P = D_t^2 - 2D_x^2$ and $\lambda = 1$, the operator $P_1 = D_t^2 - 2D_x^2 - 1$ is not GH. To see this, notice that $P_1(m,n) = m^2 - 2n^2 - 1$, and that the equation $m^2 - 2n^2 = 1$ is Pell's equation which has infinitely many solutions in integers, hence (1) cannot be satisfied. There is a direct way of showing that P_1 is not GH: take $(m_1, n_1), (m_2, n_2), \ldots$ to be solutions of $m^2 - 2n^2 - 1 = 0$, with $n_k \longrightarrow +\infty$, and notice that u defined by $u = \sum_{k=1}^{\infty} (m_k + n_k)^{-5} \exp(im_k t + in_k x)$ satisfies $u \in C^2_{2\pi}$, $u \notin C^2_{2\pi}$, $Pu = 0 \in C^\infty_{2\pi}$.

The next result will tell us how to decide, for a given $\lambda \in \mathbf{C}$, whether or not the special partial differential operator on $\mathbf{T}^2$ given by $P_\lambda = D_t^2 - 2D_x^2 - \lambda$ is GH.

THEOREM 2. *The operator* $P_\lambda = D_t^2 - 2D_x^2 - \lambda$, $\lambda \in \mathbf{C}$, *on* $\mathbf{T}^2$, *is not* GH *if and only if* λ *satisfies one of the following conditions:*

(i) $\lambda = \pm 1$

(ii) $\lambda \in \mathbf{Z} \setminus \{-1, 0, 1\}$ *and in the prime factorization* $\lambda = p_1^{r_1} \cdots p_s^{r_s}$ *one has* $p_j \equiv \pm 1 \pmod 8$, *whenever* p_j *and* r_j *are odd.*

Instead of presenting a proof of Theorem 2 we just recall two number-theoretical results which are needed for it:

- ([4], p. 108) if the equation $m^2 - 2n^2 = \lambda$ has integral solutions and p is a prime such that $p \mid \lambda$ then either $p \mid m$ (and thus also $p \mid n$) or $p \equiv \pm 1 \pmod 8$.
- ([5], p. 161) if p is an odd prime and $p \equiv \pm 1 \pmod 8$ then $m^2 - 2n^2 = p$ has integral solutions.

Notice also that when $\lambda \in \mathbf{C} \setminus \mathbf{Z}$ we have $|P_\lambda(m,n)| \geq \mathrm{dist}(\lambda, Z) > 0$, hence P_λ is GH in this case.

In Theorem 2 the set $\{\lambda \in \mathbf{R}\ ;\ P_\lambda \text{ is not GH}\}$ is a subset of $\mathbf{Z}$ hence it has Lebesgue measure zero. More generally, we have

THEOREM 3. *Let P be a partial differential operator, with constant coefficients, acting on the two-dimensional torus. Then, for almost all* $\lambda \in \mathbf{R}$ *(or* $\lambda \in \mathbf{C}$*), the operator* $P_\lambda = P - \lambda$ *is* GH.

PROOF. We must show that the set C consisting of all $\lambda \in \mathbf{R}$ for which $P_\lambda = P - \lambda$ is not GH has measure zero.

By the result of Greenfield and Wallach we have

$$C = \left\{ \lambda \in \mathbf{R}\ ;\ \begin{matrix} \text{there exists } L > 0 \text{ such that } |P_\lambda(m,n)| \geq (|m| + |n|)^{-L} \text{ for all} \\ (m,n) \in \mathbf{Z}^2 \text{ with } |m| + |n| \geq L \end{matrix} \right\}.$$

We will only prove that $C \cap [0,1]$ has measure zero; an easy modification yields the same result for $C \cap [k, k+1]$, for all integers k.

Let $A = \{(m,n) \in \mathbf{Z}^2 - \{0\} \; ; \; |P(m,n)| < 2\}$. We may write $A = \{a_1, a_2, \ldots\}$ where $a_j = (m_j, n_j)$ and $|m_j| + |n_j| \leq |m_{j+1}| + |n_{j+1}|$, $j = 1, 2, \ldots$.

Set, for each $k = 1, 2, \ldots$,

$$f_k(\lambda) = \sum_{j=1}^{k} |P_\lambda(m_j, n_j)|^{-1/2} |(m_j, n_j)|^{-3}$$

Set $g_j(\lambda) = |P_\lambda(m_j, n_j)|^{-1/2}$, $\lambda \in [0,1]$, $j = 1, 2, \ldots$.

Each g_j is integrable on $[0,1]$ and $\int_0^1 g_j(\lambda)\, d\lambda \leq 8$; to see this, we analyze three cases, according to which of the following intervals $P(m_j, n_j)$ belongs: $(-2,0)$, $[0,1]$, $(1,2)$. In the second case it is convenient to split the integral into the intervals $[0, P(m_j, n_j)]$ and $[P(m_j, n_j), 1]$ (the other cases are easier). We get

$$\int_0^1 g_j \leq 2\big(|P(m_j, n_j)|^{-1/2} + |1 - P(m_j, n_j)|^{1/2}\big) < 8.$$

The sequence (f_k) increases to the function

$$f(\lambda) = \sum_{j=1}^{\infty} |P(m_j, n_j) - \lambda|^{-1/2} |(m_j, n_j)|^{-3}$$

The Monotone Convergence Theorem implies that

$$\begin{aligned} \int_0^1 f(\lambda)\, d\lambda &= \sum_{j=1}^{\infty} |(m_j, n_j)|^{-3} \int_0^1 |P(m_j, n_j) - \lambda|^{-1/2}\, d\lambda \\ &\leq 8 \sum_{j=1}^{\infty} |(m_j, n_j)|^{-3} < \infty. \end{aligned}$$

We see that $f(\lambda)$ is finite for almost all $\lambda \in [0,1]$. Hence, for such λ, there exists $M = M(\lambda) > 0$ with

$$|P(m_j, n_j) - \lambda|^{-1/2} . |(m_j, n_j)|^{-3} \leq M \quad j = 1, 2, \ldots$$

or

$$|P(m_j, n_j) - \lambda| \geq M^{-2} |(m_j, n_j)|^{-6} \quad j = 1, 2, \ldots.$$

Now notice that if $(m,n) \notin A$ and $(m,n) \neq 0$ then $|P(m,n)| \geq 2$ and so, for $\lambda \in [0,1]$, $|P_\lambda(m,n)| \geq |P(m,n)| - |\lambda| \geq 1$.

If we set $C = \min\{1, M^{-2}\}$ we obtain

$$|P(m,n) - \lambda| \geq C |(m,n)|^{-6} \quad \forall (m,n) \in \mathbf{Z}^2 - \{0\},$$

which implies

$$|P(m,n) - \lambda| \geq |(m,n)|^{-7} \quad \text{when} \quad |(m,n)| \geq C^{-1}. \qquad \blacksquare$$

We add a few final remarks.

Pell's equation was used in [2] to produce an example of a non-GH operator in $\mathbf{T}^3$.

If P is an operator with integral coefficients then $P - \lambda$ is GH for λ outside of a discrete set; however, it is not always possible to give a description as precise as the one in Theorem 2.

In the paper [9] one finds examples of operators P which are not GH but become GH when one adds to them a function $\lambda(x)$ with average equal to zero (in our examples, we achieve the same by adding non-zero constants).

As far as global solvability is concerned (*i.e.*, $P(C^\infty_{2\pi}) = C^\infty_{2\pi}$ or $P(\mathcal{D}'_{2\pi}) = \mathcal{D}'_{2\pi}$) one also has many results; for instance, the operator $P_\lambda = D_t^2 - 2D_x^2 - \lambda$, $\lambda \in \mathbf{C}$, is globally solvable in $\mathbf{T}^2$ if and only if P_λ is GH and $\lambda \neq 0$. More generally, if P is GH then P is globally solvable up to finite codimension.

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Departamento de Matemática
Universidade Federal de São Carlos
Caixa Postal 676–13.565-905
São Carlos, SP
Brazil
e-mail: apbergam@icmsc.usp.ansp.br

Instituto de Ciências Matemáticas de São Carlos
Universidade de São Paulo
Caixa Postal 668–13.560-970
São Carlos, SP
Brazil

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