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ANALYSIS OF A DISCRETIZED FRACTIONAL-ORDER PREY–PREDATOR MODEL UNDER WIND EFFECT

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Abstract

While constructing mathematical models, scientists usually consider biotic factors, but it is crystal-clear that abiotic factors, such as wind, are also important as biotic factors. From this point of view, this paper is devoted to the investigation of some bifurcation properties of a fractional-order prey–predator model under the effect of wind. Using fractional calculus is very popular in modelling, since it is more effective than classical calculus in predicting the system's future state and also discretization is one of the most powerful tools to study the behaviour of the models. In this paper, first of all, the model is discretized by using a piecewise discretization approach. Then, the local stability of fixed points is considered. We show using the centre manifold theorem and bifurcation theory that the system experiences a flip bifurcation and a Neimark–Sacker bifurcation at a positive fixed point. Finally, numerical simulations are given to demonstrate our results.

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1. Introduction

Differential equations with fractional orders are widely used in science, and this extensive use has recently sparked much research interest. The so-called fractional-order integral or derivative operators are the main producers of fractional-order differential equations. Due to the nonlocal nature of their integral and derivative operators, fractional-order operators are more effective than other classical deterministic operators at predicting the future state of the system, because they depend not only on

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the current state, but also on all of its past states. The most popular and widely used fractional-order operators in applications of fractional-order differential equations are the Riemann–Liouville, Caputo and Grünwald–Letnikov kinds [28]. Now, let us give the definition of fractional-order integration and differentiation [44, 46].

DEFINITION 1.1. The fractional integral (or the Riemann Liouville integral) of order $\beta \in \mathbb{R}^+$ of the function $f(t)$, $t > 0$, is defined by

$$I_a^{\beta} f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds.$$

The fractional derivative of order $\alpha \in (n-1, n)$ is defined by the following two (nonequivalent) ways.

(1) Riemann–Liouville fractional derivative: take the fractional integral of order $(n-\alpha)$, and then take the nth derivative as follows:

$$D^{\alpha} f(t) = D^n I_a^{(n-\alpha)} f(t), \quad D = \frac{d}{dt}, \quad n = 1, 2, 3, \ldots.$$

(2) Caputo-fractional derivative: take the nth derivative, and then take a fractional integral of order $(n-\alpha)$

$$D^{\alpha} f(t) = I_a^{(n-\alpha)} D^n f(t).$$

Even though it is more restrictive than the Riemann–Liouville, we consider the fractional derivative provided by Caputo in this study [11]. This is because this definition is more suitable for problems consisting of fractional differential equations with initial conditions. The following result contains the main properties from fractional calculus [45].

REMARK 1.2. Let $L^1 = L^1[a, b]$ be the class of Lebesgue integrable functions on $[a, b], a < b < \infty, \beta, \gamma \in \mathbb{R}^+$ and $\alpha \in (0, 1)$. Then:

(1) if $I_a^{\beta} : L^1 \to L^1$ and $f(x) \in L^1$, then $I_a^{\gamma} I_a^{\beta} f(x) = I_a^{\gamma+\beta} f(x)$;
(2) $\lim_{\beta \to n} I_a^{\beta} f(x) = I_a^n f(x)$ uniformly on $[a, b], n = 1, 2, 3, \ldots$, where $I_a^1 f(t) = \int_0^t f(s) ds$;
(3) $\lim_{\beta \to 0} I_a^{\beta} f(t) = f(t)$ weakly;
(4) if $f(t)$ is absolutely continuous on $[a, b]$, then $\lim_{\alpha \to 1} D^{\alpha} f(t) = df(t)/dt$.

In applied mathematics, the process of converting continuous models defined by differential equations into their discrete equivalent is known as discretization. Analysing the dynamics of the discretization is crucial for understanding how well the discrete model approximates the original continuous system. Discretization is often applied to systems to perform numerical simulations or to design algorithms that run on digital platforms. As the step size approaches zero, the discrete system converges to the original continuous system. In this limit, the discrete-time model approaches the behaviour of the differential equations, ensuring that any discrepancies between the two systems vanish. In the literature, there are a large number of nonlinear fractional differential equations without an analytical solution, so they need to be solved with the help of some numerical and discretization techniques such as the Adomian decomposition method [1, 22], differential transform method [40, 51], Euler method [34], extrapolation method [18], Grünwald–Letnikov method [30] and variational iteration method [55, 56].

In the last decade, we see a new type of discretization technique for fractional equations, called piecewise discretization, which is constructed with the help of the piecewise constant arguments [2, 19, 20, 23, 32, 33, 39, 42, 57, 58, 60, 61]. The systematic study of problems involving piecewise constant arguments began in the early 1980s with the work of Shah and Wiener [50], who introduced the term "differential equations with piecewise constant argument" (DEPCA). A comprehensive source on this class of equations is provided in [52]. Busenberg and Cooke [10] were pioneers in applying such deviating arguments to mathematical models, specifically in the study of vertically transmitted diseases, by reducing their analysis to discrete equations. The main source for DEPCA theory are the papers [17, 52]. In 1991, Györi was the first who used the piecewise constant argument for approximation [24]. After this work, there has been a significant interest in the usage of the piecewise constant argument for a numerical approach [16, 25–27]. In the late 2010s, Akhmet [4] introduced the equation

$$x'(t) = f(t, x(t), x\gamma(t))),$$

where $\gamma(t)$ is a piecewise constant argument of generalized type, that is, given $(t_k)_{k\in\mathbb{Z}}$ and $(\zeta_k)_{k\in\mathbb{Z}}$ such that $t_k < t_{k+1}$ for all $k \in \mathbb{Z}$ with $\lim_{k\to\pm\infty} t_k = \pm\infty$ and $t_k \le \zeta_k \le t_{k+1}$, if $t \in I_k = [t_k, t_{k+1})$, then $\gamma(t) = \zeta_k$. These type of equations are called "differential equations with piecewise constant argument of generalized" (DEPCAG) type. They have continuous solutions, even when $\gamma(t)$ is not, producing a discrete equation. Several aspects of this equation have garnered much interest, see [3, 5, 13–15] and the references therein.

In this paper, we propose to use a piecewise discretization method. For this aim, let us first explain the usage of the method for any fractional-order differential equation. Agarwal et al. [2] considered the equation

$$D^\alpha x(t) = f\left(x\left(r\left[\frac{t}{r}\right]\right)\right), \quad x(t) = x_0, \quad t \le 0, \tag{1.1}$$

as a discretized version of the equation

$$\begin{cases} D^\alpha x(t) = f(x(t)), & t > 0, \\ x(0) = x_0, & t \le 0, \end{cases}$$

where r is the discretization parameter. Let $t \in [0, r)$, then $t/r \in [0, 1)$. So from (1.1),

$$D^\alpha x(t) = f(x_0), \ t \in [0, r).$$

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Thus,

$$x_1(t) = x_0 + \frac{t^\alpha}{\Gamma(1+\alpha)} f(x_0).$$

Let $t \in [r, 2r)$, so $t/r \in [1, 2)$. Thus,

$$D^\alpha x(t) = f(x_1(r)), \quad t \in [r, 2r),$$

whose solution is

$$x_2(t) = x_1(r) + \frac{(t-r)^\alpha}{\Gamma(1+\alpha)} f(x_1(r)).$$

Let $t \in [2r, 3r)$, then $t/r \in [2, 3)$. Thus, we obtain

$$D^\alpha x(t) = f(x_2(r)), \quad t \in [2r, 3r).$$

So,

$$x_3(t) = x_2(2r) + \frac{(t-2r)^\alpha}{\Gamma(1+\alpha)} f(x_2(2r)).$$

Repeating the process, when $t \in [nr, (n+1)r)$,

$$D^\alpha x(t) = f(x_n(nr)), \quad t \in [nr, (n+1)r),$$

which yields

$$x_{n+1}(t) = x_n(nr) + \frac{(t-nr)^\alpha}{\Gamma(1+\alpha)} f(x_n(nr)).$$

Letting $t \to (n+1)r$ in the above, the corresponding difference equation is obtained as

$$y_{n+1} = y_n + \frac{r^\alpha}{\Gamma(\alpha+1)} f(y_n) \tag{1.2}$$

with $x(nr) = y_n$.

Prey and predator live in the same ecosystem, and both of them have their own, but mostly the same, biotic factors, such as speed, stealth and camouflage (to hide while pursuing the prey or to hide from the predator). They also possess good senses of smell, sight and hearing (to locate the prey or to detect the predator), and poison (to kill the prey or to spray when approached or bitten), among other traits. In mathematical biology, these factors are usually considered as the main subjects that affect the model. However, recently, there has been a significant increase in the number of studies exploring the impact of abiotic effects. These include the influence of predator-induced fear [6, 8, 36, 37], climate change [48, 49, 54], seasonal variations [12, 41, 47] and wind [7, 43]. These elements can of course be increased.

Now, let us turn the wheels to our problem. Barman et al. [7] investigated the effect of wind in a prey–predator model. They constructed two types of functional responses to describe the whole dynamics of the considered system under the fact

TABLE 1. Description of the variables and parameters in (1.3).

Description of the variables and parameters	
x	Prey population density at time t
y	Predator population density at time t
r	Intrinsic growth rate of prey population
k	Environmental carrying capacity of the prey species
b	Hindrance rate in prey capturing for predator species
d	Natural mortality rate of predator
ω	Strength of wind flow
c_1	Prey consumption rate by predator
c_2	Food conversion rate from prey to predator

that wind may either decrease or increase the predation rate of predators. They showed positivity and boundedness of the solutions of the systems and examined the stability of the equilibrium points. Also for particular cases, it is demonstrated that wind can change the stability of the equilibrium point through Hopf bifurcation. Furthermore, considering the fact that wind flow cannot be constant for a time period, they developed the discussed system and investigated this system with the help of numerical simulations. In their work, they considered

$$\begin{cases} \dfrac{dx}{dt} = rx\Big(1 - \dfrac{x}{k}\Big) - \dfrac{c_1xy}{1 + \omega + bx + b\omega x/(1 + \omega)}, \\ \dfrac{dy}{dt} = \dfrac{c_2xy}{1 + \omega + bx + b\omega x/(1 + \omega)} - dy, \end{cases} \tag{1.3}$$

with the initial conditions

$$x(0) = x_0 \quad \text{and} \quad y(0) = y_0. \tag{1.4}$$

The variables and all positive parameters in this model are defined in Table 1.

Our aim is to investigate the fractional-order discrete version of this problem to see the effect of both the fractional order and the discretization parameter in the occurrence of flip and Neimark–Sacker bifurcation.

In recent years, there has been a considerable amount of work on flip and Neimark–Sacker bifurcations: in [31], a discrete-time predator–prey system is considered, and the chaos control is investigated. The effect of prey refuge proportional to the predator in a discrete-time prey–predator model is studied in [38]. The authors consider the dynamical behaviour of a discrete-time prey–predator system with Leslie type in [9], and in [59], a discrete Lotka–Volterra-type predator–prey system with refuge effect is analysed. The reader is also referred to the references therein.

Before stating our main problem, let us first apply nondimensionalization to (1.3) to use one of the most important benefits: reduction of the numbers of the parameters. For this aim, let us define the variables

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$$x = kP, \quad y = \frac{rQ}{c_1}, \quad t = \frac{\tau}{r}. \tag{1.5}$$

Then, the derivatives become

$$\frac{dx}{dt} = kr\frac{dP}{d\tau}, \quad \frac{dy}{dt} = \frac{r^2}{c_1}\frac{dQ}{d\tau}. \tag{1.6}$$

Substituting (1.5) and (1.6) in the initial value problem (1.3)–(1.4) gives the nondimensionalized form

$$\begin{cases} \dfrac{dP}{d\tau} = P(1-P) - \dfrac{PQ}{a_1 + a_2 P}, \\ \dfrac{dQ}{d\tau} = \dfrac{b_1 PQ}{a_1 + a_2 P} - b_2 Q, \end{cases}$$

with

$$P(0) = P_0, \quad Q(0) = Q_0,$$

where

$$a_1 = 1 + \omega, \quad a_2 = bk + \frac{b\omega k}{1+\omega}, \quad b_1 = \frac{c_2 k}{r}, \quad b_2 = \frac{d}{r}, \quad P_0 = \frac{x_0}{k}, \quad Q_0 = \frac{y_0 c_1}{r}.$$

Now, let us consider this initial value problem with fractional derivatives and apply the piecewise discretization process. For $\alpha \in (0,1)$ and $h > 0$ as the discretization parameter, it can be written as

$$\begin{cases} D^\alpha P(\tau) = P\left(\left[\frac{\tau}{h}\right]h\right)\left(1 - P\left(\left[\frac{\tau}{h}\right]h\right)\right) - \dfrac{P\left(\left[\frac{\tau}{h}\right]h\right)Q\left(\left[\frac{\tau}{h}\right]h\right)}{a_1 + a_2 P\left(\left[\frac{\tau}{h}\right]h\right)}, \\ D^\alpha Q(\tau) = \dfrac{b_1 P\left(\left[\frac{\tau}{h}\right]h\right)Q\left(\left[\frac{\tau}{h}\right]h\right)}{a_1 + a_2 P\left(\left[\frac{\tau}{h}\right]h\right)} - b_2 Q\left(\left[\frac{\tau}{h}\right]h\right), \end{cases} \tag{1.7}$$

with

$$P(0) = P_0, \quad Q(0) = Q_0.$$

The equations in (1.7) are both in the form of (1.1), so the corresponding difference equation, from (1.2), is

$$\begin{cases} M_{n+1} = M_n + \dfrac{\rho}{\Gamma(\alpha+1)}\left(M_n(1 - M_n) - \dfrac{M_n N_n}{a_1 + a_2 M_n}\right), \\ N_{n+1} = N_n + \dfrac{\rho}{\Gamma(\alpha+1)}\left(\dfrac{b_1 M_n N_n}{a_1 + a_2 M_n} - b_2 N_n\right), \end{cases} \tag{1.8}$$

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where $\rho = h^\alpha$,

$$P(nh) = M_n, \quad Q(nh) = N_n$$

with the initial conditions

$$M_0 = P_0 \quad \text{and} \quad N_0 = Q_0.$$

The presentation in this paper is organized as follows. Section 1 is devoted to the introduction of the stated problem. Preliminaries needed for the paper are given in Section 2. Stability and bifurcation analysis are investigated in Sections 3 and 4, respectively. Numerical examples are stated in Section 5 and, finally, Section 6 presents the conclusion.

2. Preliminaries

In this section, we recall some definitions and lemmas that will be useful in the later sections.

Consider the difference system

$$\begin{cases} M_{n+1} = f(M_n, Nn), \\ N_{n+1} = g(M_n, Nn). \end{cases}$$

The Jacobian of this system at any fixed point $(\bar{M}, \bar{N})$ is

$$J = \begin{pmatrix} \dfrac{\partial f}{\partial M_n} & \dfrac{\partial f}{\partial N_n} \\ \dfrac{\partial g}{\partial M_n} & \dfrac{\partial g}{\partial N_n} \end{pmatrix}\Bigg|_{(\bar{M},\bar{N})},$$

and the corresponding characteristic equation is

$$\lambda^2 - \mathrm{Tr}(J) \cdot \lambda + \det(J) = 0. \tag{2.1}$$

DEFINITION 2.1 [31]. Let λ_1 and λ_2 be the characteristic roots of (2.1). A fixed point $(\bar{M}, \bar{N})$ is called:

(i) sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, and it is locally asymptotically stable;
(ii) source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and it is locally unstable;
(iii) saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ or $|\lambda_1| < 1$ and $|\lambda_2| > 1$;
(iv) nonhyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

LEMMA 2.2 [29]. *Let $F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are constants. Suppose $F(1) > 0$, and λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then:*

(i) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $C < 1$;
(ii) $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *if and only if* $F(-1) = 0$, *and* $B \neq 0, 2$;
(iii) $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) < 0$;

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(iv) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $C > 1$*;*
(v) λ_1 *and* λ_2 *are the conjugate complex roots and* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $B^2 - 4C < 0$ *and* $C = 1$.

DEFINITION 2.3 [35]. The bifurcation associated with the appearance of $\mu_1 = -1$ is called a *flip* (or period-doubling) bifurcation, where μ_1 is an eigenvalue of the nonhyperbolic fixed point of the the system

$$x_{n+1} = f(x_n, \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R} \tag{2.2}$$

with smooth f.

THEOREM 2.4 [35]. *Suppose that a one-dimensional system*

$$x_{n+1} = f(x_n, \alpha), \quad x \in \mathbb{R}, \quad \alpha \in \mathbb{R},$$

with smooth f*, has at* $\alpha = 0$ *the fixed point* $x_0 = 0$ *and let* $\mu = f_x(0, 0) = -1$. *Assume that the following nondegeneracy conditions are satisfied:*

(i) $\frac{1}{2}(f_{xx}(0, 0))^2 + \frac{1}{3}f_{xxx}(0, 0) \neq 0$*;*
(ii) $f_{x\alpha}(0, 0) \neq 0$.

Then, there is a flip bifurcation in the system.

DEFINITION 2.5 [35]. The bifurcation corresponding to the presence of $\mu_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$, is called a *Neimark–Sacker bifurcation*, where $\mu_{1,2}$ are the eigenvalues of the nonhyperbolic fixed point of the system (2.2).

THEOREM 2.6 [35]. *Suppose that the system*

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} f(X_n, Y_n, \delta) \\ g(X_n, Y_n, \delta) \end{pmatrix},$$

with a parameter δ*, has a pair of complex conjugate eigenvalues* $\lambda_{1,2} = r(\delta)e^{\pm i\theta(\delta)}$ *at the fixed point* $(0, 0)$*, where* δ_c *is the critical parameter value,* $r(\delta_c) = 1$ *and* $\theta(\delta_c) = \theta_0$. *Assume that the following conditions are satisfied:*

(i) $d(r(\delta))/d\delta|_{\delta=\delta_c} \neq 0$*;*
(ii) $e^{ik\theta_0} \neq 1$ *for* $k = 1, 2, 3, 4$ *and* $a(\delta_c) = Re(e^{i\theta_0}c_1(\delta_c)) \neq 0$.

Then, there is a neighbourhood of origin in which a unique closed curve bifurcates from the origin as δ *passes through the critical value* δ_c*. It should be mentioned here that the sign of* $a(\delta_c)$ *determines the direction of the appearance of the invariant curve in a generic system exhibiting the Neimark–Sacker bifurcation: if* $a(\delta_c) < 0$*, then there is a supercritical Neimark–Sacker bifurcation which is stable; if* $a(\delta_c) > 0$*, then there is a subcritical Neimark–Sacker bifurcation which is unstable.*

3. Stability analysis

The difference system (1.8) has three equilibrium points given by

$$E_0 = (0,0), \quad E_1 = (1,0) \quad \text{and} \quad E_* = (M_*, N_*) \tag{3.1}$$

with

$$M_* = \frac{a_1 b_2}{b_1 - a_2 b_2}, \quad N_* = \frac{a_1 b_1 (b_1 - b_2(a_1 + a_2))}{(b_1 - a_2 b_2)^2}. \tag{3.2}$$

When simulating biological systems, positivity ensures that the model can make meaningful predictions about real-world behaviour. Negative values in simulations would lead to outcomes that are not interpretable or applicable to the biological phenomenon being studied. To ensure the positivity of the point $E_* = (M_*, N_*)$, it is assumed that

$$b_1 > b_2(a_1 + a_2). \tag{3.3}$$

Now, let us investigate the stability of each point.

3.1. Stability of E_0

THEOREM 3.1. $E_0 = (0,0)$ *is:*

(i) *saddle if* $0 < \rho < (2\Gamma(\alpha+1))/b_2$;
(ii) *source if* $\rho > (2\Gamma(\alpha+1))/b_2$,

where $\rho = h^\alpha$, *h is the discretization parameter and* α *is the order of the equation.*

PROOF. The Jacobian matrix at E_0 is

$$J(E_0) = \begin{pmatrix} \dfrac{\rho}{\Gamma(\alpha+1)} + 1 & 0 \\ 0 & 1 - \dfrac{\rho b_2}{\Gamma(\alpha+1)} \end{pmatrix}. \tag{3.4}$$

The characteristic polynomial corresponding to (3.4) is calculated as

$$F(\lambda) = \lambda^2 + \left(\frac{(b_2 - 1)\rho}{\Gamma(\alpha+1)} - 2\right)\lambda + \frac{(\Gamma(\alpha+1) + \rho)(\Gamma(\alpha+1) - b_2\rho)}{\Gamma(\alpha+1)^2},$$

whose eigenvalues are

$$\lambda_1 = 1 - \frac{b_2\rho}{\Gamma(\alpha+1)}, \quad \lambda_2 = 1 + \frac{\rho}{\Gamma(\alpha+1)} > 1.$$

Hence, we have

$$1 - \frac{b_2\rho}{\Gamma(\alpha+1)} \begin{cases} \in (-1,1) & \textit{if } 0 < \rho < \dfrac{2\Gamma(\alpha+1)}{b_2}, \\ < -1 & \textit{if } \rho > \dfrac{2\Gamma(\alpha+1)}{b_2}. \end{cases}$$

From Definition 2.1 and Lemma 2.2, the proof is complete. □

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3.2. Stability of E_1

THEOREM 3.2. *Assume that* $b_1 < (a_1 + a_2)b_2$*. Then,* $E_1 = (1, 0)$ *is:*

(i) *sink if* $0 < \rho < \min\{\delta_1, \delta_2\}$*;*
(ii) *saddle if* $\min\{\delta_1, \delta_2\} < \rho < \max\{\delta_1, \delta_2\}$*;*
(iii) *source if* $\rho > \max\{\delta_1, \delta_2\}$,

where

$$\delta_1 = 2\Gamma(\alpha+1), \quad \delta_2 = \frac{2(a_1+a_2)\Gamma(\alpha+1)}{(a_1+a_2)b_2 - b_1}.$$

PROOF. The Jacobian matrix at E_1 is obtained as

$$J(E_1) = \begin{pmatrix} 1 - \dfrac{\rho}{\Gamma(\alpha+1)} & -\dfrac{\rho}{\Gamma(\alpha+1)(a_1+a_2)} \\ 0 & 1 + \dfrac{(b_1 - (a_1+a_2)b_2)\rho}{(a_1+a_2)\Gamma(\alpha+1)} \end{pmatrix},$$

which has the characteristic polynomial

$$F(\lambda) = \lambda^2 + \Big(\frac{((a_1+a_2)b_2 + a_1 + a_2 - b_1)\rho}{(a_1+a_2)\Gamma(\alpha+1)} - 2\Big)\lambda + \frac{(\Gamma(\alpha+1) - \rho)((a_1+a_2)(\Gamma(\alpha+1) - b_2\rho) + b_1\rho)}{(a_1+a_2)\Gamma(\alpha+1)^2}.$$

Hence, the following eigenvalues are found:

$$\lambda_1 = 1 - \frac{\rho}{\Gamma(\alpha+1)}, \quad \lambda_2 = 1 - \frac{((a_1+a_2)b_2 - b_1)\rho}{(a_1+a_2)\Gamma(\alpha+1)}.$$

Since $b_1 < (a_1 + a_2)b_2$,

$$1 - \frac{\rho}{\Gamma(\alpha+1)} \begin{cases} \in (-1, 1) & \text{if } 0 < \rho < 2\Gamma(\alpha+1), \\ < -1 & \text{if } \rho > 2\Gamma(\alpha+1), \end{cases}$$

and

$$1 - \frac{((a_1+a_2)b_2 - b_1)\rho}{(a_1+a_2)\Gamma(\alpha+1)} \begin{cases} \in (-1, 1) & \text{if } 0 < \rho < \dfrac{2(a_1+a_2)\Gamma(\alpha+1)}{((a_1+a_2)b_2 - b_1)\rho}, \\ < -1 & \text{if } \rho > \dfrac{2(a_1+a_2)\Gamma(\alpha+1)}{((a_1+a_2)b_2 - b_1)\rho}. \end{cases}$$

So, Definition 2.1 and Lemma 2.2 complete the proof. □

3.3. Stability of E_*

THEOREM 3.3. *Assume that* $b_1(a_1 - a_2) + a_2b_2(a_1 + a_2) > 0$ *and* (3.3) *is true. Then, the positive equilibrium point* E_* *of system* (1.8) *is:*

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(i) *sink fixed point if one of the following conditions holds:*

$$0 < \rho < \rho_1 \quad \textit{when } \xi_2 > 4\xi_1;$$
$$0 < \rho < \frac{1}{\xi_1} \quad \textit{when } \xi_2 \leq 4\xi_1,$$

(ii) *source fixed point if one of the following conditions holds:*

$$\rho > \rho_2 \quad \textit{when } \xi_2 > 4\xi_1;$$
$$\rho > \frac{1}{\xi_1} \quad \textit{when } \xi_2 \leq 4\xi_1,$$

(iii) *saddle fixed point if the following condition holds:*

$$\rho_1 < \rho < \rho_2 \quad \textit{when } \xi_2 > 4\xi_1,$$

where $\rho = h^\alpha$ *and*

$$\begin{aligned} \xi_1 &= \frac{(b_1 - a_2 b_2)(b_1 - b_2(a_1 + a_2))}{\Gamma(\alpha+1)(b_1(a_1 - a_2) + a_2 b_2(a_1 + a_2))}, \\ \xi_2 &= \frac{b_2(b_1(a_1 - a_2) + a_2 b_2(a_1 + a_2))}{\Gamma(\alpha+1) b_1 (b_1 - a_2 b_2)}, \\ \rho_1 &= \frac{\xi_2 - \sqrt{\xi_2^2 - 4\xi_1\xi_2}}{\xi_1\xi_2}, \\ \rho_2 &= \frac{\xi_2 + \sqrt{\xi_2^2 - 4\xi_1\xi_2}}{\xi_1\xi_2}. \end{aligned} \tag{3.5}$$

PROOF. The Jacobian matrix at E_*, given in (3.1) and (3.2), is

$$J_* = \begin{pmatrix} 1 - \dfrac{\rho b_2((a_1 - a_2)b_1 + a_2(a_1 + a_2)b_2)}{\Gamma(\alpha+1) b_1 (b_1 - a_2 b_2)} & -\dfrac{\rho b_2}{\Gamma(\alpha+1) b_1} \\ \dfrac{\rho(b_1 - (a_1 + a_2)b_2)}{\Gamma(\alpha+1)} & 1 \end{pmatrix}, \tag{3.6}$$

whose characteristic equation is

$$\lambda^2 + \Phi\lambda + \Psi = 0, \tag{3.7}$$

where

$$\Phi = \Big(\frac{b_2((a_1 - a_2)b_1 + a_2(a_1 + a_2)b_2)\rho}{b_1(b_1 - a_2 b_2)\Gamma(\alpha+1)} - 2\Big),$$
$$\Psi = 1 + \frac{b_2\rho(\rho(b_1 - a_2 b_2)(b_1 - b_2(a_1 + a_2)) - \Gamma(\alpha+1)(b_1(a_1 - a_2) + a_2 b_2(a_1 + a_2)))}{b_1 \Gamma(\alpha+1)^2 (b_1 - a_2 b_2)}.$$

To apply Lemma 2.2(i), consider the left-hand side of (3.7) as a function of λ, that is,

$$F(\lambda) = \lambda^2 + \Phi\lambda + \Psi, \tag{3.8}$$

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or

$$F(\lambda) = \lambda^2 + (\rho\xi_2 - 2)\lambda + (\rho^2\xi_1\xi_2 - \rho\xi_2 + 1) \tag{3.9}$$

with $h^\alpha = \rho$, and ξ_1 *and* ξ_2 are given in (3.5). Equation (3.8) is called the characteristic polynomial corresponding to the positive equilibrium point E_*.

Taking $\lambda = 1$ and $\lambda = -1$ above, and considering Φ and Ψ, $F(1)$ and $F(-1)$ are calculated as

$$F(1) = \frac{b_2(b_1 - (a_1 + a_2)b_2)\rho^2}{b_1\Gamma(\alpha + 1)^2},$$
$$F(-1) = \frac{b_2(b_1 - (a_1 + a_2)b_2)}{b_1\Gamma^2(\alpha + 1)}\rho^2 - \frac{b_2((a_1 - a_2)b_1 + a_2(a_1 + a_2)b_2)}{b_1(b_1 - a_2b_2)\Gamma(\alpha + 1)}\rho + 4.$$

From (3.3), it is clear that $F(1) > 0$. To ensure the positivity of $F(-1)$, it is written as

$$\begin{aligned} F(-1) &= \xi_1\xi_2\rho^2 - 2\xi_2\rho + 4 \\ &= (\rho - \rho_1)(\rho - \rho_2), \end{aligned}$$

where ξ_1, ξ_2, ρ_1, ρ_2 are given in (3.5). So, the roots of $F(-1)$ are obtained as

$$\rho = \rho_1 = \frac{\xi_2 - \sqrt{\xi_2^2 - 4\xi_1\xi_2}}{\xi_1\xi_2} \quad \text{and} \quad \rho = \rho_2 = \frac{\xi_2 + \sqrt{\xi_2^2 - 4\xi_1\xi_2}}{\xi_1\xi_2}.$$

First, we assume that $\xi_2 > 4\xi_1$, then ρ_1 and ρ_2 are real and distinct. Under conditions (3.3) and $a_1 > a_2$, it is obtained that ρ_1 and ρ_2 are positive with $0 < \rho < \rho_1$ or $\rho > \rho_2$, and this implies that $F(-1) > 0$. Moreover, when $0 < \rho < \rho_1$, we get $\rho < \rho_1 < 1/\xi_1 < \rho_2$ and $C = \xi_1\xi_2\rho^2 - \xi_2\rho + 1 < 1$. When $\rho > \rho_2$, it is found $\rho > 1/\xi_1$ and then $C > 1$.

Second, if $\xi_2 = 4\xi_1$ and $\xi_2 < 4\xi_1$, then $F(-1)$ has real roots $\rho_1 = \rho_2 = 1/\xi_1$ and does not have real roots, respectively, and for each case, $F(-1)$ becomes positive. Also, if $0 < \rho < 1/\xi_1$, then $C < 1$ or if $\rho > 1/\xi_1$, then $C > 1$.

So, from Definition 2.1 and Lemma 2.2, the proof is complete. □

4. Bifurcation analysis

This section is devoted to the investigation of the flip and Neimark–Sacker bifurcation analysis of the equilibrium point E_* given in (3.1)–(3.2). Here, we choose ρ as a bifurcation parameter.

4.1. Flip bifurcation In this part, we study flip bifurcation for the system (1.8). We prove that under certain conditions, the flip bifurcation arises from the positive fixed point E_*.

THEOREM 4.1. *If the following conditions are satisfied by the characteristic polynomial* (3.8)*, then one may conclude that there is a flip bifurcation at E_*. When $\xi_2 > 4\xi_1$, then $F(-1)$ has the roots $\rho = \rho_1$ and $\rho = \rho_2$ such that*

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$$\rho \neq \frac{2}{\xi_2} \quad \textit{and} \quad \rho \neq \frac{4}{\xi_2},$$

where ρ_1 *and* ρ_2 *are given in* (3.5).

PROOF. From Lemma 2.2(ii), it is known that if

$$F(-1) = 0 \quad \text{and} \quad \rho\xi_2 - 2 \neq \{0, 2\},$$

then there may be flip bifurcation at the equilibrium point E_*. From Theorem 3.3, it is known that when $\xi_2 > 4\xi_1$, then $F(-1)$ has two real roots, which means that if we take $\rho = \rho_1$ or ρ_2, then we get $F(-1) = 0$. Moreover, considering $\rho \neq 2/\xi_2$ and $\rho \neq 4/\xi_2$ leads us to $\rho\xi_2 - 2 \neq \{0, 2\}$, and thus the proof is complete. □

However, let us take

$$\rho = \rho_1 = \frac{\xi_2 - \sqrt{\xi_2^2 - 4\xi_1\xi_2}}{\xi_1\xi_2}, \tag{4.1}$$

then the characteristic roots of (3.9) are found as $\lambda_1 = -1$ and $\lambda_2 = 3 - \rho_1\xi_2$, with $|\lambda_2| \neq 1$.

We will now focus on the analytic construction of the flip bifurcation, or, to put it another way, use Theorem 2.4 to establish the following theorem (Theorem 4.2) at the point E_*. However, to apply Theorem 2.4, we must first apply to our system (1.8) the centre manifold theorem [9, 35, 53].

THEOREM 4.2. *If the following conditions are satisfied, then there is a flip bifurcation for the system* (1.8)*:*

(i) $\xi_2 > 4\xi_1$;
(ii) $2m_1 + 2(h_1m_2 + m_5) \neq 0$;
(iii) $m_3 \neq 0$.

Furthermore, if $2m_1 + 2(h_1m_2 + m_5)$ *is positive, then the flip bifurcation is supercritical, otherwise it becomes subcritical. Here,* ξ_1 *and* ξ_2 *are given in* (3.5), *and* h_1, m_1, m_2, m_3, m_5 *are given in the following proof in* (4.6) *and* (4.8).

PROOF. For the application of the centre manifold theorem, let us define the right-hand side of the equations in system (1.8) as

$$\begin{cases} F(M_n, N_n) = M_n + \dfrac{\rho}{\Gamma(\alpha+1)}\left(M_n(1 - M_n) - \dfrac{M_nN_n}{a_1 + a_2M_n}\right), \\ G(M_n, N_n) = N_n + \dfrac{\rho}{\Gamma(\alpha+1)}\left(\dfrac{b_1M_nN_n}{a_1 + a_2M_n} - b_2N_n\right). \end{cases}$$

Expanding the Taylor series of these functions around the fixed point

$$E_* = (M_*, N_*) = \left(\frac{a_1b_2}{b_1 - a_2b_2}, \frac{a_1b_1(b_1 - b_2(a_1 + a_2))}{(b_1 - a_2b_2)^2}\right)$$

gives us the following equivalent system of (1.8):

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 1 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} f_1(X_n, Y_n, \tilde{\rho}, \bar{\rho}) \\ g_1(X_n, Y_n, \tilde{\rho}, \bar{\rho}) \end{pmatrix}, \tag{4.2}$$

where

$$\begin{aligned} B_{11} &= \frac{b_1\Gamma(\alpha+1)(b_1 - a_2b_2) - b_2\rho(a_1(a_2b_2 + b_1) + a_2(a_2b_2 - b_1))}{b_1\Gamma(\alpha+1)(b_1 - a_2b_2)}, \\ B_{12} &= -\frac{b_2\rho}{b_1\Gamma(\alpha+1)}, \\ B_{21} &= \frac{\rho(b_1 - b_2(a_1 + a_2))}{\Gamma(\alpha+1)}, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} f_1(X_n, Y_n, \tilde{\rho}, \bar{\rho}) = {} & \frac{(\bar{\rho} + \tilde{\rho})(b_1 - a_2b_2)^2}{a_1b_1^2\Gamma(\alpha+1)} X_nY_n - \frac{2a_2(\bar{\rho} + \tilde{\rho})(a_2b_2 - b_1)^3}{a_1^2b_1^3\Gamma(\alpha+1)} X_n^2Y_n \\ & - \frac{2(\bar{\rho} + \tilde{\rho})(a_1(-a_2^2b_2^2 + a_2b_1b_2 + b_1^2) - a_2(b_1 - a_2b_2)^2)}{a_1b_1^2\Gamma(\alpha+1)} X_n^2 \\ & - \frac{6a_2^2(\bar{\rho} + \tilde{\rho})(b_1 - a_2b_2)^2(b_2(a_1 + a_2) - b_1)}{a_1^2b_1^3\Gamma(\alpha+1)} X_n^3 + H.O.T., \\ g_1(X_n, Y_n, \tilde{\rho}, \bar{\rho}) = {} & \frac{(\bar{\rho} + \tilde{\rho})(b_1 - a_2b_2)^2}{a_1b_1\Gamma(\alpha+1)} X_nY_n + \frac{2a_2(\bar{\rho} + \tilde{\rho})(a_2b_2 - b_1)^3}{a_1^2b_1^2\Gamma(\alpha+1)} X_n^2Y_n \\ & + \frac{2a_2(\bar{\rho} + \tilde{\rho})(b_1 - a_2b_2)(b_2(a_1 + a_2) - b_1)}{a_1b_1\Gamma(\alpha+1)} X_n^2 \\ & - \frac{6a_2^2(\bar{\rho} + \tilde{\rho})(b_1 - a_2b_2)^2(b_2(a_1 + a_2) - b_1)}{a_1^2b_1^2\Gamma(\alpha+1)} X_n^3 + H.O.T., \end{aligned} \tag{4.4}$$

with $\rho = \rho_1$ as in (4.1), where ξ_1, ξ_2 are given in (3.5), and H.O.T. represents the higher order terms.

In this expansion, we take $\bar{\rho} = \rho_1$ and $M_n - M_* = X_n$, $N_n - N_* = Y_n$, $\rho - \bar{\rho} = \tilde{\rho}$, which transform the fixed point E_* to the origin and the bifurcation parameter's critical value to zero, that is, $\rho_c = 0$. Next, we construct the matrix

$$V = \begin{pmatrix} -2 & 2 - \bar{\rho}\xi_2 \\ B_{21} & B_{21} \end{pmatrix},$$

whose columns are the eigenvectors corresponding to the eigenvalues of J_*, given in (3.6), when $\rho = \bar{\rho} = \rho_1$, which are $\lambda_1 = -1$ and $\lambda_2 = 3 - \rho_1\xi_2$. Here, ρ_1 is given in (4.1) with ξ_1 and ξ_2 as in (3.5), and B_{21} is given in (4.3). Using the transformation

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = V \begin{pmatrix} U_n \\ V_n \end{pmatrix}$$

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in (4.2), the following system is obtained:

$$\begin{pmatrix} U_{n+1} \\ V_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3-\rho_1\xi_2 \end{pmatrix}\begin{pmatrix} U_n \\ V_n \end{pmatrix} + \begin{pmatrix} f_2(U_n,V_n) \\ g_2(U_n,V_n) \end{pmatrix} \quad \text{with} \tag{4.5}$$

$$\begin{aligned} f_2(U_n,V_n) &= m_1U_n^3 + m_2U_nV_n^2 + m_3U_n^2V_n + m_4V_n^3 + m_5U_n^2 \\ &\quad + m_6U_nV_n + m_7V_n^2 + H.O.T., \\ g_2(U_n,V_n) &= n_1U_n^3 + n_2U_nV_n^2 + n_3U_n^2V_n + n_4V_n^3 + n_5U_n^2 + n_6U_nV_n + n_7V_n^2 + H.O.T., \end{aligned}$$

where

$$\begin{aligned} m_1 &= (a_1^2b_1^3\Gamma(\alpha+1))^{-1}8a_2(\bar{\rho}+\tilde{\rho})(b_1-a_2b_2)^2(b_1T_{12}-T_{11})(a_2(b_2(6a_1+B_{21})-6b_1) \\ &\quad + 6a_2^2b_2 - b_1B_{21}), \\ m_2 &= (a_1^2b_1^3\Gamma(\alpha+1))^{-1}(a_1^2b_1^3\Gamma(\alpha+1))^{-1}2a_2(\bar{\rho}+\tilde{\rho})(r_2\bar{\rho}-2)(b_1-a_2b_2)^2(b_1T_{12}-T_{11}) \\ &\quad \times(r_2\bar{\rho}(a_2(18a_1b_2-18b_1+b_2B_{21})+18a_2^2b_2-b_1B_{21}+2a_2(-18a_1b_2+18b_1 \\ &\quad + b_2B_{21})-36a_2^2b_2-2b_1B_{21})a_1+B_{21}))-18a_2^2b_2+b_1B_{21}), \\ m_3 &= (a_1^2b_1^3\Gamma(\alpha+1))^{-1}8a_2(\bar{\rho}+\tilde{\rho})(b_1-a_2b_2)^2(b_1T_{12}-T_{11})(\bar{\rho}+\tilde{\rho})(r_2\bar{\rho}(a_2(b_2(9a_1 \\ &\quad + B_{21})-9b_1)+9a_2^2b_2-b_1B_21)+a_2(18b_1-b_2(18a_1+B_{21}))-18a_2^2b_2+b_1B_{21}), \\ m_4 &= (a_1^2b_1^3\Gamma(\alpha+1))^{-1}(2a_2(\bar{\rho}+\tilde{\rho})(r_2\bar{\rho}-2)^2(b_1-a_2b_2)^2(b_1T_{12}-T_{11}) \\ &\quad \times(3a_2r_2\bar{\rho}(b_2(a_1+a_2)-b_1)+a_2(-6a_1b_2+6b_1+b_2B_{21})-6a_2^2b_2-b_1B_{21})), \\ m_5 &= -(a_1b_1^2\Gamma(\alpha+1))(2(\bar{\rho}+\tilde{\rho})(4a_1(-a_2^2b_2^2T_{11}+b_1^2(T_{11}-a_2b_2T_{12}) \\ &\quad + a_2b_1b_2(a_2b_2T_{12}+T_{11}))+(4a_2+B_{21})(b_1-a_2b_2)^2(b_1T_{12}-T_{11}))), \\ m_6 &= -(a_1b_1^2\Gamma(\alpha+1))^{-1}((\bar{\rho}+\tilde{\rho})(r_2\bar{\rho}(8a_1(-a_2^2b_2^2T_{11}+b_1^2(T_{11}-a_2b_2T_{12}) \\ &\quad + a_2b_1b_2(a_2b_2T_{12}+T_{11}))+(8a_2+B_{21})(b_1-a_2b_2)^2(b_1T_{12}-T_{11})) \\ &\quad - 16(a_1(-a_2^2b_2^2T_{11}+b_1^2(T_{11}-a_2b_2T_{12})+a_2b_1b_2(a_2b_2T_{12}+T_{11})) \\ &\quad + a_2(b_1-a_2b_2)^2(b_1T_{12}-T_{11})))), \\ m_7 &= -(a_1b_1^2\Gamma(\alpha+1))^{-1}((\bar{\rho}+\tilde{\rho})(\xi_2\bar{\rho}-2)(2\xi_2\bar{\rho}(a_1(-a_2^2b_2^2T_{11}+b_1^2(T_{11}-a_2b_2T_{12}) \\ &\quad + a_2b_1b_2(a_2b_2T_{12}+T_{11}))+a_2(b_1-a_2b_2)^2(b_1T_{12}-T_{11})), \\ &\quad - 4a_1(-a_2^2b_2^2T_{11}+b_1^2(T_{11}-a_2b_2T_{12})+a_2b_1b_2(a_2b_2T_{12}+T_{11})) \\ &\quad + (4a_2-B_{21})(-(b_1-a_2b_2)^2)(b_1T_{12}-T_{11}))), \end{aligned}$$

$$\begin{aligned} n_1 &= (a_1^2b_1^3\Gamma(\alpha+1))^{-1}(8a_2(\bar{\rho}+\tilde{\rho})(b_1-a_2b_2)^2(b_1T_{22}-T_{21})(a_2(b_2(6a_1+B_{21})-6b_1) \\ &\quad + 6a_2^2b_2-b_1B_{21})), \\ n_2 &= (a_1^2b_1^3\Gamma(\alpha+1))^{-1}(2a_2(\bar{\rho}+\tilde{\rho})(r_2\bar{\rho}-2)(b_1-a_2b_2)^2(b_1T_{22}-T_{21}), \\ &\quad \times(r_2\bar{\rho}(a_2(18a_1b_2-18b_1+b_2B_{21})+18a_2^2b_2-b_1B_{21}) \\ &\quad + 2a_2(-18a_1b_2+18b_1+b_2B_{21})-36a_2^2b_2-2b_1B_{21})), \end{aligned}$$

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$$
\begin{aligned}
n_3 &= (a_1^2 b_1^3 \Gamma(\alpha+1))^{-1}(8a_2(\bar{\rho}+\tilde{\rho})(b_1-a_2b_2)^2(b_1T_{22}-T_{21})(r_2\bar{\rho}(a_2(b_2(9a_1+B_{21})-9b_1) \\
&\quad + 9a_2^2 b_2 - b_1 B_{21}) + a_2(18b_1 - b_2(18a_1+B_{21})) - 18a_2^2 b_2 + b_1 B_{21})), \\
n_4 &= (a_1^2 b_1^3 \Gamma(\alpha+1))^{-1}(2a_2(\bar{\rho}+\tilde{\rho})(r_2\bar{\rho}-2)^2(b_1-a_2b_2)^2(b_1T_{22}-T_{21}) \\
&\quad \times (3a_2 r_2 \bar{\rho}(b_2(a_1+a_2)-b_1) + a_2(-6a_1b_2 + 6b_1 + b_2B_{21}) - 6a_2^2 b_2 - b_1 B_{21})), \\
n_5 &= -((a_1 b_1^2 \Gamma(\alpha+1))^{-1}(2(\bar{\rho}+\tilde{\rho})(4a_1(-a_2^2b_2^2T_{21} + b_1^2(T_{21}-a_2b_2T_{22}) \\
&\quad + a_2b_1b_2(a_2b_2T_{22}+T_{21})) + (4a_2+B_{21})(b_1-a_2b_2)^2(b_1T_{22}-T_{21}))), \\
n_6 &= -(a_1 b_1^2 \Gamma(\alpha+1))^{-1}((\bar{\rho}+\tilde{\rho})(r_2\bar{\rho}(8a_1(-a_2^2b_2^2T_{21} + b_1^2(T_{21}-a_2b_2T_{22}) \\
&\quad + a_2b_1b_2(a_2b_2T_{22}+T_{21})) + (8a_2+B_{21})(b_1-a_2b_2)^2(b_1T_{22}-T_{21})) \\
&\quad - 16(a_1(-a_2^2b_2^2T_{21} + b_1^2(T_{21}-a_2b_2T_{22}) + a_2b_1b_2(a_2b_2T_{22}+T_{21})) \\
&\quad + a_2(b_1-a_2b_2)^2(b_1T_{22}-T_{21})))), \\
n_7 &= -(a_1 b_1^2 \Gamma(\alpha+1))^{-1}((\bar{\rho}+\tilde{\rho})(\xi_2\bar{\rho}-2)(2\xi_2\bar{\rho}(a_1(-a_2^2b_2^2T_{21} + b_1^2(T_{21}-a_2b_2T_{22}) \\
&\quad + a_2b_1b_2(a_2b_2T_{22}+T_{21})) + a_2(b_1-a_2b_2)^2(b_1T_{22}-T_{21})) \\
&\quad - 4a_1(-a_2^2b_2^2T_{21} + b_1^2(T_{21}-a_2b_2T_{22}) + a_2b_1b_2(a_2b_2T_{22}+T_{21})) \\
&\quad + (4a_2-B_{21})(-(b_1-a_2b_2)^2)(b_1T_{22}-T_{21}))),
\end{aligned} \tag{4.6}
$$

with

$$
\begin{aligned}
T_{11} &= \frac{B_{21}}{-4B_{21} + B_{21}\bar{\rho}\xi_2}, \\
T_{12} &= \frac{-2+\bar{\rho}\xi_2}{-4B_{21} + B_{21}\bar{\rho}\xi_2}, \\
T_{21} &= \frac{-B_{21}}{-4B_{21} + B_{21}\bar{\rho}\xi_2}, \\
T_{22} &= \frac{-2}{-4B_{21} + B_{21}\bar{\rho}\xi_2},
\end{aligned}
$$

and B_{21} is given in (4.3). Applying centre manifold theorem [9, 35, 53] to the system (4.5), one may obtain the one-dimensional system

$$
U_{n+1} = -U_n + \tilde{\rho} m_3 U_n + (m_1 + \tilde{\rho} h_1 m_4 + \tilde{\rho} m_6) U_n^2 + (h_1 m_2 + m_5 + \tilde{\rho} m_4) U_n^3 \tag{4.7}
$$

with

$$
h_1 = \frac{m_1}{1-\lambda_2} \quad \text{and} \quad h_2 = \frac{-m_3}{1+\lambda_2}. \tag{4.8}
$$

Now, let us apply Theorem 2.4 to (4.7). Considering the right-hand side of (4.7) as $f(U_n, \tilde{\rho})$, we calculate that

$$\frac{\partial f}{\partial U_n}(0,0) = -1,$$
$$\frac{1}{2}\Big(\frac{\partial^2 f}{\partial U_n^2}(0,0)\Big)^2 + \frac{1}{3}\frac{\partial^3 f}{\partial U_n^3}(0,0) = 2m_1^2 + 2(h_1m_2 + m_5) \neq 0,$$
$$\frac{\partial^2 f}{\partial U_n \partial \tilde{\rho}} = m_3 \neq 0.$$

So, from Theorem 2.4, we may conclude that there exists a flip bifurcation for the system (1.8) under the conditions (i), (ii), (iii) in Theorem 4.2. So the proof of Theorem 4.2 is complete. □

4.2. Neimark–Sacker bifurcation This section focuses on the construction of the Neimark–Sacker bifurcation at the point

$$E_* = (M_*, N_*) = \Big(\frac{a_1b_2}{b_1 - a_2b_2}, \frac{a_1b_1(b_1 - b_2(a_1 + a_2))}{(b_1 - a_2b_2)^2}\Big)$$

with the help of condition (v) in Lemma 2.2.

THEOREM 4.3. *If*

$$\rho = \frac{1}{\xi_1} \quad \textit{and} \quad \frac{\xi_2}{4} < \xi_1, \tag{4.9}$$

then there exists a Neimark–Sacker bifurcation for the equilibrium point E_*.

PROOF. It is known that if the characteristic roots of the Jacobian matrix at E_* are complex conjugate with modulus 1, then there may be a Neimark–Sacker bifurcation at this point. In our problem, the characteristic equation is given by

$$\lambda^2 - (2 - \rho\xi_2)\lambda + (\xi_1\xi_2\rho^2 - \xi_2\rho + 1) = 0.$$

Under (4.9), conditions given in Lemma 2.2(v) are satisfied and the proof is complete. □

To show the existence of the Neimark–Sacker bifurcation, we apply Theorem 2.6. For this aim, let us obtain the characteristic roots and the bifurcation parameter explicitly under the condition $b_1(a_1 - a_2) + a_2b_2(a_1 + a_2) > 0$. The complex characteristic roots are in the form

$$\lambda_{1,2}(\rho) = \alpha(\rho) \pm i\beta(\rho) = r(\rho)e^{\pm i\theta(\rho)}$$

with

$$\alpha(\rho) = \frac{2 - \xi_2\rho}{2}, \qquad \beta(\rho) = \frac{\rho\sqrt{\xi_2(4\xi_1 - \xi_2)}}{2},$$
$$r(\rho) = \sqrt{\alpha^2(\rho) + \beta^2(\rho)}, \quad \theta(\rho) = \arctan\Big(\frac{\beta(\rho)}{\alpha(\rho)}\Big).$$

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Note that when $\rho = 1/\xi_1$,

$$|\lambda_{1,2}| = r(\rho) = 1, \quad \theta(\rho) = \arctan\left(\frac{\sqrt{\xi_2(4\xi_1 - \xi_2)}}{2 - \xi_2\rho}\right) = \theta_0,$$

and our critical bifurcation parameter is $\rho_c = 1/\xi_1$ and

$$\frac{d(r(\rho))}{d\rho}\bigg|_{\rho_c = 1/\xi_1} = \frac{\xi_2}{2} > 0,$$

because of the condition $a_1 > a_2$. Furthermore, since $\xi_2 > 0$, it is seen that $(\lambda_{1,2}(\rho_c))^k = e^{\pm ik\theta=0} \neq 1$ for $k = 1, 2, 3, 4$. So, we conclude that Neimark–Sacker bifurcation occurs as for the fixed point E_* as ρ passes $\rho_c = 1/\xi_1$ from left to right.

To construct the normal form of this bifurcation, let us consider the system (4.2) again as follows:

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 1 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} f_1(X_n, Y_n, \tilde{\rho}, \bar{\rho}) \\ g_1(X_n, Y_n, \tilde{\rho}, \bar{\rho}) \end{pmatrix}, \tag{4.10}$$

where B_{11}, B_{12}, B_{21} and $f_1(X_n, Y_n, \tilde{\rho}, \bar{\rho})$, $g_1(X_n, Y_n, \tilde{\rho}, \bar{\rho})$ are given in (4.3) and (4.4), respectively. With this system, we transform the fixed point E_* to the origin and the bifurcation parameter $\rho_c = \bar{\rho}$ to zero.

Now, let us consider the following Jacobian matrix:

$$J(\bar{\rho}) = J\left(\frac{1}{\xi_1}\right) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} \quad \text{where}$$

$$\begin{aligned}
j_{11} &= \frac{1}{b_1(b_1 - a_2b_2)^2(b_1 - b_2(a_1 + a_2))}(b_1^2b_2(-a_1^2 + 2a_1a_2(b_2 + 1) + a_2^2(3b_2 - 1)) \\
&\quad - a_2^2b_2^3(a_1 + a_2)^2 - b_1^3b_2(a_1 + 3a_2) - a_2b_1b_2^2(a_1 + a_2)(2a_1 + a_2(b_2 - 2)) + b_1^4), \\
j_{12} &= -\frac{b_2(b_1(a_1 - a_2) + a_2b_2(a_1 + a_2))}{b_1(b_1 - a_2b_2)(b_1 - b_2(a_1 + a_2))}, \\
j_{21} &= \frac{b_1(a_1 - a_2) + a_2b_2(a_1 + a_2)}{b_1 - a_2b_2}, \\
j_{22} &= 1.
\end{aligned}$$

We now create a matrix

$$V = \begin{pmatrix} -j_{12} & 0 \\ j_{11} - \mu & \omega \end{pmatrix},$$

whose columns are made up of the real and imaginary parts of an eigenvector that correspond to $\lambda_1 = \mu - i\omega$, where

$$\mu = 1 - \frac{\xi_2}{2\xi_1} \quad \text{and} \quad \omega = \frac{\sqrt{\xi_2(4\xi_1 - \xi_2)}}{2\xi_1}.$$

Next, we apply the transformation

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = V \begin{pmatrix} U_n \\ V_n \end{pmatrix}$$

to the system (4.10) and procure the following system:

$$\begin{pmatrix} U_{n+1} \\ V_{n+1} \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \begin{pmatrix} U_n \\ V_n \end{pmatrix} + \begin{pmatrix} H^1(U_n, V_n) \\ H^2(U_n, V_n) \end{pmatrix},$$

where

$$\begin{aligned} H^1(U_n, V_n) &= h_{11} U_n^3 + h_{12} U_n^2 V_n + h_{13} U_n^2 + h_{14} U_n V_n, \\ H^2(U_n, V_n) &= h_{21} U_n^3 + h_{22} U_n^2 V_n + h_{23} U_n^2 + h_{24} U_n V_n, \end{aligned}$$

with

$$\begin{aligned} h_{11} &= -\frac{1}{a_1^2 b_1^6 \xi_1^4 \Gamma(\alpha+1)^4} (2a_2 b_2^2 (b_1 - a_2 b_2)^2 (b_2 (a_1 (b_1 - 2a_2 b_2) + 2a_2 (b_1 - a_2 b_2)) \\ &\quad + b_1 (\mu - 1) \xi_1 \Gamma(\alpha+1)(b_1 - a_2 b_2))), \\ h_{12} &= \frac{2a_2 b_2^2 \omega (b_1 - a_2 b_2)^3}{a_1^2 b_1^5 \xi_1^3 \Gamma(\alpha+1)^3}, \\ h_{13} &= -\frac{b_2}{a_1 b_1^4 \xi_1^3 \Gamma(\alpha+1)^3} ((b_2 (a_1 (a_2^2 b_2^2 - 2a_2 b_1 b_2 + 3b_1^2) + a_2 (b_1 - a_2 b_2)^2) \\ &\quad + b_1 (\mu - 1) \xi_1 \Gamma(\alpha+1)(b_1 - a_2 b_2)^2)), \\ h_{14} &= -\frac{b_2 \omega (b_1 - a_2 b_2)^2}{a_1 b_1^3 \xi_1^2 \Gamma(\alpha+1)^2}, \end{aligned}$$

and

$$\begin{aligned} h_{21} &= \frac{1}{a_1^2 b_1^5 \xi_1^4 \Gamma(\alpha+1)^4} (2a_2 b_2^2 (b_1 - a_2 b_2)^2 (b_2 (a_1 (b_1 - 2a_2 b_2) + 2a_2 (b_1 - a_2 b_2)) \\ &\quad + b_1 (\mu - 1) \xi_1 \Gamma(\alpha+1)(b_1 - a_2 b_2))), \\ h_{22} &= \frac{2a_2 b_2^2 \omega (a_2 b_2 - b_1)^3}{a_1^2 b_1^4 \xi_1^3 \Gamma(\alpha+1)^3}, \\ h_{23} &= \frac{b_2}{a_1 b_1^3 \xi_1^3 \Gamma(\alpha+1)^3} ((b_2 (a_1 (b_1 - a_2 b_2)^2 + a_2 (a_2^2 b_2^2 - 2a_2 b_1 b_2 - 3b_1^2)) \\ &\quad + b_1 (\mu - 1) r_1 \Gamma(\alpha+1)(b_1 - a_2 b_2)^2)), \\ h_{24} &= \frac{b_2 \omega (b_1 - a_2 b_2)^2}{a_1 b_1^2 \xi_1^2 \Gamma(\alpha+1)^2}. \end{aligned}$$

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However, the direction of this bifurcation is calculated with the help of the formula [21]

$$a\Big(\frac{1}{\xi_1}\Big) = -Re\Big(\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}\zeta_{20}\zeta_{11}\Big) - \frac{1}{2}|\zeta_{11}|^2 - |\zeta_{02}|^2 + Re(\bar{\lambda}\zeta_{21}) \tag{4.11}$$

with the following functions, which are all calculated at the point $(0, 0, 1/\xi_1)$:

$$\begin{aligned}
\zeta_{20} &= \tfrac{1}{8}[(H^1_{UU} - H^1_{VV} + 2H^2_{UV}) + i(H^2_{UU} - H^2_{VV} - 2H^1_{UV})],\\
\zeta_{11} &= \tfrac{1}{4}[(H^1_{UU} + H^1_{VV}) + i(H^2_{UU} + H^2_{VV})],\\
\zeta_{02} &= \tfrac{1}{8}[(H^1_{UU} - H^1_{VV} - 2H^2_{UV}) + i(H^2_{UU} - H^2_{VV} + 2H^1_{UV})],\\
\zeta_{21} &= \tfrac{1}{16}[(H^1_{UUU} + H^1_{UVV} + H^2_{UUV} + H^2_{VVV}) + i(H^2_{UUU} + H^2_{UVV} - H^1_{UUV} - H^1_{VVV})],
\end{aligned}$$

and it is said that the bifurcation is *supercritical* if $a(1/\xi_1) < 0$ and *subcritical* if $a(1/\xi_1) > 0$. As a result of the above discussion, we state the following theorem.

THEOREM 4.4. *If $\xi_2/4 < \xi_1$ and $a(1/\xi_1) \neq 0$, then system (1.8) has a Neimark–Sacker bifurcation at the fixed point $E_* = (M_*, N_*)$ and this bifurcation is supercritical if $a(1/\xi_1) < 0$ and subcritical if $a(1/\xi_1) > 0$.*

5. Examples

This section provides two numerical examples to strengthen and broaden the conclusions drawn from the theory in the other sections.

EXAMPLE 5.1. This example is constructed by choosing the parameters in [7]. Here, we obtain the system

$$\begin{cases}
M_{n+1} = M_n + 2.11101\Big(M_n(1-M_n) - \dfrac{M_nN_n}{26+39.2308M_n}\Big),\\
N_{n+1} = N_n + 2.11101\Big(\dfrac{8M_nN_n}{26+39.2308M_n} - 0.12N_n\Big)
\end{cases}$$

with the initial conditions

$$M_0 = 0.1 \quad \text{and} \quad N_0 = 0.1.$$

The equilibrium point for this system is determined to be $E_* = (0.947664, 3.30649)$. As the first condition in Theorem 3.3 holds, we conclude that this point is a sink as illustrated in Figure 1(a) and the phase portrait of the system is given in Figure 1(b).

However, since the conditions of Theorem 4.2 are satisfied, we have established that there exists a flip bifurcation when ρ exceeds the critical value 1.93915. This can also be observed in Figures 2(a) and 2(b), where the flip bifurcation takes place within the range $[0, 3]$, which includes the critical value $\rho_c = 1.93915$.

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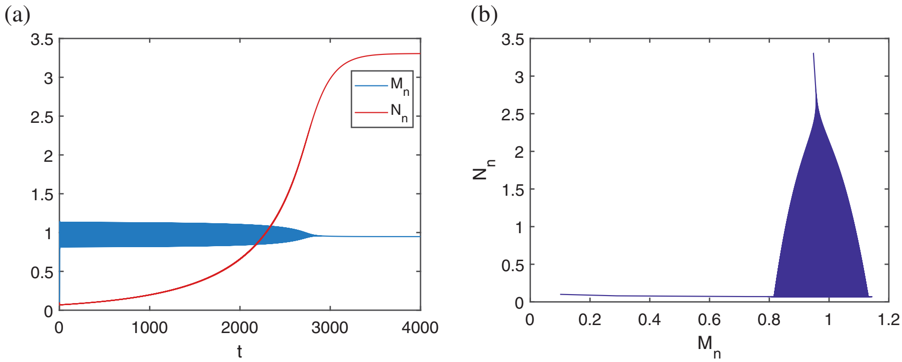

FIGURE 1. (a) Stability of $E_* = (0.947664, 3.30649)$ of system (1.8) and (b) phase portrait for (M_n, N_n).

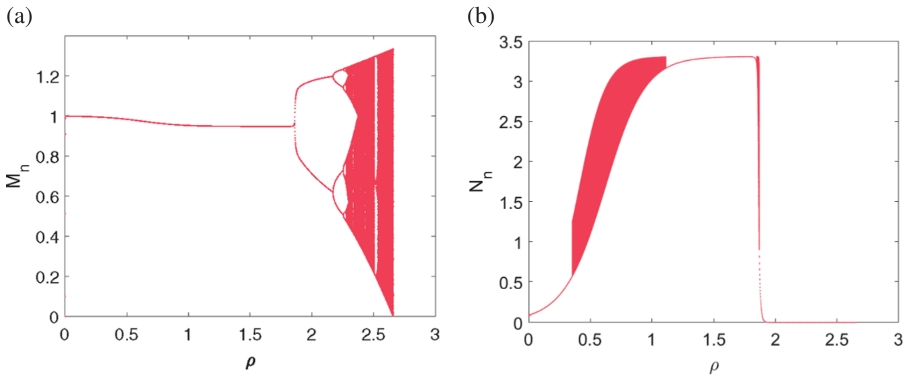

FIGURE 2. Bifurcation diagrams in $\rho \in [0, 3]$ with initial condition $(M_0, N_0) = (0.1, 0.1)$.

EXAMPLE 5.2. Let us consider the following system:

$$\begin{cases} M_{n+1} = M_n + 1.5957\Big(M_n(1 - M_n) - \dfrac{M_n N_n}{26 + 39.2308 M_n}\Big), \\ N_{n+1} = N_n + 1.5957\Big(\dfrac{8 M_n N_n}{26 + 39.2308 M_n} - 0.06 N_n\Big) \end{cases}$$

with the initial conditions

$$M_0 = 0.1 \quad \text{and} \quad N_0 = 0.1.$$

The equilibrium $E_* = (0.276294, 26.6608)$ of this system is a sink, as it satisfies the first condition of Theorem 3.3. This can be seen in Figure 3(a) and the phase portrait of the system is given in Figure 3(b). Meanwhile, since the conditions in Theorem 2.6 are true, there exists a Neimark–Sacker bifurcation at $\rho_c = 1.83218$. This bifurcation is supercritical in view of (4.11), which is calculated as $-0.000042416 < 0$.

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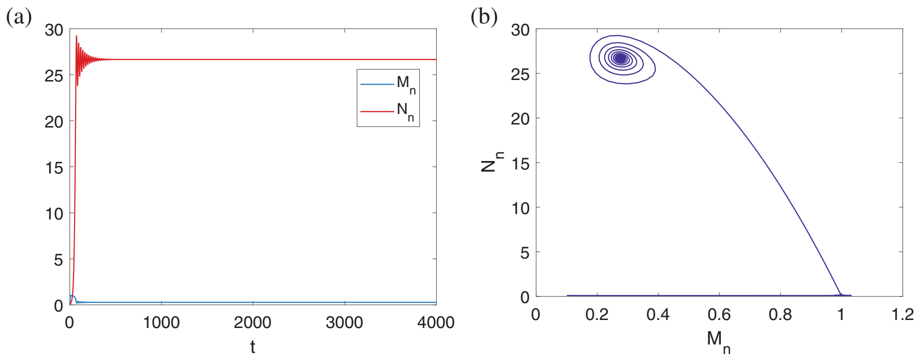

FIGURE 3. (a) Stability of $E_* = (0.276294, 26.6608)$ of system (1.8) and (b) phase portrait for (M_n, N_n).

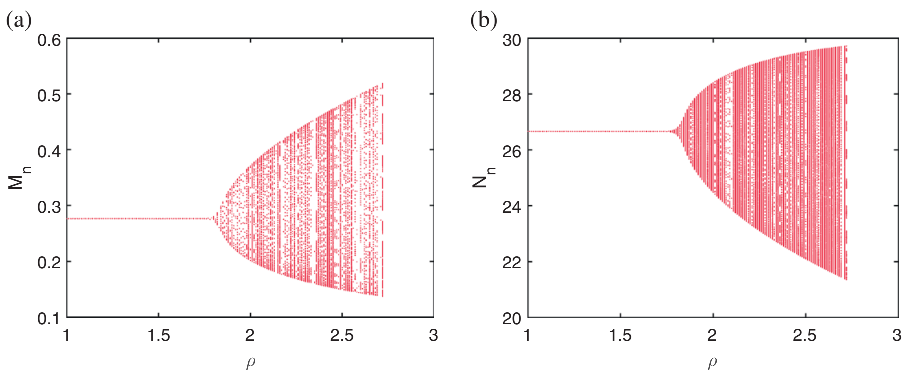

FIGURE 4. Bifurcation diagrams in $\rho \in [0, 3]$ with initial condition $(M_0, N_0) = (0.1, 0.1)$.

This behaviour is presented in Figures 4(a) and 4(b), which occurs within the range $[0, 3]$.

6. Conclusion

This paper explores the bifurcation properties of a fractional-order prey–predator model influenced by the wind, which is one of the most important abiotic factors often overlooked in ecological studies. The model was discretized using a piecewise discretization approach, allowing a detailed analysis of its discrete-time dynamics. We choose $\rho = h^\alpha$ as the bifurcation parameter that consists of both the discretization parameter h and the fractional order α of the system. The local stability of fixed points was investigated, and it was shown using the centre-manifold theorem and bifurcation theory that the system experiences both flip and Neimark–Sacker bifurcations at a positive fixed point. Numerical simulations were provided to illustrate and validate the theoretical results.

This work underscores the importance of including the abiotic factors such as wind in ecological models and demonstrates the advantages of fractional-order systems for capturing complex dynamics. Future research could focus on extending the model to include additional ecological interactions or exploring control strategies for practical applications. Further, validating the theoretical findings with empirical data could enhance the model's relevance and applicability to the real-world ecological scenarios.

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References

[1] A. Afreen and A. Raheem, "Study of a nonlinear system of fractional differential equations with deviated arguments via Adomian decomposition method", *Int. J. Appl. Comput. Math.* **8** (2022) Article ID: 269; doi:10.1007/s40819-022-01464-5.

[2] R. P. Agarwal, A. El-Sayed and S. M. Salman, "Fractional-order Chua's system: discretization, bifurcation and chaos", *Adv. Difference Equ.* **2013** (2013) 1–13; doi:10.1186/1687-1847-2013.

[3] M. Akhmet, *Nonlinear hybrid continuous/discrete-time models*, Volume 8 in *Atlantis Stud. Math. Eng. Sci.* (ed. C. K. Chui) (Atlantis Press, Amsterdam–Paris, 2011).

[4] M. U. Akhmet, "Integral manifolds of differential equations with piecewise constant argument of generalized type", *Nonlinear Anal. Theory Methods Appl.* **66** (2007) 367–383; doi:10.1016/j.na.2005.11.

[5] M. U. Akhmet, D. Aruğaslan and E. Yılmaz, "Stability in cellular neural networks with a piecewise constant argument", *J. Comput. Appl. Math.* **233** (2010) 2365–2373; doi:10.1016/j.cam.2009.10.021.

[6] D. Barman, J. Roy and S. Alam, "Trade-off between fear level induced by predator and infection rate among prey species", *J. Appl. Math. Comput.* **64** (2020) 635–663; doi:10.1007/s12190-020-01372-1.

[7] D. Barman, J. Roy and S. Alam, "Impact of wind in the dynamics of prey–predator interactions", *Math. Comput. Simul.* **191** (2022) 49–81; doi:10.1016/j.matcom.2021.07.022.

[8] D. Barman, J. Roy, H. Alrabaiah, P. Panja, S. P. Mondal and S. Alam, "Impact of predator incited fear and prey refuge in a fractional order prey predator model", *Chaos Solitons Fractals* **142** (2021) Article ID: 110420; doi:10.1016/j.chaos.2020.110420.

[9] P. Baydemir, H. Merdan, E. Karaoglu and G. Sucu, "Complex dynamics of a discrete-time prey–predator system with Leslie type: stability, bifurcation analyses and chaos", *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **30** (2020) Article ID: 2050149; doi:10.1142/s0218127420501497.

[10] S. Busenberg and K. L. Cooke, "Models of vertically transmitted diseases with sequentialcontinuous dynamics", in: *Nonlinear phenomena in mathematical sciences* (ed. V. Lakshmikantham) (Elsevier, London, 1982) 179–187; doi:10.1016/B978-0-12-434170-8.50028-5.

[11] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent—II", *Geophys. J. Int.* **13** (1967) 529–539; doi:10.1111/j.1365-246X.1967.tb02303.x.

[12] R. Charles, O. D. Makinde and M. Kuńgaro, "A review of the mathematical models for the impact of seasonal weather variation and infections on prey predator interactions in Serengeti ecosystem", *Open J. Ecol.* **12** (2022) 718–732; doi:10.4236/oje.2022.1211041.

[13] K. S. Chiu, "Stability of oscillatory solutions of differential equations with a general piecewise constant argument", *Electron. J. Qual. Theory Differ. Equ.* **88** (2011) 1–15; doi:10.14232/ejqtde.2011.1.88.

[14] K. S. Chiu, "Periodicity and stability analysis of impulsive neural network models with generalized piecewise constant delays", *Discrete Contin. Dyn. Syst. Ser. B* **27** (2022) 659–689; doi:10.3934/dcdsb.2021060.

[15] K. S. Chiu and M. Pinto, "Variation of parameters formula and Gronwall inequality for differential equations with a general piecewise constant argument", *Acta Math. Sin. (Engl. Ser.)* **27** (2011) 561–568; doi:10.1007/s10255-011-0107-5.

[16] K. L. Cooke and I. Györi, "Numerical approximation of the solutions of delay differential equations on an infinite interval using piecewise constant arguments", *Comput. Math. Appl.* **28** (1994) 81–92; doi:10.1016/0898-1221(94)00095-6.

[17] K. L. Cooke and J. Wiener, "A survey of differential equations with piecewise continuous arguments", in: *Delay differential equations and dynamical systems: proceedings of a conference in honor of Kenneth Cooke, Claremont, California, January 13–16, 1990* (eds. S. Busenberg and M. Martelli) (Springer, Berlin–Heidelberg, 2006) 1–15; doi:10.1007/bfb0083475.

[18] K. Diethelm and G. Walz, "Numerical solution of fractional order differential equations by extrapolation", *Numer. Algorithms* **1997**(16) (1997) 231–253; doi:10.1023/a:1019147432240.

[19] Z. F. El Raheem and S. M. Salman, "On a discretization process of fractional-order logistic differential equation", *J. Egyptian Math. Soc.* **22** (2014) 407–412; doi:10.1016/j.joems.2013.09.001.

[20] A. M. A. El-Sayed and S. M. Salman, "On a discretization process of fractional-order Riccati differential equation", *J. Fract. Calc. Appl.* **4** (2013) 251–259; https://jfca.journals.ekb.eg/article_283803_1c7327b0a18a6fa1304a51f51c55585f.pdf.

[21] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Volume 742 of *Appl. Math. Sci.* (Springer Science & Business Media, New York, 2013).

[22] P. Guo, "The Adomian decomposition method for a type of fractional differential equations", *J. Appl. Math. Phys.* **7** (2019) 2459–2466; doi:10.4236/jamp.2019.710166.

[23] F. Gürcan, G. Kaya and S. Kartal, "Conformable fractional order Lotka–Volterra predator–prey model: discretization, stability and bifurcation", *J. Comput. Nonlinear Dyn.* **14** (2019) Article ID: 111007; doi:10.1115/1.4044313.

[24] I. Györi, "On approximation of the solutions of delay differential equations by using piecewise constant arguments", *Int. J. Math. Math. Sci.* **14** (1991) 111–16; doi: 10.1155/S016117129100011X.

[25] I. Györi and F. Hartung, "Numerical approximation of neutral differential equations on infinite interval", *J. Difference Equ. Appl.* **8** (2002) 983–999; doi:10.1080/10236190290015263.

[26] I. Györi and F. Hartung, "On numerical approximation using differential equations with piecewiseconstant arguments", *Period. Math. Hungar.* **56** (2008) 55–69; doi:10.1007/s10998-008-5055-5.

[27] I. Györi, F. Hartung and J. Turi, "Numerical approximations for a class of differential equations with time-and state-dependent delays", *Appl. Math. Lett.* **8** (1995) 19–24; doi:10.1016/0893-9659(95)00079-6.

[28] L. V. C. Hoan, M. A. Akinlar, M. Inc, J. F. Gómez-Aguilar, Y. M. Chu and B. Almohsen, "A new fractional-order compartmental disease model", *Alex. Eng. J.* **59** (2020) 3187–3196; doi:10.1016/j.aej.2020.07.040.

[29] Z. Hu, Z. Teng and L. Zhang, "Stability and bifurcation analysis of a discrete predator–prey model with nonmonotonic functional response", *Nonlinear Anal. Real World Appl.* **12** (2011) 2356–2377; doi:10.1016/j.nonrwa.2011.02.009.

[30] U. L. Ihtisham, A. Nigar and K. S. Nisar, "An optimal control strategy and Grunwald–Letnikov finite-difference numerical scheme for the fractional-order COVID-19 model", *Math. Model. Numer. Simul. Appl.* **2** (2022) 108–116; doi:10.53391/mmnsa.2022.009.

[31] F. Kangalgil and S. Işık, "Controlling chaos and Neimark–Sacker bifurcation in a discrete-time predator–prey system", *Hacet. J. Math. Stat.* **49** (2020) 1761–1776; doi:10.15672/hujms.531024.

[32] S. Kartal and F. Gürcan, “Discretization of conformable fractional differential equations by a piecewise constant approximation”, *Int. J. Comput. Math.* **96** (2019) 1849–1860; doi:10.1080/00207160.2018.1536782.

[33] G. Kaya, S. Kartal and F. Gürcan, “Dynamical analysis of a discrete conformable fractional order bacteria population model in a microcosm”, *Physica A* **547** (2020) Article ID: 123864; doi:10.1016/j.physa.2019.123864.

[34] V. A. Kim and R. I. Parovik, “Application of the explicit Euler method for numerical analysis of a nonlinear fractional oscillation equation”, *Fractals Fract.* **6** (2022) Article ID: 274; doi:10.3390/fractalfract6050274.

[35] Y. A. Kuznetsov, I. A. Kuznetsov and Y. Kuznetsov, *Elements of applied bifurcation theory*, Volume 112 of *Appl. Math. Sci.* (Springer, New York, 1998).

[36] J. Liu, Y. Cai, J. Tan and Y. Chen, “Dynamical behaviours of a delayed diffusive ecoepidemiological model with fear effect”, *Chaos Solitons Fractals* **161** (2022) Article ID: 112349; doi:10.1016/j.chaos.2022.112349.

[37] Q. Liu and D. Jiang, “Influence of the fear factor on the dynamics of a Stochastic predator–prey model”, *Appl. Math. Lett.* **112** (2021) Article ID: 106756; doi:10.1016/j.aml.2020.106756.

[38] G. S. Mahapatra, P. K. Santra and E. Bonyah, “Dynamics on effect of prey refuge proportional to predator in discrete-time prey–predator model”, *Complexity* **2021** (2021) 1–12; doi:10.1155/2021/6209908.

[39] A. E. Matouk, A. A. Elsadany, E. Ahmed and H. N. Agiza, “Dynamical behavior of fractional order Hastings–Powell food chain model and its discretization”, *Commun. Nonlinear Sci. Numer. Simul.* **27** (2015) 153–167; doi:10.1016/j.cnsns.2015.03.004.

[40] D. Nazari and S. Shahmorad, “Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions”, *J. Comput. Appl. Math.* **234** (2010) 883–891; doi:10.1016/j.cam.2010.01.053.

[41] T. Oksanen, L. Oksanen, K. E. M. Vuorinen, C. Wolf, A. Mäkynen, J. Olofsson, W. J. Ripple, R. Virtanen and T. A. Utsi, “The impact of thermal seasonality on terrestrial endotherm food web dynamics: a revision of the exploitation ecosystem hypothesis”, *Ecography* **43** (2020) 1859–1877; doi:10.1111/ecog.05076.

[42] H. S. Panigoro, M. Rayungsari and A. Suryanto, “Bifurcation and chaos in a discrete-time fractional-order logistic model with Allee effect and proportional harvesting”, *Int. J. Dyn. Control.* (2023) 1–15; doi:10.1007/s40435-022-01101-5.

[43] P. Panja, “Impacts of wind and anti-predator behaviour on predator-prey dynamics: a modelling study”, *Int. J. Comput. Sci. Math.* **15** (2022) 396–407; doi:10.1504/ijcsm.2022.125906.

[44] I. Podlubny, *Fractional differential equations*, Volume 198 of Math. Sci. Eng. (Academic Press, San Diego, CA, 1999).

[45] F. A. Rihan, “Numerical modeling of fractional-order biological systems”, *Abstr. Appl. Anal.* **2013** (2013) 1–11; doi:10.1155/2013/816803.

[46] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives: theory and applications* (Gordon and Breach Science Publishers, Switzerland–Philadelphia, PA, 1993).

[47] A. M. C. Sauve, R. A. Taylor and F. Barraquand, “The effect of seasonal strength and abruptness on predator–prey dynamics”, *J. Theoret. Biol.* **491** (2020) Article ID: 110175; doi:10.1016/j.jtbi.2020.110175.

[48] Y. Sekerci, “Adaptation of species as response to climate change: predator–prey mathematical model”, *AIMS Math.* **5** (2020) 3875–3898; doi:10.3934/math.2020251.

[49] Y. Sekerci, “Climate change effects on fractional order prey–predator model”, *Chaos Solitons Fractals* **134** (2020) Article ID: 109690; doi:10.1016/j.chaos.2020.109690.

[50] S. M. Shah and J. Wiener, “Advanced differential equations with piecewise constant argument deviations”, *Int. J. Math. Math. Sci.* **6** (1983) 671–703; doi:10.1155/S0161171283000599.

[51] K. L. Wang and K. J. Wang, “A modification of the reduced differential transform method for fractional calculus”, *Therm. Sci.* **22** (2018) 1871–1875; doi: 10.2298/TSCI1804871W.

[52] J. Wiener, *Generalized solutions of functional differential equations* (World Scientific, Singapore, 1993).

[53] S. Wiggins and M. Golubitsky, *Introduction to applied nonlinear dynamical systems and chaos*, Volume 2 of *Texts in Appl. Math.* (Springer, New York, 2003).

[54] C. C. Wilmers, E. Post and A. Hastings, "The anatomy of predator–prey dynamics in a changing climate", *J. Animal Ecol.* (2007) 1037–1044; doi:10.1111/j.1365-2656.2007.01289.x.

[55] G. C. Wu, "A fractional variational iteration method for solving fractional nonlinear differential equations", *Comput. Math. Appl.* **61** (2011) 2186–2190; doi:10.1016/j.camwa.2010.09.010.

[56] G. C. Wu and D. Baleanu, "Variational iteration method for fractional calculus – a universal approach by Laplace transform", *Adv. Difference Equ.* **2013** (2013) 1–9; doi:10.1186/1687-1847-2013-18.

[57] B. Xin, W. Peng and L. Guerrini, "A continuous time Bertrand duopoly game with fractional delay and conformable derivative: modeling, discretization process, Hopf bifurcation, and chaos", *Front. Phys.* **7** (2019) Article ID 84; doi:10.3389/fphy.2019.00084.

[58] B. Xin, W. Peng, Y. Kwon and Y. Liu, "Modeling, discretization, and hyperchaos detection of conformable derivative approach to a financial system with market confidence and ethics risk", *Adv. Difference Equ.* **2019** (2019) 1–14; doi:10.1186/s13662-019-2074-8.

[59] S. Yíldíz, S. Bilazeroglu and H. Merdan, "Stability and bifurcation analyses of a discrete Lotka–Volterra type predator–prey system with refuge effect", *J. Comput. Appl. Math.* **422** (2023) Article ID 114910; doi:10.1016/j.cam.2022.114910.

[60] F. Yousef, B. Semmar and K. Al Nasr, "Dynamics and simulations of discretized caputo-conformable fractional-order Lotka–Volterra models", *Nonlinear Eng.* **11** (2022) 100–111; doi:10.1515/nleng-2022-0013.

[61] F. Yousef, B. Semmar and K. Al Nasr, "Incommensurate conformable-type three-dimensional Lotka–Volterra model: discretization, stability, and bifurcation", *Arab. J. Basic Appl. Sci.* **29** (2022) 113–120; doi:10.1080/25765299.2022.2071524.