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RESEARCH ARTICLE

On families of K3 surfaces with real multiplication

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Abstract
We exhibit large families of K3 surfaces with real multiplication, both abstractly, using lattice theory, the Torelli theorem and the surjectivity of the period map, as well as explicitly, using dihedral covers and isogenies.

Contents

1. Introduction

Most of the theory of complex K3 surfaces is governed by the Hodge structure on the second cohomology group. The symmetries of this Hodge structure lead to the concepts of real multiplication (RM) and complex multiplication (CM). We already have a decent understanding of CM, in particular for the existence problems which we consider here, by work of Taelman [Tae]. However, RM remains rather mysterious, with only very few abstract constructions and even fewer concrete examples so far (cf. 2.4) and no analogue of Taelman's result. The present paper aims to remedy this by developing new general methods which can be used to construct families of K3 surfaces with RM (and with CM). Here, a family of K3 surfaces has RM (or CM) by a field F if the very general member X in the family has $F = \mathrm{End}_{\mathrm{Hod}}(T_{X,\mathbb{Q}})$. The K3 surfaces considered in this paper are all algebraic.

It is a special feature of K3 surfaces with RM that they always come in deformation families, as opposed to being isolated, as is quite frequent in the CM case (cf. Remark 2.2). More precisely, if a K3 surface X of Picard number ρ has RM by a field F of degree $m = [F : \mathbb{Q}]$, then $l := (22 - \rho)/m$ is an integer and X deforms in a family of K3 surfaces with RM by F of dimension $l-2$, with $l-2 \geq 1$; see 2.6. As $l \geq 3$, one has $m = 2, 3, \ldots, 7$, and the maximal l is $10, 7, 5, 4, 3, 3$, respectively. A high interest lies in exhibiting families of the maximal dimension. Using Taelman's results on K3 surfaces with CM, we prove the following theorem which produces maximal families for any totally real field F of degree 2 or 5:

1.1. Theorem

Let F be a totally real field of degree $1 < m \leq 5$. Let $l = \lfloor \frac{20}{m} \rfloor$ for $m \neq 4$ resp. $l = 4$ for $m = 4$. Then there is an $(l-2)$-dimensional family of K3 surfaces with RM by F.

A variation of this method allows us to find maximal families also for the remaining degrees $m = 3, 4, 6, 7$ in Theorems 3.15 and 3.19, but only under certain conditions on F or for specific fields. In the case of real quadratic fields, we give an alternative proof of the existence of maximal families, with more explicit K3 surfaces and with an explicit description of the action of F on the transcendental lattice, in Theorem 3.10.

To get explicit examples of families of K3 surfaces with RM, our first approach uses K3 surfaces X with a purely non-symplectic automorphism σ of order m so that X has CM by the cyclotomic field $\mathbb{Q}(\zeta_m)$. We deform the cyclic covering $X \to X/\langle\sigma\rangle$ in such a way that the action of the totally real subfield of $\mathbb{Q}(\zeta_m)$ on the transcendental cohomology deforms with X. These deformations are not Galois coverings, but their monodromy group is the dihedral group D_m of order $2m$. Such deformations have also been studied in [EM+], with applications to Teichmüller curves in $\mathcal{M}_g$ among others. This provides a concrete implementation of the ideas that go into the abstract proof of Theorem 3.5. We obtain the following results.

1.2. Theorem

Let ρ denote the rank of $\mathrm{Pic}(X)$ for a very general K3 surface X in a family, so $d = \dim_{\mathbb{Q}} T_{X,\mathbb{Q}} = 22 - \rho$.

(5) The 7-dimensional family of degree 2 K3 surfaces in §5.9 has $\rho = 2$ and RM by $\mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\zeta_5 + \zeta_5^{-1})$.
(7) The 3-dimensional family of elliptic K3 surfaces in §5.3 has $\rho = 4$ and RM by the cubic field $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$.
(9) The 2-dimensional family of elliptic K3 surfaces in §5.5 has $\rho = 10$ and RM by the cubic field $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$.
(11) The 2-dimensional family of elliptic K3 surfaces in §5.7 has $\rho = 2$ and RM by the degree 5 field $\mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1})$.

Our second approach, in Sections 6 and 7, exploits isogenies between elliptic K3 surfaces to exhibit self-maps of K3 surfaces, a topic of independent interest. Such a self-map is then shown to induce RM or CM.

1.3. Theorem

Let ρ denote the rank of $\mathrm{Pic}(X)$ for a very general K3 surface X in a family, so $d = \dim_{\mathbb{Q}} T_{X,\mathbb{Q}} = 22 - \rho$.

(2) The 4-dimensional family of elliptic K3 surfaces in Proposition 6.2 has $\rho = 10$ and RM by $\mathbb{Q}(\sqrt{2})$.
(3) The 3-dimensional family of elliptic K3 surfaces in Proposition 7.2 has $\rho = 10$ and RM by $\mathbb{Q}(\sqrt{3})$.

In fact, the very same approach gives also new large families with CM (see Propositions 6.2, 7.2) and subfamilies with larger CM fields (see 6.7). There is also a single example with RM by the field $\mathbb{Q}(\sqrt{7})$ (see 7.6).

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1.4. Organization of the paper

Section 2 reviews basics of Hodge theory and how they apply to the moduli of K3 surfaces with RM or CM. We then prove our abstract results such as Theorem 1.1 in Section 3 using Taelman's work and lattice theory.

Turning to concrete settings, Section 4 explains how Dickson polynomials can be used to deform K3 surfaces admitting non-symplectic automorphisms in the realm of D_m-covers. As a special feature, the construction provides a cycle on $X \times X$ which induces the RM-action on $H^2(X)$; see 4.8. The construction gives rise to explicit examples in Section 5 which prove Theorem 1.2.

The last two sections take a different approach based on isogenies of elliptic surfaces, say $X \dashrightarrow X'$. We exploit a systematic way to ensure that $X \cong X'$, thus endowing the surfaces with rational self-maps. These are then verified to induce RM (or CM). In detail, the 2-isogenies in Section 6 and the 3-isogenies in Section 7 lead to a proof of Theorem 1.3.

2. Hodge structures and moduli

2.1. Hodge structures

A complex K3 surface X defines a simple polarized weight two Hodge structure

$$T_X := \mathrm{Pic}(X)^\perp \subset H^2(X,\mathbb{Z}), \quad \text{let} \quad T_{X,\mathbb{Q}} := T_X \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

The polarized Hodge structure T_X has $\dim T_X^{2,0} = 1$. A $\mathbb{Q}$-linear map $a : T_{X,\mathbb{Q}} \to T_{X,\mathbb{Q}}$ is an endomorphism of this Hodge structure if its complexification $a_{\mathbb{C}}$ preserves the Hodge decomposition $T_{X,\mathbb{C}} = \oplus T_X^{p,q}$. Zarhin [Zar, Thm. 1.5.1] showed that the $\mathbb{Q}$-algebra of these endomorphisms

$$F := \mathrm{End}_{\mathrm{Hod}}(T_{X,\mathbb{Q}}) = \left\{a \in \mathrm{End}(T_{X,\mathbb{Q}}) : \; a_{\mathbb{C}}(T_X^{p,q}) \subset T_X^{p,q}\right\}$$

is either a totally real field or a CM field. In the latter case, we say that X has complex multiplication (CM), but the notion of real multiplication (RM) is usually reserved for those K3 surfaces where $F \neq \mathbb{Q}$ and F is totally real.

Since F is a field, $T_{X,\mathbb{Q}}$ is an F-vector space, and we write

$$d = \dim_{\mathbb{Q}} T_{X,\mathbb{Q}}, \qquad m := [F:\mathbb{Q}], \qquad l := d/m = \dim_F T_{X,\mathbb{Q}}.$$

Since $F \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_\sigma \mathbb{C}$, where σ runs over the complex embeddings of F, the action of F on T_X is diagonalizable. The polarization on $T_{X,\mathbb{Q}}$, which is induced by the intersection form on H^2, has the property

$$(ax, y) = (x, \bar{a}y) \qquad \forall\, x, y \in T_{X,\mathbb{Q}}, \quad a \in F\,, \tag{2.1}$$

where $\bar{a}$ denotes the complex conjugate of a; of course, $\bar{a} = a$ in case F is totally real.

2.2. Remark

Some authors (for example, [Tae, Def. 9]) define X to have CM by a CM field F only if moreover, $l = \dim_F T_{X,\mathbb{Q}} = 1$; this setting is of special interest because it comes with the extra feature that X can be defined over some number field, like the singular K3 surfaces highlighted in 2.3. For recent results, cf. [Bay]. Our focus is, however, on exhibiting *families* of K3 surfaces with RM or CM, so we allow for $l > 1$, in agreement with [Zar].

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2.3. Complex multiplication (CM)

A notable example of a K3 surface X with CM is a singular K3 surface (i.e., X has maximal Picard number $\rho(X) = 20$). The CM arises here from the positive definite quadratic form Q on the transcendental lattice T_X; in fact, the CM field is $F = \mathbb{Q}(\sqrt{-\det(Q)})$. These K3 surfaces form isolated points in moduli.

Other examples of K3 surfaces with CM are easily exhibited by considering surfaces with a purely non-symplectic automorphism σ of order $n > 2$, so we require that σ acts on a holomorphic 2-form ω as multiplication by a primitive n-th root of unity ζ_n:

$$\sigma^*\omega = \zeta_n \omega.$$

The action of σ^* on $T_{X,\mathbb{Q}}$ gives this $\mathbb{Q}$-vector space the structure of a $\mathbb{Q}(\zeta_n)$-vector space, and thus, X has CM by a field F containing $\mathbb{Q}(\zeta_n)$. We shall use this in 4.5 to find examples of RM.

A CM field $F \subset \mathrm{End}_{\mathrm{Hod}}(T_X)$ defines an eigenspace decomposition of $T_{X,\mathbb{C}}$ for the action of F. There are $m = [F : \mathbb{Q}]$ eigenspaces, each of dimension l. Let $V \subset T_{X,\mathbb{C}}$ be the eigenspace containing $H^{2,0}(X)$. The surjectivity of the period map and the Torelli theorem imply that deforming the subspace $H^{2,0}$ in V deforms X, with the given action of F, in a family of dimension

$$l - 1 = \dim_{\mathbb{C}} \mathbb{P}V = \frac{\mathrm{rank}(T_X)}{[F : \mathbb{Q}]} - 1. \tag{2.2}$$

In the presence of a purely non-symplectic automorphism of order n, the degree of F over $\mathbb{Q}$ is given by $\phi(n)$, where ϕ is the Euler totient function. In particular, X is isolated in moduli when $\mathrm{rank}(T_X) = \phi(n)$.

2.4. Real multiplication (RM)

We shall now compare the CM setting with the RM case following [vG]. If X is a K3 surface with RM by a totally real field F, then $T_X \otimes \mathbb{R}$ admits a decomposition into eigenspaces for the F-action, each of real dimension equal to $l = \mathrm{rank}(T_X)/[F : \mathbb{Q}]$. These eigenspaces are parametrized by the real embeddings $\sigma : F \hookrightarrow \mathbb{R}$. We denote by $T_\sigma \subset T_X \otimes \mathbb{R}$ the eigenspace on which $a \in F$ acts as multiplication by $\sigma(a)$. Consider the special eigenspace T_ϵ whose complexification contains the holomorphic 2-form ω; that is,

$$H^{2,0}(X) \subset T_\epsilon \otimes \mathbb{C}.$$

Then, by complex conjugation, also $H^{0,2}(X) \subset T_\epsilon \otimes \mathbb{C}$ (thus, T_ϵ has signature $(2, l-2)$ whereas the other eigenspaces are negative definite). The main overall restriction on RM structures is the following:

2.5. Lemma [vG, Lemma 3.2]

In the RM case, one has $l = \dim_{\mathbb{R}} T_\epsilon \geq 3$.

Indeed, otherwise, T_ϵ would be positive definite of dimension two which implies that $T_{X,\mathbb{Q}}$ admits extra endomorphisms, just like for singular K3 surfaces, and, in fact, X has CM (compare the example in Proposition 7.7).

2.6. Moduli

As a consequence, a K3 surface with RM (by a field $F \neq \mathbb{Q}$) has a transcendental lattice of rank at least $6 = 2 \cdot 3$, so the Picard number satisfies

$$\rho(X) \leq 16. \tag{2.3}$$

The deformation space of K3 surfaces with given F-action on $T_{X,\mathbb{Q}}$ has dimension $l-2$ (see the proof of [vG, Lemma 3.2]):

$$l-2 \;=\; \dim_{\mathbb{R}} T_\epsilon - 2 \;=\; \frac{\operatorname{rank}(T_X)}{[F:\mathbb{Q}]} \;-\; 2 \;\geq\; 1. \tag{2.4}$$

In particular, a K3 surface with RM is never isolated in moduli.

2.7. *Families of K3 surfaces with RM*

To conclude, if an algebraic K3 surface X has RM by a field F, one has $d \leq 21$; hence, $l \geq 3$ implies that $m = [F : \mathbb{Q}] \leq 7$. Besides the condition that $m \leq 7$, we do not know of previous general results on the problem which totally real fields can be obtained as $\mathrm{End}_{\mathrm{Hod}}(T_{X,\mathbb{Q}})$ for a K3 surface X, besides the abstract ones with $[F : \mathbb{Q}] = 2$ and $l = 3$ in [vG, Example 3.4] and the impressive explicit examples in the papers [EJ1], [EJ2], [EJ3] and [EJ4] for real quadratic fields and high Picard rank.

2.8. *Cyclotomic fields*

In this paper, we consider in particular totally real subfields $F \subset \mathbb{Q}(\zeta_n)$ of the cyclotomic field of n-th roots of unity $(n > 2)$. This CM number field is a Galois extension of $\mathbb{Q}$ with Galois group the group of units $(\mathbb{Z}/n\mathbb{Z})^\times$ in $\mathbb{Z}/n\mathbb{Z}$, and it has degree $\phi(n)$ over $\mathbb{Q}$, where ϕ is again Euler's totient function. Complex conjugation on $Gal(\mathbb{Q}(\zeta_n))$ is given $-1 \in (\mathbb{Z}/n\mathbb{Z})^\times$, and thus, the totally real subfields of $\mathbb{Q}(\zeta_n)$ correspond to the subgroups $H < (\mathbb{Z}/n\mathbb{Z})^\times$ with $-1 \in H$; in particular, any such field is contained in the maximal totally real subfield $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$, of degree $\phi(n)/2$ over $\mathbb{Q}$.

3. General results on RM

3.1. *Abstract deformations from CM to RM*

Given a K3 surface X with CM by a field $E = \mathrm{End}_{\mathrm{Hod}}(T_{X,\mathbb{Q}})$ of degree m over $\mathbb{Q}$, the totally real subfield $F \subset E$, of degree $m/2$ over $\mathbb{Q}$, also acts on $T_{X,\mathbb{Q}}$. In case $l = \dim_E(T_{X,\mathbb{Q}}) \geq 2$, one has $\dim_F(T_{X,\mathbb{Q}}) = 2l \geq 4$, so the obstruction to RM in Lemma 2.5 is not present, and therefore, X is a member of a $2l-2$-dimensional family of K3 surfaces with RM by F.

3.2. *Proposition*

Let X be a K3 surface with CM by the field E and assume that $l := \dim_E(T_{X,\mathbb{Q}}) \geq 2$.

Then there exists a $2l-2$-dimensional family of K3 surfaces with real multiplication by the totally real subfield F of E. These K3 surfaces are deformations of X, and for the very general X_η in this family, there are isometries $T_{X_\eta} \cong T_X$ and $\mathrm{Pic}(X_\eta) \cong \mathrm{Pic}(X)$.

3.3. *Proof*

This follows from 2.4; we provide some details (cf. [vG, Proof of Lemma 3.2]). The inclusion $F \hookrightarrow \mathrm{End}_{\mathrm{Hod}}(T_{X,\mathbb{Q}})$ induces a splitting of the real vector space $T_{X,\mathbb{R}}$ as a direct sum of $[F : \mathbb{Q}]$ eigenspaces for the action of $F_{\mathbb{R}} := F \otimes_{\mathbb{Q}} \mathbb{R}$, each of dimension $2l$. Let

$$V = T_\epsilon \subset T_{X,\mathbb{R}}$$

be the eigenspace with $H^{2,0}(X) \subset V_{\mathbb{C}}$. The intersection form on $H^2(X,\mathbb{Z})$ induces a bilinear form $(\cdot,\cdot)_V$ on V of signature $(2+, (2l-2)-)$. The deformations X_η of X with $F \subset \mathrm{End}_{\mathrm{Hod}}(X_\eta)$ are parametrized by the one-dimensional subspaces $\langle\omega_\eta\rangle$ of $V_{\mathbb{C}}$ with $(\omega_\eta,\omega_\eta)_V = 0$ and $(\omega_\eta,\overline{\omega_\eta})_V > 0$. In this way,

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one obtains a $2l-2$-dimensional family of deformations of X. Since $2l-2>0$ by assumption, the very general member in this family has $F=\mathrm{End}_{\mathrm{Hod}}(X_\eta)$ since overfields of F give families with fewer moduli.

3.4. Maximal families

Recall that a K3 surface X with RM by a field F of degree $m(>1)$ has a transcendental lattice of rank $d=l\cdot m$ for some $l\geq 3$ and $d\leq 21$, and then X is a member of an $(l-2)$-dimensional family of K3 surfaces with RM by F. In case $m=2,5$, the maximal l is $20/m=10,4$, respectively, and Theorem 3.5 shows that for any totally real field of such a degree, a family of maximal dimension exists.

In case $m=3$, the maximal l is $l=7$, whereas for $m=4$, it is $l=5$, and for $m=6,7$, it is $l=3$. We will show in Theorems 3.15 and 3.19 that there exist K3 surfaces having RM with fields of these degrees with these values of l. Finding succinct conditions on which totally real fields of these degrees and these l occur as $\mathrm{End}_{\mathrm{Hod}}(X)$ for a K3 surface X goes beyond the scope of the present paper; it was subsequently achieved, using the full framework of quadratic forms, in [BvGS].

3.5. Theorem [= Theorem 1.1]

Let F be a totally real field of degree $1<m\leq 5$. Let $l=\lfloor\frac{20}{m}\rfloor$ for $m\neq 4$ resp. $l=4$ for $m=4$. Then there is an $(l-2)$-dimensional family of K3 surfaces with RM by F.

3.6. Proof

Given F, we embed it in a CM field E as follows. Let K be any CM field of degree l over $\mathbb{Q}$ such that $K\cap F=\mathbb{Q}$, the intersection taken with respect to any embeddings $F,K\hookrightarrow\mathbb{C}$. (Note that $l>2$ is even, assuring the existence of such K; one can take K to be composed of any imaginary quadratic field with almost any another field of complementary degree.) Then the composite field $E:=FK$ is a CM field of degree $ml\leq 20$ with $F\subset E$. Taelman proved that there exists a K3 surface X with $E=\mathrm{End}_{\mathrm{Hod}}(T_{X,\mathbb{Q}})$ and $\dim_E(T_{X,\mathbb{Q}})=1$ (see [Tae, Thm. 4]). Then $\dim_F(T_{X,\mathbb{Q}})=l\geq 4$, and as $l\geq 3$, this guarantees the existence of the family of the K3 surfaces that are deformations of X, with RM by F.

The examples we construct below use the following proposition. It is a converse for the results discussed in the previous section.

3.7. Proposition

Let V be a $\mathbb{Q}$-vector space with a nondegenerate bilinear form $(\cdot,\cdot)$ such that the quadratic form it defines on $V_{\mathbb{R}}$ has signature $(2,n-2)$. Let F, $F\neq\mathbb{Q}$, be a totally real number field and assume that V also has the structure of an F-vector space such that the following two conditions hold true:

- there is an eigenspace $V_\epsilon\subset V_{\mathbb{R}}$ for the F-action on $V_{\mathbb{R}}$ on which the signature of the restriction of the bilinear form is $(2,e)$ for some e;
- the adjoint property $(ax,y)=(x,ay)$ holds for all $a\in F$ and all $x,y\in V$.

Then a positive definite oriented 2-plane in V_ϵ defines a Hodge structure of K3 type on V with $F\subset\mathrm{End}_{\mathrm{Hod}}(V)$; this Hodge structure is simple if the 2-plane is very general. If $\dim_F V\geq 3$, then $F=\mathrm{End}_{\mathrm{Hod}}(V)$ for a very general V, so V has RM by F.

3.8. Proof

Let $\omega\in V_{\epsilon,\mathbb{C}}$ be an eigenvector for the rotation by $\pi/2$ in the 2-plane such that the orientation is given by $\omega+\bar\omega$, $(1/i)(\omega-\bar\omega)$. The Hodge structure on V is defined by

$$V^{2,0}=\mathbb{C}\omega,\quad V^{0,2}=\mathbb{C}\bar\omega,\quad V^{1,1}=\langle\omega,\bar\omega\rangle^{\perp}\subset V_{\mathbb{C}}.$$

Then $(V, (\cdot,\cdot))$ is a polarized weight two Hodge structure of K3 type. To see that $F \subset \mathrm{End}_{\mathrm{Hod}}(V)$, notice that $aV^{2,0} = V^{2,0}$, $aV^{0,2} = V^{0,2}$ since their bases lie in an eigenspace of a. Next, we use the adjoint property: for $x \in V^{1,1}$, one has

$$(ax, \omega) = (x, a\omega) = (x, \epsilon(a)\omega) = \epsilon(a)(x, \omega) = 0,$$

and similarly for $\bar{\omega}$. Hence, $ax \in V^{1,1}$, and we have $F \subset \mathrm{End}_{\mathrm{Hod}}(V)$. That for a very general 2-plane one has equality follows by considering the Mumford Tate group of the Hodge structure V as in [vG].

3.9. *Maximal families of K3 surfaces with RM by quadratic fields*

For any real quadratic field F, we can directly show the existence of algebraic K3 surfaces with a genus one fibration that have RM by F. Such a K3 surface X must have $d = \dim T_{X,\mathbb{Q}} \leq 20$. Thus, the maximal dimension of a family of such K3 surfaces is $l - 2 = (20/2) - 2 = 8$. The existence of families of this dimension is a consequence of Theorem 3.5. Here, we provide an alternative proof which provides families with nontrivial geometrical information on the K3 surfaces X and an explicit description of the action of F on T_X. We do not know of other families of maximal dimension, but their existence is quite likely.

3.10. *Theorem*

For any squarefree $d > 0$ and any $r > 0$, there is an 8-dimensional family of K3 surfaces with a genus one fibration such that the very general member X has

$$\mathrm{Pic}(X) \cong U(r) := \left(\mathbb{Z}^2, \left(\begin{smallmatrix} 0 & r \\ r & 0 \end{smallmatrix}\right)\right)$$

and has RM by $\mathbb{Q}(\sqrt{d})$.

3.11. *Proof*

Consider a K3 surface with $\mathrm{Pic}(X) = U(r)$. Then $T_X = U \oplus U(r) \oplus E_8^2$, where E_8 is the unique unimodular even negative definite lattice of rank 8. The linear system of an isotropic vector in $\mathrm{Pic}(X)$ endows X with a genus one fibration which very generally only admits multisections of degree divisible by r. The theorem for a squarefree $d > 0$ thus follows as an application of the surjectivity of the period map and the Torelli theorem once we endow $T_{X,\mathbb{Q}}$ with a suitable action by $\mathbb{Q}(\sqrt{d})$. This can be achieved as follows. On any $U(r)$ $(r \in \mathbb{Z}, r \neq 0)$, we define an endomorphism

$$\tau : U(r) \longrightarrow U(r), \qquad (u, v) \longmapsto (dv, u),$$

such that $\tau^2 = d$; that is, τ^2 acts as multiplication by d. Notice that, for the diagonal action, the eigenspace

$$(U_{\mathbb{R}} + U(r)_{\mathbb{R}})^{\tau=\sqrt{d}} = \mathbb{R}(1, \sqrt{d}, 0, 0) \oplus \mathbb{R}(0, 0, 1, \sqrt{d})$$

has indeed signature $(2, 0)$, as required by 2.4. On E_8^2, we consider the same map, $(u, v) \mapsto (dv, u)$ now with $u, v \in E_8$. The diagonal action of these maps $M : T_{X,\mathbb{Q}} \to T_{X,\mathbb{Q}}$ is such that M^2 acts as multiplication by d and thus defines an action of $\mathbb{Q}(\sqrt{d})$ which satisfies $q(Mx, y) = q(x, My)$, where $q(\cdot,\cdot)$ is the polarization on $T_{X,\mathbb{Q}}$. The family of K3 surfaces is the one whose periods lie in an eigenspace of M in $T_{X,\mathbb{C}}$.

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3.12. Remark

Alternatively, one can use the following method for E_8^2. To find real multiplication structures by all $\sqrt{d}$ on a given lattice L, it suffices to assume that $\mathrm{Aut}(L)$ contains 4 anti-commuting involutions $g_1, \ldots, g_4$. Then (in $\mathrm{End}(L)$)

$$(a_1 g_1 + \ldots + a_4 g_4)^2 = a_1^2 + \ldots + a_4^2 \qquad \forall\, a_i \in \mathbb{Z},$$

so the endomorphism $M = a_1 g_1 + \ldots + a_4 g_4$, which satisfies 2.1, endows $L_{\mathbb{Q}}$ with the structure of $\mathbb{Q}(\sqrt{\sum a_i^2})$ vector space. Using Lagrange's four-square theorem, this gives RM on L by any real $\mathbb{Q}(\sqrt{d})$. To conclude, one verifies that the required anti-commuting involutions can be found in the Weyl group $W(E_8)$. We omit the details.

3.13. Remark

The same argument as in 3.11 applies to imaginary quadratic fields (i.e., to $d < 0$), resulting in 9-dimensional maximal families of K3 surfaces with CM. By [BvGS, Prop. 12.1], any such family exclusively comprises K3 surfaces admitting genus one fibrations (but the analogous statement for RM does not hold).

3.14. Maximal families of K3 surfaces with RM by fields of degree 3, 4, 6, 7

For a K3 surface with RM by the field F, after identifying $T_{\mathbb{Q}} = F^l$, the property 2.1 is equivalent to the property that the intersection form on $H^2(X, \mathbb{Q})$, restricted to $T_{X,\mathbb{Q}}$, is given by

$$(x, y) = \mathit{Trace}_{F/\mathbb{Q}}({}^t x \Delta y), \qquad x, y \in F^l,$$

for some $l \times l$ matrix Δ with coefficients in F. It is easy to see that any such bilinear form has the property 2.1, and for the converse, one can argue as in the proof of [BL, Proposition 9.2.3] (cf. [BvGS] for details).

To find the examples, we will restrict ourselves to the case that $T_{X,\mathbb{Q}}$ is $\mathbb{Q}$-isometric to $(\mathbb{Q}^d, \mathrm{I}_{p,q})$, where $\mathrm{I}_{p,q}$ is the diagonal matrix with p diagonal coefficients equal to $+1$ and q diagonal coefficients equal to -1. Besides 'well-known' fields given in Theorem 3.19, we found many more examples using the same methods. However, only for cyclic cubic fields did we find a general lattice criterion for the construction of K3 surfaces with RM (which is now superseded by the results from [BvGS] that extensively use quadratic forms). We discuss the fields in question first to demonstrate our approach.

3.15. Theorem

Let F be a totally real cyclic cubic field with class number one. Then there is a 7-dimensional family of K3 surfaces with RM by F.

3.16. Proof

As we have seen, it suffices to find an action of F on $T_{X,\mathbb{Q}}$, satisfying (2.1) and the signature condition from 2.4, for some algebraic K3 surface X with Picard rank one. As $\mathrm{Pic}(X) = \mathbb{Z}h$ with $h^2 = e$ for an even positive integer e, one finds that $T_X \cong \mathbb{Z}v \oplus U^2 \oplus E_8(-1)^2$ with $v^2 = -e$. In case $e = k^2$ is a square, $(1/k)v \in T_{X,\mathbb{Q}}$ has square -1. Hence, the lattice generated by $(1/k)v$ and $U^2 \oplus E_8(-1)^2$ is odd and unimodular, and thus, since $\dim T_{X,\mathbb{Q}} = 21$, we have isometries

$$\begin{aligned} T_{X,\mathbb{Q}} &\cong \langle 1\rangle^2 \oplus \langle -1\rangle^{19} \\ &\cong (\langle 1\rangle \oplus \langle -1\rangle^2)^2 \oplus (\langle -1\rangle^3)^5\,. \end{aligned} \tag{3.1}$$

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Let $\mathcal{O}_F$ be the ring of integers of F; it is a free $\mathbb{Z}$-module of rank $3 = [F : \mathbb{Q}]$. For $y \in F$, the values of $Trace_{F/\mathbb{Q}}(xy)$ are in $\mathbb{Z}$ for all $x \in \mathcal{O}_F$ iff $y \in \mathfrak{d}^{-1}$, where $\mathfrak{d} \subset \mathcal{O}_F$ is an ideal called the different. So the dual lattice of $\mathcal{O}_F$ with respect to the trace form is the fractional ideal $\mathfrak{d}^{-1} \subset F$. By assumption, $\mathfrak{d}$ is a principal ideal, and we choose a generator $\delta \in \mathcal{O}_F$. Then the bilinear form b_δ on $\mathcal{O}_F$ defined by

$$b_\delta : \mathcal{O}_F \times \mathcal{O}_F \longrightarrow \mathbb{Z}, \qquad b_\delta(x, y) := Trace_{F/\mathbb{Q}}(\delta^{-1}xy)$$

is unimodular, and thus, there is an isometry $(F, b_\delta) \cong (\mathbb{Q}^3, \mathrm{I}_{p,q})$ for certain $p + q = 3$. For any unit $u \in \mathcal{O}_F$, also $u\delta$ is a generator, and we use this to find bilinear forms $b_{u\delta}$ with the correct signatures to match (3.1).

Recall that $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^3$, where $x \otimes \lambda$ maps to $\lambda(\sigma_1(x), \dots, \sigma_3(x))$, where the σ_i are the three embeddings $F \hookrightarrow \mathbb{R}$. As

$$Tr_{F/\mathbb{Q}}(\delta^{-1}x^2) = \sum \sigma_i(\delta^{-1})\sigma_i(x)^2,$$

the signature of the $\mathbb{R}$-linear extension of this quadratic form is determined by the signs of the $\sigma_i(\delta)$. It follows from [AF] (recently extended to higher degree fields in [BVV, Cor. 4.3.5]) that F has full unit signature rank. That is, for any p, q with $p + q = 3$, there is a $u \in \mathcal{O}_F^\times$ such that $\sigma_i(u\delta)$ assumes p positive and q negative values for $i = 1, \dots, 3$. In particular, we may assume that b_δ is negative definite and that $b_{u\delta}$ has signature $(1+, 2-)$ for a certain unit u.

The isometry $T_{X,\mathbb{Q}} \cong (F, b_{u\delta})^2 \oplus (F, b_\delta)^5$ and the diagonal action of F on the right-hand side gives the desired action of F on $T_{X,\mathbb{Q}}$.

3.17. Remark

The crucial property that $\mathfrak{d}$ is a principal ideal also holds generally whenever there is a single element $\alpha \in F$ such that $\mathcal{O}_F = \mathbb{Z}[\alpha]$. For these cases, the above proof applies whenever the totally real cyclic cubic field F has odd class number since it allows one to use [BVV, Cor. 4.3.5] again.

3.18. Example

In the totally real cubic subfield $F = \mathbb{Q}(\alpha)$ of $\mathbb{Q}(\zeta_7)$ with $\alpha = \zeta + \zeta^{-1}$, one can take $\delta = -\alpha^2 - 3\alpha - 4$ totally negative, so $(F, b_\delta) \cong (\mathbb{Q}^3, \mathrm{I}_{0,3})$ and $u\delta = 2\alpha^2 - \alpha - 6$ such that $(F, b_{u\delta}) \cong (\mathbb{Q}^3, \mathrm{I}_{1,2})$.

Using the same approach as above, we now cover real fields of the remaining degrees:

3.19. Theorem

For $m = 4, 6, 7$ and $l = 5, 3, 3$ respectively, there exist $(l - 2)$-dimensional families of K3 surfaces such that the very general member X has RM by the field F_m, where F_m is defined by the polynomial f_m or g_m in Table 1.

3.20. Proof

We start by setting up the K3 surfaces in question by specifying their Picard and transcendental lattices.

For $m = 7$, we take exactly the same lattices as in the proof of Theorem 3.15.

For $m = 4$, let $r \in \mathbb{N}$ and $\mathrm{Pic}(X) = U(r)$ as in Theorem 3.10. Then, as a quadratic space,

$$T_{X,\mathbb{Q}} \cong U \oplus \langle 4r^2 \rangle \oplus \langle -4r^2 \rangle \oplus E_8^2 \cong \langle 1 \rangle^2 \oplus \langle -1 \rangle^{18}.$$

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Table 1. *Totally real fields with suitable generators of the different ideal.*

$f_4 = x^4 + x^3 - 6x^2 - x + 1,$ $\delta = -14\alpha^3 - 19\alpha^2 + 76\alpha + 46,$ $u\delta = (3\alpha^3 - 2\alpha^2 - 26\alpha - 5)/2,$	
$g_4 = x^4 - 6x^2 + 4,$ $\delta = 3\alpha^3 - 2\alpha^2 - 14\alpha + 16,$ $u\delta = \alpha^3 - 2\alpha^2 - 8\alpha + 16,$	
$f_6 = x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1,$ $\delta = 10\alpha^5 + 4\alpha^4 - 55\alpha^3 - 7\alpha^2 + 72\alpha - 21,$ $u\delta = -2\alpha^5 + 7\alpha^4 + 11\alpha^3 - 22\alpha^2 - 4\alpha - 1,$	
$g_7 = x^7 + x^6 - 18x^5 - 35x^4 + 38x^3 + 104x^2 + 7x - 49,$ $\delta = (48274\alpha^6 - 25217\alpha^5 - 830561\alpha^4 - 425142\alpha^3 + 2481943\alpha^2 + 1241898\alpha - 1553433)/7$ $u\delta = (138\alpha^6 - 582\alpha^5 - 1418\alpha^4 + 6270\alpha^3 + 1455\alpha^2 - 15145\alpha + 7749)/7$	
$f_7 = x^7 - 2x^6 - 5x^5 + 9x^4 + 7x^3 - 10x^2 - 2x + 1,$ $\delta = 2\alpha^6 - 3\alpha^5 - 13\alpha^4 + 7\alpha^3 + 15\alpha^2 - 3\alpha - 6,$ $u\delta = 4\alpha^6 - 13\alpha^5 - 2\alpha^4 + 33\alpha^3 - 11\alpha^2 - 20\alpha - 6$	

For $m = 6$, let $r \in \mathbb{N}$ and $\mathrm{Pic}(X) = U \oplus \langle -4r^2\rangle^2$. Then $T_X \cong \langle 4r^2\rangle^2 \oplus E_8^2$, so as a quadratic space,

$$T_{X,\mathbb{Q}} \cong \langle 1\rangle^2 \oplus E_8^2 \cong \langle 1\rangle^2 \oplus \langle -1\rangle^{16}.$$

In summary, each case has $\dim T_{X,\mathbb{Q}} = lm$, and thus,

$$T_{X,\mathbb{Q}} \cong (\langle 1\rangle \oplus \langle -1\rangle^{m-1})^2 \oplus (\langle -1\rangle^m)^{l-2}.$$

As in the proof of Theorem 3.15, it remains to find a totally negative generator δ of $\mathfrak{d}$ and a unit $u \in \mathcal{O}_F^\times$ such that $(F, b_{u\delta})$ is hyperbolic.

This can be achieved with the help of Magma [BCP] as follows. The table below gives a defining polynomial f_m or g_m of the field F_m. Let $\alpha \in F_m$ be a root of f_m resp. g_m; then we write our choices of δ and $u\delta$ as linear combinations of the α^i.

The fields defined by f_3, f_6 are the totally real subfields of $\mathbb{Q}(\zeta_7)$ and $\mathbb{Q}(\zeta_{13})$, respectively, with $\alpha = \zeta + \zeta^{-1}$.

The field defined by f_4 is the degree 4 totally real subfield of $\mathbb{Q}(\zeta_{17})$, with $\alpha = \zeta + \zeta^4 + \zeta^{-4} + \zeta^{-1}$, thus cyclic over $\mathbb{Q}$, while g_4 defines the biquadratic field $\mathbb{Q}(\sqrt{2}, \sqrt{5}) = \mathbb{Q}(\alpha)$ for $\alpha = \zeta^3 + \zeta^{13} + \zeta^{-13} + \zeta^{-3}$ inside $\mathbb{Q}(\zeta_{40})$.

The field defined by f_7 is not a Galois extension, but the field defined by g_7 is Galois as it is the degree 7 totally real subfield of $\mathbb{Q}(\zeta_{43})$, with $\alpha = \zeta + \zeta^6 + \zeta^7 + \zeta^{-6} + \zeta^{-7} + \zeta^{-1}$.

In each case, we have by the choice of δ, u that

$$T_{X_\mathbb{Q}} \cong (F, b_{u\delta})^2 \oplus (F, b_\delta)^{l-2}.$$

Endowing this with the natural diagonal F-action gives the theorem.

3.21. Remark

In Sections 5, 6 and 7 we will exhibit explicit families of RM K3 surfaces using jacobian elliptic fibrations. However, the dimensions of the families will sometimes be smaller then the maximum allowed for by the degree of the field and the Picard number of the general member.

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4. Dickson polynomials and D_n-type covers $\mathbb{P}^1 \to \mathbb{P}^1$

4.1. The dihedral group D_n

Let D_n be the dihedral group of order $2n$. We denote by $\tau, \sigma \in D_n$ an element of order two, n respectively, so that

$$D_n = \langle \sigma, \tau \mid \sigma^n = \tau^2 = 1, \quad \sigma\tau = \tau\sigma^{-1} \rangle.$$

We refer to [CP] for general results on D_n-covers of algebraic varieties. The D_n-action on certain varieties allows us to obtain K3 surfaces with RM by the totally real subfield of $\mathbb{Q}(\zeta_n)$. For this, we need the deformations of cyclic covers provided by the Dickson polynomials.

4.2. Dickson polynomials and D_n-type covers of $\mathbb{P}^1$

The Dickson polynomial of degree n with parameter a is the (unique) degree n polynomial $p_{n,a}(x) \in \mathbb{Z}[a][x]$ satisfying, in the Laurent ring $\mathbb{Z}[a][v, v^{-1}]$,

$$p_{n,a}(v + a/v) = v^n + (a/v)^n.$$

In particular, for $a = 0$, we get $p_{n,0} = x^n$, so $p_{n,0} : \mathbb{P}^1_x \to \mathbb{P}^1_u$, $u = x^n$, is a cyclic cover. One easily verifies that

$$p_{n,a^2}(ax) = a^n p_{n,1}(x).$$

The following Dickson polynomials will be used in this paper:

$$\begin{aligned}
p_{3,a} &= x^3 - 3ax, \\
p_{5,a} &= x^5 - 5ax^3 + 5a^2x, \\
p_{7,a} &= x^7 - 7ax^5 + 14a^2x^3 - 7a^3x, \\
p_{9,a} &= x^9 - 9ax^7 + 27a^2x^5 - 30a^3x^3 + 9a^4x, \\
p_{11,a} &= x^{11} - 11ax^9 + 44a^2x^7 - 77a^3x^5 + 55a^4x^3 - 11a^5x.
\end{aligned}$$

4.3. Lemma

For any $n > 2$ and for any nonzero $a \in \mathbb{C}$, the map defined by a degree n Dickson polynomial $p_{n,a}$,

$$f : \mathbb{P}^1_x \longrightarrow \mathbb{P}^1_u, \qquad x \longmapsto u := p_{n,a}(x),$$

is a degree n covering with monodromy group D_n. This covering is totally ramified over $\infty \in \mathbb{P}^1_u$.

4.4. Proof

For a nonzero a, we define an action of D_n on $\mathbb{P}^1_v$ by

$$\sigma, \tau : \mathbb{P}^1_v \longrightarrow \mathbb{P}^1_v, \qquad \sigma : v \longmapsto \zeta_n v, \qquad \tau : v \longmapsto a/v.$$

Consider the composition

$$\tilde{f} : \mathbb{P}^1_v \longrightarrow \mathbb{P}^1_x \longrightarrow \mathbb{P}^1_u, \qquad u := v^n + (a/v)^n,$$

where the first map has degree 2 and is given by $x = v + a/v$. The induced map $\mathbb{P}^1_x \to \mathbb{P}^1_u$ is thus given by $u = p_{n,a}(x)$; hence, it is the map f. Notice that $\mathbb{P}^1_x = \mathbb{P}^1_v/\tau$ and $\mathbb{P}^1_u = \mathbb{P}^1_v/D_n$. For $n > 2$, the subgroup

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$\langle\tau\rangle \subset D_n$ is not a normal subgroup, and thus, the degree n cover $f : \mathbb{P}^1_x \to \mathbb{P}^1_u$ is not a Galois cover. The Galois closure of f is $\tilde{f}$, a D_n-cover; hence, f has monodromy group D_n.

4.5. Real multiplication on elliptic K3 surfaces

Lemma 4.3 provides, for any nonzero $a \in \mathbb{C}$, a deformation of a cyclic degree n cover of $\mathbb{P}^1$ to a cover with monodromy group D_n. We apply these deformations to cyclic covers of prime degree of jacobian elliptic surfaces.

Let $\overline{\mathcal{E}} \to \mathbb{P}^1_u$ be an elliptic surface and assume that its Weierstrass model is defined by a minimal Weierstrass equation (with $\alpha, \beta \in \mathbb{C}[u]$):

$$\overline{\mathcal{E}} : \quad Y^2 = X^3 + \alpha(u)X + \beta(u). \tag{4.1}$$

Then $\overline{\mathcal{E}}$ is rational if $\deg(\alpha) \leq 4$ and $\deg(\beta) \leq 6$. (cf. [SS, Prop. 5.51]). Similarly, $\overline{\mathcal{E}}$ is a K3 surface if it is not rational, $\deg(\alpha) \leq 8$ and $\deg(\beta) \leq 12$ and the fibration is relatively minimal. In particular, if for $n > 1$ the equation

$$\mathcal{E} : \quad Y^2 = X^3 + \alpha(x^n)X + \beta(x^n) \tag{4.2}$$

defines a K3 surface, then (4.1) defines a rational surface which is the quotient $\mathcal{E}/\sigma_0$ (with quotient map that sends $x \mapsto u = x^n$) by the purely non-symplectic automorphism σ_0 of order n given by

$$\sigma_0 : \mathcal{E} \longrightarrow \mathcal{E}, \qquad \sigma_0(X, Y, x) = (X, Y, \zeta_n x). \tag{4.3}$$

That σ_0 is non-symplectic can also be seen by computing σ_0^* of the regular 2-form $dX \wedge dx/Y$ on $\mathcal{E}$.

4.6. Proposition

Let $n \in \{5, 7, 11\}$ and $a \in \mathbb{C}^\times$. Assume that $\mathcal{E}$, defined by (4.2), is a K3 surface. Then the deformation $\mathcal{E}_a$ of $\mathcal{E}$ defined by the Weierstrass equation

$$\mathcal{E}_a : \quad Y^2 = X^3 + \alpha(p_{n,a}(x))X + \beta(p_{n,a}(x)) \tag{4.4}$$

is again an elliptic K3 surface, and the totally real subfield $F = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$ of $\mathbb{Q}(\zeta_n)$ acts by Hodge endomorphisms on $T_{\mathcal{E}_a}$, so $F \subset \mathrm{End}_{\mathrm{Hod}}(T_{\mathcal{E}_a})$.

4.7. Proof

To see this, let $\tilde{\mathcal{E}}_a$ be the (relatively minimal) elliptic surface obtained as the pull-back of $\mathcal{E}_a$ along the double cover $\tilde{f} : \mathbb{P}^1_v \to \mathbb{P}^1_x$ defined by $x := v + a/v$. It has a Weierstrass equation

$$\tilde{\mathcal{E}}_a : \quad Y^2 = X^3 + \alpha(v^n + (a/v)^n)X + \beta(v^n + (a/v)^n).$$

Then D_n acts on the Weierstrass model via its action on $\mathbb{P}^1_v$ – that is, by

$$\sigma(X, Y, v) = (X, Y, \zeta_n v), \quad \tau(X, Y, v) = (X, Y, a/v).$$

This action extends to $\tilde{\mathcal{E}}_a$, and $\tilde{\mathcal{E}}_a/D_n$ is birational to the rational surface $\overline{\mathcal{E}}$.

Consider the Hodge substructure $T_{\mathcal{E}_a,\mathbb{Q}} \subset H^2(\mathcal{E}_a, \mathbb{Q})$ defined by the transcendental lattice of the K3 surface $\mathcal{E}_a$; it is simple and has $\dim_{\mathbb{C}} T^{2,0}_{\mathcal{E}_a} = 1$. After pull-back to a desingularization of the base change and push-foward along blow-downs to obtain the relative minimal model $\tilde{\mathcal{E}}_a$, one obtains a Hodge

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substructure $T_a \subset H^2(\tilde{\mathcal{E}}_a, \mathbb{Q})$ with an isomorphism of Hodge structures $T_a \xrightarrow{\cong} T_{\mathcal{E}_a,\mathbb{Q}}$. It suffices to show that $F \subset \mathrm{End}_{\mathrm{Hod}}(T_a)$.

Since the rational map $\tilde{\mathcal{E}}_a \to \mathcal{E}_a$ is birational to the quotient by τ, one has $T_a \subset H^2(\tilde{\mathcal{E}}_a, \mathbb{Q})^{\tau^*}$, the subspace of τ^*-invariants, and $H^{2,0}(\tilde{\mathcal{E}}_a)^{\tau^*} \cong H^{2,0}(\mathcal{E}_a)$. Since $\mathcal{E}_a$ is a K3 surface, we see that $H^{2,0}(\tilde{\mathcal{E}}_a)^{\tau^*}$ is one-dimensional, and hence, T_a is the unique simple Hodge substructure of $H^2(\tilde{\mathcal{E}}_a, \mathbb{Q})^{\tau^*}$ with nonzero $(2,0)$-component.

Since $\sigma\tau = \tau\sigma^{-1}$, the endomorphisms $\sigma^* + (\sigma^{-1})^*$ and τ^* of $H^2(\tilde{\mathcal{E}}_a, \mathbb{Q})$ commute. Thus, $\sigma^* + (\sigma^{-1})^*$ defines an endomorphism of the subspace of τ^*-invariants $H^2(\tilde{\mathcal{E}}_a, \mathbb{Q})^{\tau^*}$. This endomorphism maps T_a into itself since the automorphisms σ, σ^{-1} preserve the Hodge structure, and by the unicity of T_a. In particular, $\sigma^* + (\sigma^{-1})^* \in \mathrm{End}_{\mathrm{Hod}}(T_a)$.

Since $(\sigma^*)^n$ is the identity on $H^2(\tilde{\mathcal{E}}_a, \mathbb{Q})$ and n is prime, the subalgebra of $\mathrm{End}(H^2(\tilde{\mathcal{E}}_a, \mathbb{Q}))$ it generates is a quotient of $\mathbb{Q}[T]/(T^n - 1) \cong \mathbb{Q} \times \mathbb{Q}(\zeta_n)$. The subalgebra generated by $\sigma^* + (\sigma^{-1})^*$ is therefore a quotient of $\mathbb{Q} \times F$ and $F \ncong \mathbb{Q}$ since $n > 3$. To show that T_a is an F-vector space, it suffices to show that $\sigma^* + (\sigma^{-1})^*$ acting on T_a has no eigenvalue $\lambda \in \mathbb{Q}$. An eigenspace of $\lambda \in \mathbb{Q}$ is a Hodge substructure of T_a, which contradicts that T_a is simple, unless it is all of T_a. Since the eigenvalues of σ^* can only be ζ_n^k, $k = 0, \ldots, n-1$, and $n > 3$ is an odd prime, this implies that $\lambda = 2$ and that σ^* induces the identity map on T_a. As τ^* is also the identity on T_a, we find that D_n acts trivially on T_a. But then T_a is isomorphic to a Hodge substructure of $\tilde{\mathcal{E}}/D_n$. However, this is a rational surface whereas $T_a^{2,0} \neq 0$.

The Hodge endomorphism $\sigma^* + (\sigma^{-1})^*$ thus generates a subalgebra of $\mathrm{End}_{\mathrm{Hod}}(T_a)$ which is isomorphic to F. In particular, F acts by Hodge endomorphisms on $T_a \cong T_{\mathcal{E}_a,\mathbb{Q}}$.

4.8. Cycles inducing the real multiplication

The real multiplication by $\zeta_m + \zeta_m^{-1}$ on $T_{\mathcal{E}_a,\mathbb{Q}}$ is a $\mathbb{Q}$-linear endomorphism which is induced by the corresponding endomorphism of $H^2(\tilde{\mathcal{E}}_a, \mathbb{Q})$. For $k \in \{0, 1, \ldots, n-1\}$, let

$$\Gamma_k := \{(x, \sigma^k(x)) \in \tilde{\mathcal{E}}_a \times \tilde{\mathcal{E}}_a : x \in \tilde{\mathcal{E}}_a\}$$

be the graph of the order n automorphism $\sigma \in D_n$ of $\tilde{\mathcal{E}}_a$.

Let $[\Gamma_k] \in H^4(\tilde{\mathcal{E}}_a \times \tilde{\mathcal{E}}_a, \mathbb{Q})$ be the cohomology class of the subvariety Γ_k. Using the Künneth formula

$$[\Gamma_k] = \sum_i [\Gamma_k]_{2d-i} \in \bigoplus_i H^{2d-i}(\tilde{\mathcal{E}}_a, \mathbb{Q}) \otimes H^i(\tilde{\mathcal{E}}_a, \mathbb{Q}),$$

and Poincaré duality $H^{2d-i}(\tilde{X}', \mathbb{Q}) \cong H^i(\tilde{X}', \mathbb{Q})^*$, one finds

$$H^{2d-i}(\tilde{X}', \mathbb{Q}) \otimes H^i(\tilde{X}', \mathbb{Q}) \cong \mathrm{End}(H^i(\tilde{X}', \mathbb{Q})).$$

The endomorphism of $H^2(\tilde{\mathcal{E}}_a, \mathbb{Q})$ defined by the Künneth component $[\Gamma_k]_2$ lies in $\mathrm{End}_{\mathrm{Hod}}(H^2(\tilde{\mathcal{E}}_a, \mathbb{Q}))$ since it has Hodge type $(2,2)$ and one has $[\Gamma_k]_2 = (\sigma^{-k})^*$; the inverse is due to the definition of the action of the group D_n on the cohomology, which is defined by $g \cdot v := (g^{-1})^* v$ to assure that $g \cdot (h \cdot v) = (gh) \cdot v$.

In particular, the action of $\zeta_n + \zeta_n^{-1}$ on $H^2(\tilde{\mathcal{E}}_a, \mathbb{Q})$ is induced by the cycle $\Gamma_1 + \Gamma_{-1}$ on $\tilde{\mathcal{E}}_a \times \tilde{\mathcal{E}}_a$. This cycle induces a cycle on $\mathcal{E}_a \times \mathcal{E}_a$ which defines the real multiplication on $T_{X,\mathbb{Q}}$.

4.9. Remark

In general, if X is a K3 surface with RM by $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$, then there is a priori no reason to assume that the real multiplication is induced by a cycle with two irreducible components as we found for the $\mathcal{E}_a$. This suggests that such K3 surfaces are quite special among those with RM. This is confirmed by the fact that in various examples, we do not find maximal families of RM K3 surfaces (the dimension of the deformation space is given by Proposition 3.2). Assuming the Hodge conjecture, there must be a cycle

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inducing the RM, but the general member of such a maximal family probably has a more complicated cycle than in the case we considered here.

4.10. Remark

Dickson polynomials have the special feature of being permutation polynomials for certain finite fields, only depending on the degree (but not on a). The elliptic K3 surfaces considered in Proposition 4.6 therefore satisfy certain congruences for their point counts over finite fields; this relates to the approach towards RM taken in [EJ3, Thm 1.1].

5. Examples

5.1. The case where n is prime

We consider elliptic surfaces with purely non-symplectic automorphisms of order n. In case n is a prime number, we use the deformations given by Dickson polynomials as in Proposition 4.6 to construct explicit families of elliptic K3 surfaces with RM. A slight modification allows us to also handle the case $n = 9$. In case $n = 5$, we find a much larger family using a variation of Proposition 4.6 in Section 5.9.

5.2. The case $n = 5$, approached via elliptic fibrations

Consider the rational elliptic surfaces given by the Weierstrass form

$$S: \quad y^2 = x^3 + a_1 x + a_2, \qquad a_i \in k[t], \quad \deg(a_i) \le i. \tag{5.1}$$

This family is 3-dimensional since the a_i have $2 + 3 = 5$ coefficients, but there are the scalings $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$ and $t \mapsto \mu t$ to take into account. The elliptic fibration on S has a singular fibre of Kodaira type IV* at ∞ and generally Mordell–Weil lattice MWL $\cong A_2^\vee$ (see [SS, Table 8.2, No. 27]). In fact, solving for $x =$ const. such that the RHS of (5.1) is a perfect square leads exactly to 6 sections of height $2/3$, corresponding to the minimal vectors of $A_2^\vee$.

Base change by $t = s^5$ gives rise to a 3-dimensional family of K3 surfaces with

- ∘ a non-symplectic automorphism σ_0 of order 5,
- ∘ (generally) a singular fibre of type IV at $s = \infty$, so NS $\supset U \oplus A_2$,
- ∘ Mordell–Weil lattice MWL $\supseteq A_2^\vee(5)$, the original Mordell–Weil lattice of S scaled by 5,

so $\rho \ge 6$ with very general equality by 2.3. One can show that this family corresponds to the second family in [AST, Table 2], listed under $S(\sigma) = H_5 \oplus A_4$ (the invariant lattice under σ_0^* acting on $H^2(X, \mathbb{Z})$, isometric to the very general Néron–Severi lattice).

We can deform the above family by considering the K3 surface X_a obtained as the base change of S by $t = p_{5,a}(s)$ for $a \in \mathbb{C}$. This results in a 4-dimensional family of K3 surfaces with the same very general Néron–Severi lattice (since the sections and reducible fibre deform). By Proposition 4.6, one has $\mathbb{Q}(\sqrt{5}) \subset \mathrm{End}_{\mathrm{Hod}}(T_{X_a,\mathbb{Q}})$. To see that this is an equality very generally, assume that there is a strictly larger field F such that $F \subset \mathrm{End}_{\mathrm{Hod}}(T_{X_a,\mathbb{Q}})$ very generally. Then $m = [F : \mathbb{Q}] \ge 4$, so by 2.3, 2.6, the 4-dimensional family would force $\mathrm{rank}(T_X) \ge m \cdot (4+1) = 20$, which is impossible since $\rho \ge 6$.

5.3. The case $n = 7$

According to [AST, §6], there are two 2-dimensional families of K3 surfaces admitting a non-symplectic automorphism of order 7 (and this is the maximal dimension of such families). The dimension of the $\mathbb{Q}(\zeta_7)$-vector space $T_{X,\mathbb{Q}}$ is then $l = 3$ for the general X in either family. By Proposition 3.2, there exist

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two $2l - 2 = 4$-dimensional families of K3 surfaces with RM by the cubic field $F = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$. Using one of these families and Proposition 4.6, we find an explicit 3-dimensional family of elliptic K3 surfaces with RM by F.

5.4. *Proof of Theorem 1.2 (7)*

One of the families from [AST] is given by the Weierstrass forms

$$y^2 = x^3 + (b_1 t^7 + b_0)x + (c_1 t^7 + c_0), \qquad b_i, c_i \in \mathbb{C} \tag{5.2}$$

(this is a 2-dimensional family once we account for scalings). Generally, such a fibration has only one reducible singular fibre (of Kodaira type III, located at $t = \infty$). There is also a section $(x(t), y(t))$ with $x = -c_1/b_1$, of height $7/2$. Hence, a general X in the family has $\mathrm{Pic}(X) = U \oplus K_7$, a lattice of rank $\rho = 4$ in the notation of [AST], and $T_X = U^2 \oplus A_6 \oplus E_8$.

We deform (5.2) by replacing t^7 by $p_{7,a}(t)$, with $a \in \mathbb{C}$ as in (4.4), to obtain a 3-dimensional family of K3 surfaces X_a with $F = \mathbb{Q}(\zeta_7 + \zeta_7^{-1}) \subset \mathrm{End}_{\mathrm{Hod}}(T_{X_a,\mathbb{Q}})$. If $F \neq E := \mathrm{End}_{\mathrm{Hod}}(T_{X_a,\mathbb{Q}})$, then E must be a field of degree at least 6, and $\dim_E T_{X_a,\mathbb{Q}} \leq 3$; hence, there would be at most $3 - 1 = 2$ moduli if E is CM or $3 - 2 = 1$ moduli if E is totally real, contradicting the count of 3 moduli we found.

As the singular fibre types stay the same generally and the section obviously deforms, we infer that for the general deformation, $\mathrm{Pic}(X_a) = U \oplus K_7$. Thus, the remaining claim of Theorem 1.2 (7) about the very general Picard number follows.

5.5. *The case* $n = 9$

A complete classification of the K3 surfaces X with a non-symplectic automorphism σ of order 9 w.r.t. the fixed locus of σ is given in [ACV]. We use deformations of a 1-dimensional family to find an explicit 2-dimensional family with RM by the degree 3 totally real field $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$.

5.6. *Proof of Theorem 1.2 (9)*

A one-dimensional family (denoted by D2 in [ACV]) of elliptic K3 surfaces with very general $\rho = 10$ (hence, $d = \dim_{\mathbb{Q}}(T_{X,\mathbb{Q}}) = 12$) admitting a purely non-symplectic automorphism of order 9 is given by

$$y^2 = x^3 + bx + c_1 t^9 + c_0, \qquad b, c_1, c_0 \in \mathbb{C}. \tag{5.3}$$

For very general X, one has $\mathrm{Pic}(X) = U \oplus A_2^4$, but this becomes visible on the above fibration only indirectly – namely, through the fibre of type I_0^* at $t = \infty$ and through the Mordell–Weil lattice $\mathrm{MWL}(X) = D_4^\vee(3)$, which is induced from the rational elliptic surface given by $s = t^3$, which is intermediate to the cyclic cover given by $u = t^9$. (See [SS, Table 8.2, No. 9] for the intermediate rational elliptic surface.)

We deform (5.3) by replacing t^9 by $p_{9,a}$ ($a \in \mathbb{C}$). To show that the Picard lattice is preserved by the deformation, note that the fibre at ∞ is clearly preserved. As for the Mordell–Weil lattice, it is well-known that

$$p_{mn,a}(t) = p_{m,a^n}(p_{n,a}(t)).$$

Presently, with $m = n = 3$, this implies that also the deformation factors through a rational elliptic surface with $\mathrm{MWL} = D_4^\vee$. Hence, we get the same very general Pic as before.

To show that one obtains a 2-dimensional family of elliptic K3 surfaces with RM by $F = \mathbb{Q}(\zeta_9 + \zeta_9^{-1})$, one modifies the proof of Proposition 4.6 by splitting $H^2(\tilde{\mathcal{E}}_a, \mathbb{Q})$ into three summands that are D_n-representations on which σ acts with eigenvalues 1, primitive cube roots of unity and primitive nineth-roots of unity, respectively. One shows that T_a lies in the last summand using that the intermediate

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D_3-cover is rational.

The family is maximal since the Picard lattice is preserved by the deformation.

5.7. *The case* $n = 11$

According to [AST, §7], there are two 1-dimensional families of K3 surfaces admitting a non-symplectic automorphism of order 11 (and this is the maximum dimension of such families). Since $[\mathbb{Q}(\zeta_{11}) : \mathbb{Q}] = 10$, the dimension of the $\mathbb{Q}(\zeta_{11})$-vector space $T_{X,\mathbb{Q}}$ is 2 for the general X in either family. By Proposition 3.2, there exist two $2l - 2 = 2$-dimensional families of K3 surfaces with RM by $F = \mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1})$. Using one of these families and Proposition 4.6, we find an explicit 2-dimensional family of elliptic K3 surfaces with RM by F.

5.8. *Proof of Theorem 1.2 (11)*

We consider the family of elliptic K3 surfaces given by the Weierstrass form

$$y^2 = x^3 + bx + (c_1 t^{11} + c_0), \qquad b, c_1, c_0 \in \mathbb{C}. \tag{5.4}$$

This is a 1-dimensional family once we account for scalings, and thus, for the general X, we find $\dim_{\mathbb{Q}(\zeta_{11})} T_{X,\mathbb{Q}} \geq 2$. For dimension reasons, we must then have equality, and so the general X has Picard number two, and thus, $\mathrm{Pic}(X) = U$ and $T_X = U^2 \oplus E_8^2$. Generally, there is only one additive singular fibre (of Kodaira type II, located at $t = \infty$), all other fibres having Kodaira type I_1.

We deform (5.4) by replacing t^{11} by $p_{11,a}$ ($a \in \mathbb{C}$) as in (4.4) to obtain a 2-dimensional family of K3 surfaces with RM by $F = \mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1})$ as before; note that this is a maximal family since $l = \dim_F T_{X,\mathbb{Q}} = 4$, and thu,s there are $l - 2 = 2$ moduli. Theorem 1.2 (11) follows.

5.9. *Proof of Theorem 1.2 (5)*

In [AST, §5], one finds a description of a family $\mathcal{A}$ (case 5A) of K3 surfaces with a non-symplectic automorphism σ of order five. Whereas in Proposition 4.6 we found such deformations by base change of an elliptic fibration, we now have to consider a generalization of the proof which is based on a D_n-type deformation of the quotient map $X \mapsto X/\sigma$ for a general member of this family.

The general member in the family $\mathcal{A}$ has Picard rank two, and the Picard lattice H_5 is generated by the classes of two smooth rational curves (cf. [AST, §1]). The very general transcendental rational Hodge structure T_X is thus a $\mathbb{Q}(\zeta_5)$-vector space of dimension $l = (22 - 2)/4 = 5$, and the family has $l - 1 = 4$ moduli.

The general member $\mathcal{A}_p$ of the family $\mathcal{A}$ is defined as the double cover of $\mathbb{P}^2$, with homogeneous coordinates $(x_0 : x_1 : x_2)$, branched over a smooth sextic curve C_p, where $p \in \mathbb{C}[x_0, x_1]$ is a degree 6 polynomial with six distinct zeroes in $\mathbb{P}^1_{(x_0:x_1)}$, not divisible by x_1:

$$C_p : \qquad p(x_0, x_1) + x_1 x_2^5 = 0. \tag{5.5}$$

(Changing the coordinate x_0, p can be put in the form $x_0(x_0 - x_1)\prod_{i=1}^4 (x_0 - \lambda_i x_1)$, thus making the 4 moduli apparent. We can also observe the generators of $\mathrm{Pic}(X)$ as the components of the pull-back of the line $\{x_1 = 0\} \subset \mathbb{P}^2$.) The automorphism σ on $\mathcal{A}_p$ is induced by the automorphism

$$\bar{\sigma} : \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \qquad (x_0 : x_1 : x_2) \longmapsto (x_0 : x_1 : \zeta_5 x_2).$$

These surfaces were described in [GP, 7.2] as product-quotient surfaces. The quotient $Y = \mathbb{P}^2/\langle\bar{\sigma}\rangle$ is the (singular) weighted projective space $\mathbb{P}(1, 1, 5)$ which is isomorphic to a cone in $\mathbb{P}^6$ over a rational normal curve of degree 5. After blowing up the fixed point $(0 : 0 : 1) \in \mathbb{P}^2$ and the singular point of

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$Y \cong \mathbb{P}(1,1,5)$, one obtains a cyclic degree 5 cover of the Hirzebruch surfaces $\mathbb{F}_1 \longrightarrow \mathbb{F}_5$.

We deform the family $\mathcal{A}$ by replacing x_2^5 in (5.5) by the Dickson polynomial $p_{5,a}(x_2)$, where $a \in \mathbb{C}[x_0, x_1]$ is homogeneous of degree 2, so that $p_{5,a}(x_2) \in \mathbb{C}[x_0, x_1, x_2]$ is homogeneous of degree 5. One obtains a covering map $\mathbb{F}_1 \longrightarrow \mathbb{F}_5$ with monodromy group D_5. This induces a degree 5 covering between the double covers, and similar to the proof of Proposition 4.6, one finds that any $\mathcal{A}_p$ deforms to a K3 with RM by $\mathbb{Q}(\sqrt{5})$.

Since the transcendental rational Hodge structure of a general deformation still has dimension 20 over $\mathbb{Q}$, it has dimension $l = 20/2 = 10$ over $\mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\zeta_5 + \zeta_5^{-1})$. Therefore, the deformations of K3 surfaces in the family $\mathcal{A}$ with RM by $\mathbb{Q}(\sqrt{5})$ have $l - 2 = 8$ moduli. However, the D_n-type deformations depend on 3 parameters, the coefficients of a, so we get $4 + 3 = 7$ moduli for the D_n-type deformations.

Since these K3 surfaces have Picard number 2, they should be (very special) members of an 8-dimensional family of K3 surfaces with RM by $\mathbb{Q}(\sqrt{5})$. Unfortunately, we do not know the general member in such an 8-dimensional family explicitly.

5.10. Relation between Section 5.2 and Theorem 1.2 (5)

The K3 surfaces with CM by $\mathbb{Q}(\zeta_5)$ from Section 5.2 which arise from (5.1) by a cyclic degree 5 base change form a codimension one subfamily of the CM family $\mathcal{A}$ in the proof of Theorem 1.2 (5). In terms of the standard form of the polynomial p given just below (5.5), it is given by $\lambda_1 \cdots \lambda_4 = 0$ (which imposes an A_4 singularity on (5.5)). Indeed, with this condition, (5.5) can be considered, in the affine chart $x_0 = 1$, as a quartic in x_1 over $\mathbb{C}(x_2)$; the double cover thus describes an elliptic curve (with two rational points at $x_1 = 0$) which transforms to the Weierstrass form from (5.1).

Turning to the deformations with RM by $\mathbb{Q}(\sqrt{5})$, we exhibited a 7-dimensional family in 5.9 depending on the homogeneous polynomials $p, a \in \mathbb{C}[x_0, x_1]$ of degree 4 resp. 2. Along the same lines as above, imposing an A_4 singularity at $[0,1,0]$ amounts to $\lambda_1 \cdots \lambda_4 = 0$ and $x_0 \mid a$. Thereby, we obtain a 5-dimensional family of elliptic K3 surfaces with $\rho \geq 6$ and RM by $\mathbb{Q}(\sqrt{5})$, again 1 dimension short of being maximal. The 4-dimensional family from 5.2 is encoded in the extra condition $x_0^2 \mid a$ which allows one to realize the RM K3 surfaces as base change of the rational elliptic surfaces in (5.1).

6. An approach using isogenies

Exploiting isogenies forms a classical approach toward exhibiting explicit elliptic curves with CM. More generally, it is very useful for the study of $\mathbb{Q}$-curves. We shall explore similar ideas for the elliptic fibrations over $\mathbb{P}^1$ below in order to find K3 surfaces with RM (induced by suitable rational self-maps).

6.1. Degree 2 isogenies

If E is an elliptic curve with a 2-torsion point P over a field K of characteristic $\neq 2$, then E can be converted to the standard form, with $P = (0,0)$:

$$E: \quad y^2 = x(x^2 + 2ax + b), \quad a, b \in K, \quad b(a^2 - b) \neq 0. \tag{6.1}$$

Quotienting by translation by the 2-torsion section P, E admits a 2-isogeny ([S1, III.4.5])

$$E \xrightarrow{\psi} E', \qquad (u, v) = \left(\frac{y^2}{x^2}, \frac{y(x^2 - b)}{x^2}\right)$$

to the elliptic curve E' given by

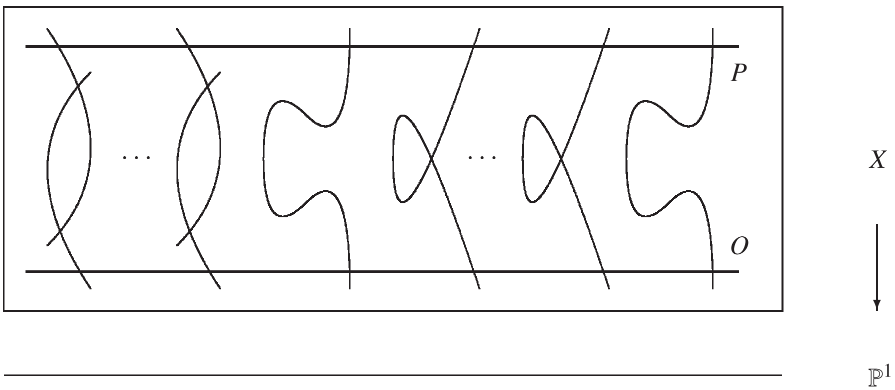
$$E': \quad v^2 = u(u^2 - 4au + 4(a^2 - b)).$$

Asking for E and E' to be isomorphic over $\bar{K}$ generally leads to the CM-curves with j-invariants 1728, −3375 and 8000 ([S2, II.2]. In the realm of elliptic surfaces (i.e., with $K = k(t)$), we can set up the fibration (6.1) to be isotrivial with general fibre one of the above three elliptic curves E. Then the surface automatically acquires CM (and in the K3 case, at least for $j \neq 1728$, it turns out to be a Kummer surface for the product of E with another elliptic curve).

However, there is more flexibility since requiring that the underlying surfaces X and X' are isomorphic does not imply that the isomorphism acts as identity on the base of the fibration; it need not even preserve the fibration. In what follows, we impose the condition that there is an automorphism σ of $\mathbb{P}^1$ such that the fibration π on X and the twisted fibration $\sigma \circ \pi'$ on X' are isomorphic as elliptic fibrations:

$$X \xrightarrow{\pi} \mathbb{P}^1 \quad \text{and} \quad X' \xrightarrow{\sigma\circ\pi'} \mathbb{P}^1. \tag{6.2}$$

For this, note that, generally, X will have singular fibres of type I_2 at the zeroes of b, and of type I_1 at the zeroes of $a^2 - b$, as displayed below.



On X', the singular fibres are interchanged, so we basically want to undo this using σ. In practice, we will take σ as an involution of $\mathbb{P}^1_t$ which we normalize to be $\sigma(t) = -t$. Then for (6.2) to give isomorphic elliptic surfaces requires that

$$a = \pm a^\sigma \quad \text{and} \quad b + b^\sigma = a^2.$$

Now we restrict to the K3 setting with $k \subset \mathbb{C}$ and draw the desired consequences for RM and CM.

6.2. *Proposition*

Let $\alpha, \beta \in \mathbb{C}[x]$ with $\deg(\beta) \leq 3$. Then,

- if $\deg(\alpha) \leq 2$, the 5-dimensional family of elliptic K3 surfaces

$$y^2 = x\left(x^2 + 2\alpha(t^2)x + \frac{1}{2}\alpha(t^2)^2 + t\beta(t^2)\right)$$

  has CM by $\mathbb{Q}(\sqrt{-2})$, and very generally, the Picard rank is $\rho = 10$;
- if $\deg(\alpha) \leq 1$, the 4-dimensional family of elliptic K3 surfaces

$$y^2 = x\left(x^2 + 2t\alpha(t^2)x + \frac{1}{2}t^2\alpha(t^2)^2 + t\beta(t^2)\right)$$

  has RM by $\mathbb{Q}(\sqrt{2})$, and very generally, the Picard rank is $\rho = 10$.

In particular, the second point implies Theorem 1.3 (2).

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6.3. Remark

The K3 surfaces in Proposition 6.2 admit rational self-maps of degree 2, as we will see in the proof. They should thus be of independent interest; cf. [Ded].

6.4. Proof of Proposition 6.2

The degree bounds ensure that the elliptic surfaces are K3 surfaces for general α, β (cf. [SS, Prop. 5.51]).

In the first case, $a^\sigma(t) = \alpha((-t)^2) = a(t)$ and $b^\sigma(t) = \frac{1}{2}\alpha((-t)^2)^2 + (-t)\beta((-t)^2)) = \frac{1}{2}\alpha(t^2)^2 - \beta(t^2))$; hence, $(b^\sigma + b)(t) = \alpha(t^2)^2 = a^2(t)$, and we can extend σ to an isomorphism

$$\begin{aligned} \varphi: \quad X' &\xrightarrow{\cong} \quad X \\ (u, v, t) &\mapsto (-2u, 2\sqrt{-2}v, -t). \end{aligned}$$

Thus, we obtain a self-map $\varphi \circ \psi$ of X of degree 2. Since the isogeny ψ preserves the regular 2-forms, $(\varphi \circ \psi)^*$ acts on $\omega = dx \wedge dt/y$ as multiplication by $\sqrt{-2}$. This proves the claimed CM by $\mathbb{Q}(\sqrt{-2})$.

In the second case, $a^\sigma = -a$ and $b^\sigma + b = a^2$, so the analogous argument applies to the isomorphism

$$\begin{aligned} \varphi': \quad X' &\xrightarrow{\cong} \quad X \\ (u, v, t) &\mapsto (2u, 2\sqrt{2}v, -t). \end{aligned}$$

Hence, $\mathbb{Q}(\sqrt{2}) \subset \mathrm{End}_{\mathrm{Hod}}(T_X)$ by inspection of the degree 2 self-map $\varphi' \circ \psi$ of X and its induced action on ω.

We continue by verifying the stated moduli dimensions. They amount to a simple parameter count, compared against the 2 degrees of freedom left by scaling on the one hand t and on the other hand admissibly (x, y) (since the Möbius transformations have to preserve the fixed points $0, \infty$ of the involution σ). Thus, the stated moduli dimensions follow.

The bounds for the Picard numbers are an immediate application of the Shioda-Tate formula: generally, at the zeroes of b, there are 8 reducible fibres of Kodaira type I_2 in the CM case of setup (1), resp. 6 fibres of Kodaira type I_2 and two fibres of type III in the RM case of setup (2), so in either case, we have

$$\mathrm{NS}(X) \supset U \oplus A_1^8$$

of rank 10 at least. But then, taking into account the moduli dimensions, $\mathrm{End}_{\mathrm{Hod}}(T_X)$ can be at most quadratic by (2.2), (2.4). Therefore, we obtain very generally $\rho = 10$ and CM by $\mathbb{Q}(\sqrt{-2})$ resp. RM by $\mathbb{Q}(\sqrt{2})$.

6.5. Remark

We emphasize that with the given Picard number (or lattice polarization), Proposition 6.2 exhibits maximal dimensional families of K3 surfaces with RM or CM, again by (2.2), (2.4).

6.6. Noether–Lefschetz loci

One can easily exhibit several Noether–Lefschetz loci of the above family. Concentrating on the RM case from Proposition 6.2, there are three fibres of types I_1, I_2, III merging to I_0^* when $t \mid \beta$ or $\deg(\beta) < 3$. This gives 3-dimensional families with very general $\rho = 12$, again with RM by $\mathbb{Q}(\sqrt{2})$ by construction.

Analogous results hold when we merge two pairs of I_2's and I_1's to I_4 and I_2 (which is easily implemented by solving for b to admit a 4-fold zero at $t = 1$ as the resulting equations are linear in the coefficients of β) or when we impose additional sections (which is tedious, but doable for height 2, for instance).

6.7. Higher CM strata

We can also find subfamilies with CM by fields of a higher degree. Notably, this occurs when $\alpha \equiv 0$ as then the generic fibre of (6.1) acquires an automorphism of order 4 and thus has CM itself. Hence, we obtain a 2-dimensional family with CM by $\mathbb{Q}(\sqrt{2}, \sqrt{-1}) = \mathbb{Q}(\zeta_8)$ (and very general $\rho = 10$ for dimension reasons).

Specializing further so that $t\beta(t^2) = p_{5,a}(t)$ resp. $p_{7,a}(t)$, we obtain isolated K3 surfaces (since we can normalize $a = 1$) with CM by $\mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{5})$ resp. $\mathbb{Q}(\sqrt{2}, \sqrt{-1}, \zeta_7 + \zeta_7^{-1})$.

Similarly, at $\alpha = 1$, $\beta = t^2$, the K3 surface admits a non-symplectic automorphism of order 3, so it has CM by $\mathbb{Q}(\sqrt{2}, \sqrt{-3})$. Since it has singular fibres of type I_0^* at $t = 0$ and III^* at $t = \infty$, we conclude that $\rho = 18$.

Along the same lines, for $\alpha = 1$, $\beta = t^3$, the K3 surface admits a non-symplectic automorphism of order 5, so there is CM by $K = \mathbb{Q}(\sqrt{2}, \zeta_5)$. As $\rho \geq 10$ by construction, $T_{X,\mathbb{Q}}$ can presently only have dimension $[K : \mathbb{Q}] = 8$ by 2.3, and thus, $\rho = 14$.

7. Higher degree isogenies

It turns out that an isogeny between elliptic surfaces with torsion points of higher order that are rational over the base does not give rise to K3 surfaces with RM (partly because those isogenies force a relatively large Picard number, whereas (2.3) shows that the Picard rank is at most 16 if a K3 has RM), but isogenies still do the job since we only need a subgroup, the kernel of the isogeny, to be rational. For brevity, we focus on the degree 3 case.

7.1. Degree 3 isogenies

Following [Top] (or [Fri, II.4 §2]), one can write an elliptic curve E admitting a 3-isogeny over a field K of characteristic $\neq 2, 3$ as

$$E: \quad y^2 = x^3 + 27a(x - 4b)^2, \qquad a, b \in K.$$

Here, the isogenous curve E' is given by

$$E': \quad v^2 = u^3 - 27^2 a(u - 108(a + b))^2,$$

and the 3-isogeny is

$$(x, y) \mapsto \left(\frac{9}{x^2}\left(2y^2 + 2ab^2 - x^3 - \frac{2}{3}ax^2\right), 27\frac{y}{x^3}(-4abx + 8ab^2 - x^3)\right).$$

Following the approach of 6.1, we obtain the analogous cases in which $E \cong E'$ in terms of auxiliary polynomials $\alpha, \beta \in k[t]$:

(1) $a = \alpha(t^2), \quad b = -\frac{1}{2}a + t\beta(t^2)$;
(2) $a = t\alpha(t^2), \quad b = -\frac{1}{2}a + \beta(t^2)$.

In the K3 setting, we derive the following two families for $k \subset \mathbb{C}$, one for each above case:

7.2. Proposition

Assume $a, b \in k[t]$ from one of the above case.

(i) If $\deg(\alpha) \leq 2$ and $\deg(\beta) \leq 1$, setup (1) leads to a 3-dimensional family of K3 surfaces with $\rho \geq 10$ and CM by $\mathbb{Q}(\sqrt{-3})$;

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(ii) if $\deg(\alpha) \leq 1$ and $\deg(\beta) \leq 2$, setup (2) leads to a 3-dimensional family of K3 surfaces with $\rho \geq 10$ and RM by $\mathbb{Q}(\sqrt{3})$.

As before, the second point implies most of Theorem 1.3 (3).

7.3. *Proof of Proposition 7.2*

The proof follows the same lines as 6.4. Note that, generally, X has

- 4 fibres of type II at the zeroes of a;
- 4 fibres of type I_3 at the zeroes of b;
- 4 fibres of type I_1 at the zeroes of $a + b$.

Hence, Shioda–Tate again gives $\rho \geq 10$ since $NS(X) \supset U \oplus A_2^4$, and the claimed very general Hodge endomorphisms algebra follows from the analogous parameter count using (2.2), (2.4).

7.4. *Proof of Theorem 1.3 (3)*

The theorem follows almost completely from Proposition 7.2. There is only the statement about the very general Picard number missing. To prove this, it suffices to exhibit a special member X of the family with $\rho(X) = 10$. This can be achieved by computing $\rho(X \otimes \bar{\mathbb{F}}_p)$ at a prime p of good reduction. By the (proven) Tate conjecture, the Picard number of $X \otimes \bar{\mathbb{F}}_p$ is encoded in the zeta function which can be computed using Magma's built-in functionality for elliptic curves over function fields [BCP], for instance.

In detail, if $\alpha = 1 + 2t$ and $\beta = 3 + 4t + t^2$, then the characteristic polynomial of Frobenius at $p = 7$ on $H^2_{\text{ét}}(X \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell(1))$ $(\ell \neq p)$ is

$$(T-1)^3(T+1)(T^2+1)(T^4+1)\left(T^{12} + \frac{8}{7}T^{10} + \frac{6}{7}T^8 + T^6 + \frac{6}{7}T^4 + \frac{8}{7}T^2 + 1\right).$$

Since the last factor is irreducible, but not integral, it cannot be cyclotomic, so we deduce $\rho(X \otimes \bar{\mathbb{F}}_p) = 10$ (since we knew that $\rho \geq 10$ anyway). This completes the proof of Theorem 1.3 (3).

7.5. *Remark*

The 3-dimensional family with CM by $\mathbb{Q}(\sqrt{-3})$ in Proposition 7.2 (i) also fails to be maximal. As in 7.4, this can be shown by exhibiting a special member X with $\rho(X) = 10$, so the maximal dimension of the deformation space is $(12/2) - 2 = 4$.

Let $\alpha = 3 + 4t + t^2$ and $\beta = 1 + 3t$. This leads to the characteristic polynomial of Frobenius at $p = 5$ on $H^2_{\text{ét}}(X \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell(1))$ $(\ell \neq p)$ being

$$(T-1)^6(T^2+T+1)^2\left(T^{12} - T^{10} + T^8 - \frac{7}{5}T^6 + T^4 - T^2 + 1\right).$$

Again, we infer that $\rho(X \otimes \bar{\mathbb{F}}_p) = 10$.

7.6. *Higher degrees*

The analogous approach for isogenies of degree 5 or 7, based on the classical Fricke parametrizations (cf. [Fri, II.4 §3]), gives 1-dimensional families with CM by $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{-7})$, which we will not

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give here. An isolated member of a one dimensional family with $\rho = 16$ and RM by $\mathbb{Q}(\sqrt{7})$ is given by

$$\begin{aligned} y^2 &= x^3 - a(t)x + b(t), \\ a(t) &= 27(t^2+13t+49)(t^2+5t+1)(t^2-49)^2, \\ b(t) &= 54(t^2+13t+49)(t^4+14t^3+63t^2+70t-7)(t^2-49)^3. \end{aligned}$$

This admits a self-map of degree 7 induced by $t \mapsto 49/t$ which respects the common factors of a and b. It acts on the regular 2-form as multiplication by $\sqrt{7}$, thus providing the RM structure.

Using 5-isogenies, we also find the following proposition:

7.7. *Proposition*

The K3 surface

$$\begin{aligned} X: \quad y^2 = x^3 &- 27\left(t^2-125\right)^2\left(t^2+10t+5\right)\left(t^2+22t+125\right)x \\ &- 54\left(t^2+4t-1\right)\left(t^2-125\right)^3\left(t^2+22t+125\right)^2 \end{aligned}$$

has Picard number 18 and CM by $\mathbb{Q}(\sqrt{5}, \sqrt{-2})$.

7.8. *Proof*

The elliptic K3 surface X admits a rational self-map g of degree 5 given by a 5-isogeny which acts as $t \mapsto 125/t$ on the base and as multiplication by $\sqrt{5}$ on the regular 2-form. By construction, we thus have $\mathbb{Q}(\sqrt{5}) \subset \mathrm{End}_{\mathrm{Hod}}(T_{X,\mathbb{Q}})$.

The singular fibre types I_5, I_1, III twice and I_0^* twice imply that

$$\mathrm{NS}(X) \supseteq U \oplus A_4 \oplus A_1^2 \oplus D_4^2.$$

By the Shioda–Tate formula, this gives $\rho(X) \geq 16$ which would still be compatible with X having RM. However, the Picard number turns out to be $\rho(X) = 18$, and consequently, X has CM, as evidenced by the following:

On the one hand, the elliptic fibration admits a section of height 2 with x-coordinate $-3\left(t^2-125\right)\left(t^2+16t+35\right)$ (and the pull-back by g^* of height 10), so $\rho(X) \geq 18$.

On the other hand, the characteristic polynomial of Frobenius on $H^2_{\text{ét}}(X \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell(1))$ at the ordinary prime $p = 11$ admits the irreducible factor

$$h = T^4 - \frac{12}{11}T^3 + \frac{18}{11}T^2 - \frac{12}{11}T + 1,$$

so $\rho(X) \leq \rho(X \otimes \bar{\mathbb{F}}_{11}) \leq 18$, yielding the claimed equality.

It follows from 2.4 that X has CM by a field F of degree 4 containing $\mathbb{Q}(\sqrt{5})$. As h splits completely over $\mathbb{Q}(\sqrt{5}, \sqrt{-2})$, by [Tae], it can only have CM by this degree 4 CM field.

7.9. *Remark*

As exploited in the proof of Proposition 7.7, all the K3 surfaces from Propositions 7.2, 7.7 and from 7.6 admit rational self-maps of degree 3, 5, 7, respectively (which are not induced from the generic fibres of some isotrivial elliptic fibration).

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