Stefan Banach (1892 – 1945)
A commemoration of his life and work

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Stefan Banach is widely regarded as the most powerful and influential mathematician to emerge in Poland in the inter-war years. He was healthy and strong, descended from highland peasantry, a fact he was fond of proclaiming, yet his early years were anything but auspicious. Abandoned at birth, he was brought up in a garret in Cracow by a laundress. From the age of 15 he had to support himself by private coaching and was especially keen on teaching mathematics, a subject in which he was self-taught. Somehow he acquired a knowledge of French, for he claimed to have read Tannery's *Introduction à la théorie des fonctions* and was familiar with the three volumes of Darboux's *Differential geometry*. Irregularly, and for a short time before the First World War, he attended lectures delivered by S. Zaremba at the Jagiellonian University, Cracow, and then enlisted at the Lwów Institute of Technology where he studied engineering. Not for the only time Banach's life was disrupted by war and he returned to Cracow, where he maintained an interest in mathematics, albeit intermittently, by reading and talking with other mathematicians.

The subsequent astonishing eruption of Polish mathematics and the meteoric rise of Stefan Banach as a mathematician have to be seen against the backcloth of historical events. Poland, partitioned between Prussia, Austria and Russia for over a century, did not acquire its independence until 1919. Before this, only Galicia, where the universities of Cracow and Lwów functioned, enjoyed a reasonable degree of autonomy. Even so, collaboration amongst Polish academics in general was very difficult. Furthermore, young people who sought higher education either went to Cracow or Lwów or abroad to countries such as Germany, France and Great Britain. This was so among those who were to become most prominent in founding the Polish Mathematical School, namely Janiszewski, who took his doctorate in Paris under the direction of Poincaré, Lebesgue and Fréchet, and Mazurkiewicz, Steinhaus and Sierpiński who studied in Göttingen.

ZAREMBA

SIERPIŃSKI

JANISZEWSKI

In 1918, Janiszewski published a seminal article 'On the needs of mathematics in Poland', which argued that Polish mathematicians should concentrate, initially at least, on a relatively narrow field of mathematics where

https://doi.org/10.2307/3618071 Published online by Cambridge University Press

they had already achieved recognition abroad, particularly in set theory, topology, the foundations of mathematics and mathematical logic. To disseminate the distinctive contributions of Polish mathematicians to the world of learning and to attract foreign authors with similar interests, Janiszewski also proposed the establishment of a journal. Thus conceived, *Fundamenta Mathematicae*, the first volume of which appeared in 1920, was soon to win universal acclaim [1].

Meanwhile, Banach was languishing in 'fortress Cracow' (the official status of Cracow during the time of war) with little hope of advancement. But chance was to play a decisive part in his life in 1916. Let Steinhaus take up the story.

> 'One summer evening, in 1916, as I was walking along the Planty (a park surrounding the city), I heard a conversation, or rather only a few words. I was so struck by the words "the Lebesgue integral" that I came nearer to the bench on which the speakers were sitting and, then and there, I made their acquaintance. The speakers, Stefan Banach and Otto Nikodým, were discussing mathematics.' [2].

This meeting had almost immediate consequences, for Steinhaus proposed a problem, concerning the average convergence of Fourier series, which he had been working on for some time. A few days later, to Steinhaus' astonishment, Banach brought a solution. Thus arose Banach's first paper which was published, jointly with Steinhaus, in the Bulletin of the Cracow Academy in 1918. In due course Steinhaus would claim that Banach was his greatest discovery.

Banach's dream of being appointed mathematical assistant at the Lwów Institute of Technology was realised in 1920 when Łomnicki gave him the post. It is worthy of note that Banach, who was averse to examinations, obtained his doctorate in 1922 although he never completed his formal studies. His thesis, which also appeared in 1922 in the third volume of *Fundamenta Mathematicae,* was entitled 'Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales', and was the first to be devoted to the theory of linear operators on arbitrary normed spaces.

Echoes of the subtle blend of algebra and topology that give functional analysis its distinctive flavour may be found in work done during the period 1880–1920 by Volterra and Fredholm on integral equations, by Hilbert on spectral theory, and by Hadamard, Fréchet, Schmidt, F. Riesz and Helly on specific function spaces. But it was Banach's dissertation which provided the definitive theoretical basis for functional analysis – a field which subsequently proved to be of paramount importance for the development of mathematical analysis. In the same year, Banach was appointed assistant professor. In recognition of his rising eminence he was awarded a number of prestigious prizes and in 1924 became a member of the Polish Academy of Sciences. In 1927 he was appointed to a full professorship at the University of Lwów.

As well as his teaching commitments Banach initiated and developed a great deal of research. In a short time he became the greatest authority on functional analysis, particularly in its most abstract and general formulations. A group of talented young students gathered round him and under his and

Steinhaus' direction there developed what was to become known as the Lwów School of Mathematics which, as early as 1929, founded its own journal, *Studia Mathematica*, devoted to functional analysis. In 1932, Banach's main work, *Théorie des opérations linéaires,* was published as the first volume of a series of Polish Mathematical Monographs; in the Preface, a sequel was promised, but fate decreed otherwise [3]. It was a comprehensive account of all the results then known about normed spaces and Banach's abstract framework enabled many problems to be solved in a general way which formerly had required special treatment and considerable ingenuity. As Dieudonné puts it in [4],

> 'These features, as well as many applications to classical analysis, gave the book a great appeal, and it had on functional analysis the same impact that van der Waerden's book had on algebra two years earlier. Analysts all over the world began to realise the power of the new methods and to apply them to a great variety of problems; Banach's terminology and notations were universally adopted, complete normed spaces became known as Banach spaces, and soon their theory was considered a compulsory part in most curricula of graduate students.'

Although functional analysis was Banach's chief field of activity he contributed also to other branches of mathematics grouped round the theory of functions of a real variable, set theory, measure theory and transformation groups. He proved, for example, theorems on functions of bounded variation and the differentiability of functions and wrote a paper on the lengths of curves and the areas of surfaces. Stanisław Ulam, a student of Banach, asserts that 'Banach's work of the early 1920s was characterised by extreme elegance and perspicacity of proofs' [5]. In the sequel we shall give a thumbnail sketch of a few of the many important theorems bearing his name: the Hahn-Banach theorem, the cornerstone of duality theory, the Banach-Steinhaus theorem, the Banach Fixed Point Theorem and the Banach-Tarski Paradox.

The complete list of Stefan Banach's publications comprises 58 items, of which six were posthumous [6, 7]. We can't help but note that many of his papers are co-authored, often with his students, especially S. Mazur. This was the result of Banach's style of work which involved frequent and prolonged visits to restaurants or coffee-houses where many results originated in discussions with his collaborators. One such establishment, The Scottish Café, eventually became the regular haunt of Banach, Mazur and Ulam, and less often of others, such as Kuratowski and Steinhaus [8]. Banach was always there, aiding and abetting mathematical activity; posing, analysing and occasionally solving problems. If no solutions were evident, Banach was likely to appear some days later with several sheets of paper torn from a notebook containing outlines of proofs he had completed meanwhile. If, as was often the case, these lacked polish, Mazur would regularly render them in a more satisfactory form; Banach would then cut out the superfluous parts and stick underneath a piece of clean paper on which he would write the amended version.

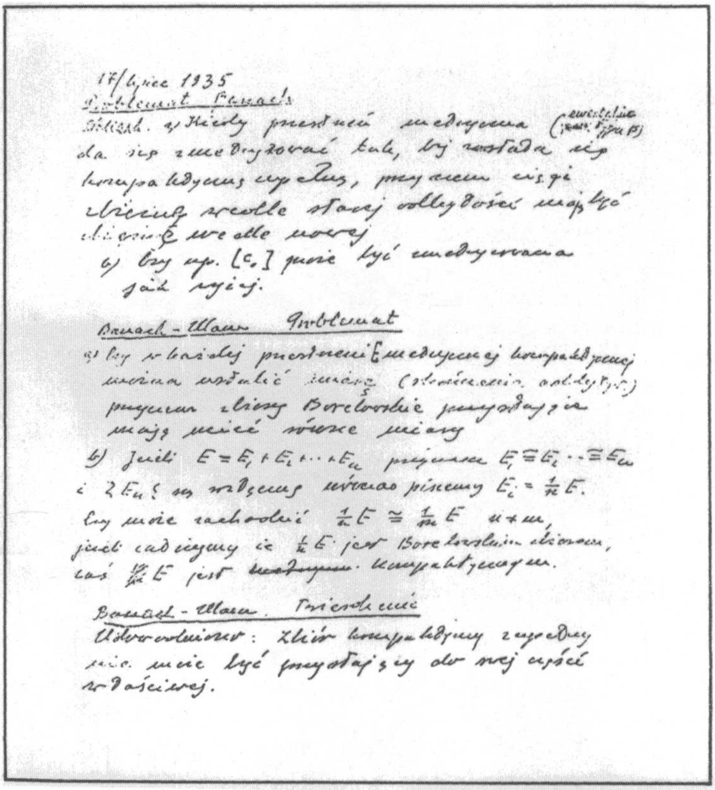
Often, however, they wrote solutions on marble tables in the Café which, unfortunately, were cleaned by the staff each morning. This, in part, seems to explain Steinhaus' observation that 'We regret to say that many valuable results of Banach and his School's work were lost to the great detriment of Polish science as a result of carelessness on the part of the School's members and, first of all, of Banach himself' [2].

Stanislaw Mazur (left) and Stanislaw Ulam, Lwów, ca 1931.

It is to the credit of Mrs Łucja Banach, whom Banach had married in the early 1920s and to whom he had dedicated his *Théorie des opérations linéaires*, that she entrusted the head waiter with a thick notebook with a stiff cover in which problems could be written down and the rewards for solving them

https://doi.org/10.2307/3618071 Published online by Cambridge University Press

recorded. These rewards varied from a small cup of black coffee to a live goose; the latter was eventually awarded to Per Enflö in 1973 for his construction of a separable Banach space without a basis – a question raised by Banach in his book and (non-trivially) equivalent to Mazur's Problem 153 in the notebook. Thus was born the famous *Scottish Book* which contained 193 items prior to June 1941 [9]. (The last involves a reference to Banach's match box problem posed by Steinhaus [10].) By all accounts Banach was imbibing copious quantities of something stronger than coffee in these sessions, one of which lasted for 17 hours. It is worth recording, too, that in addition to these informal gatherings, the Lwów section of the Polish Mathematical Society held its meetings at the University most Saturday evenings!



A page from the *Scottish Book*

And what of Banach as a university lecturer? Ulam records that his lectures were none too well prepared and, accordingly, lacked polish. He would occasionally make mistakes or omissions but it was most stimulating to watch him at the board as he struggled; invariably he pulled through. Steinhaus, too, rated him an excellent lecturer who never lost himself in particulars and avoided covering the board with numerous and complicated symbols.

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Banach's publications reflect only a part of his mathematical powers and the diversity of his interests. His faculty for proposing problems illuminating whole sections of mathematical disciplines was very great and his personal influence on other mathematicians in Lwów and elsewhere in Poland was in many cases decisive. He used to say that mathematics is marked by a specific beauty and cannot be reduced to any rigid deductive system since, eventually, it will burst any formal framework and create new principles. It was not the utilitarian but the intrinsic value of mathematical theories which mattered to him. According to Steinhaus, other workers in the field of operator theory either tended to deal with spaces which were too general, and so obtained only trivial results, or assumed too much about those spaces, which restricted the extent of the applications to a few artificial examples. 'Banach's genius reveals itself in finding the golden mean. This ability of hitting the mark proves that Banach was born a high class mathematician.' [2].

Ulam writes that 'Banach confided to me once that ever since his youth he had been especially interested in finding proofs – that is, demonstrations of conjectures. He had a subconscious system for finding hidden paths – the hallmark of his special genius.' Banach also once said to Ulam that 'Good mathematicians see analogies between theorems or theories, the very best ones see analogies between analogies.' [11].

Banach's interest in politics was minimal. Given the turbulence of the times he lived in, we can understand his oft quoted maxim that 'Hope is the mother of fools.' Nature impressed him not one whit and he rated fine arts, literature and the theatre but second-rate amusements. Banach was, first of all, a mathematician who worked with great intensity and single-mindedness. 'He combined within himself a spark of genius with an astonishing internal urge, which addressed him incessantly in the words of the poet Paul Verlaine: "There is only one thing: the ardent glory of one's craft."' [2]. Banach was not given to self-deception; he knew very well that there is but a very small percentage of people who can understand mathematics. One day he said to Steinhaus, 'I'll tell you something, old chap! Humanities are more important in secondary schools than mathematics – mathematics is too sharp an instrument, it is not made for children to play with ...' [2]. Banach was no ascetic either, but a realist, whose attitude to the intelligentsia without portfolio was one of contempt. Indeed, such was his love of the basic pleasures of life, compounded by a complete lack of thrift, that he eventually began to incur debts and found he could only extricate himself from these by writing textbooks, some of which were written together with Sierpiński and Stożek. Amazingly, he published *Mechanics for academic schools* in 1938, which Ulam adjudged a masterful presentation and worth translating into English!

The final chapter of Banach's life is one of haunting sadness. War was to have an especially devastating effect on academic life. Some mathematicians such as Eilenberg, Kac, Tarski, Ulam and Zygmund had left Poland before hostilities began; most stayed, however, and roughly half of those engaged in research perished at the hands of those parading the swastika. Banach was to pass the war years in Lwów. Having been elected President of the Polish Mathematical Society in 1939, he was then appointed Dean of the University, a

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post he held in the years 1940-1941. During this time the city was occupied by the Russians but in 1941 it fell to the Germans. Now Banach was in considerable danger and was forced to work in an institute run by a Professor Weigel, feeding lice which were used in the treatment of typhoid.

Meanwhile, Warsaw was being reduced to rubble. The thirty-third volume of *Fundamenta Mathematicae* failed to come off the presses and the German director of the printing works had the plates destroyed and the galleys and remaining manuscripts burned. On 1 September 1942 fires consumed Warsaw and devastated the offices of *Fundamenta Mathematicae* and the mathematical holdings at the University of Warsaw. Nothing remained of the archives and the first editions of *Fundamenta Mathematicae*.

The inscription on this photograph, sent to Ulam, reads:
'To the great disciple and great friend of my father, I offer this last photograph of him made in his lifetime in 1943 in Podhale during the Nazi occupation of Poland. Stefan Banach jn.'

Following the liberation of Lwów by the Soviet army, Banach returned to his work at the University but, alas, he was to succumb to lung cancer on 31 August 1945. He had been a heavy smoker. Banach was greatly affected by the atrocities the Nazis had perpetrated; pupils of his, such as Schauder, who realised the importance of Banach spaces for the boundary problems of partial differential equations, had perished, and it is heart-rending to read the thirty-third volume of *Fundamenta Mathematicae* which appeared so soon after peace was declared. The editors, Sierpiński, Kuratowski and Borsuk, dedicated this edition to the memory of those fallen colleagues who had contributed to the journal; here, too, the death of Banach was announced [12].

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Miraculously saved from the ravages of war by Banach's widow, the *Scottish Book* came into the possession of Banach's son, Dr Stefan Banach, a neurosurgeon, who presented it to the Stefan Banach International Mathematical Centre when it was established in Warsaw in 1972 [13]. The book has been continued at Wrocław where the University of Lwów was forcibly moved following the annexation of Lwów by the Soviet Union. Thus the tradition of the *Scottish Book*, so closely associated with Stefan Banach, has survived.

Banach's mathematics

Our aim is to give some snapshots of the mathematical vistas that Banach opened up. No attempt has been made to present results in their full generality but we hope that sufficient details have been given to make the account flow in a reasonably self-contained manner. Inevitably we have had to be selective; for example, we mention nothing about Banach's deepest but most technical work on weak convergence.

A few definitions are necessary to set the scene. A *norm* on a real or complex vector space X is a notion of length $\|x\|$, assigned to each vector x, which has the properties one would expect of a length, such as $\|cx\| = |c|\|x\|$ for scalars c and the triangle law $\|x+y\| \leqslant \|x\| + \|y\|$. Defining the distance between x and y as $\|x-y\|$ then renders X a metric space and permits the realistic use of geometrical imagery such as the 'open ball' $B(x; r) = \{y : \|x - y\| < r\}$ and the 'closed ball' $B[x; r]$ (with $\leqslant$ replacing $<$). Finally, X is a *Banach space* if it is also a complete metric space, i.e. every sequence (x_n) with $\|x_n - x_m\| \to 0$ as $n, m \to \infty$ has a limit x in X (for which $\|x_n - x\| \to 0$ as $n \to \infty$). There is an amusing aside here. Throughout his book, *Théorie des opérations linéaires*, Banach innocently refers to such spaces as 'espaces du type (B)': it was left to later authors to turn Banach's terminology into an acronym!

For example, if I is a closed bounded interval of $\mathbb{R}$, then $C(I)$, the space of continuous real (or complex) functions on I is a real (complex) Banach space with respect to the norm defined by $\|f\| = \max\{|f(x)|: x \in I\}$. Here, typically, all of the requisite axioms are easy to check apart from completeness which here amounts to the assertion that the limit of a 'uniformly Cauchy' sequence of continuous functions both exists pointwise and is itself continuous.

Banach's (contraction mapping) fixed point theorem, which appeared in his thesis, gives a neat illustration of Banach's ability to isolate the topological ingredients which make a proof in analysis tick. His proof is essentially the same as that of the familiar '$|f'(x)| < 1$ criterion' for the convergence of the iteration $x_{n+1} = f(x_n)$ but, by generalising the context, a welter of new applications are suggested. The theorem states that if Y is a closed subset of a Banach space Y with $T: Y \to Y$ a contraction mapping, i.e. one for which $\|T(x) - T(y)\| \leqslant k\|x - y\|$ for all x, y in Y and some fixed $k < 1$, then there is a unique fixed point (i.e. a point x in Y with $T(x) = x$).

Uniqueness is clear, for if $T(x) = x$ and $T(x') = x'$ then $\|x - x'\| = \|T(x) - T(x')\| \leqslant k\|x - x'\|$ forces $x - x' = 0$.

For *existence*, consider the sequence $x_{n+1} = T(x_n)$ with starting value x_0 in

Y chosen arbitrarily. *If* (x_n) converges, its limit x will be in the *closed* subset Y (remember that closed sets are sets that contain their limit points) and, since T is clearly continuous, the facts that $x_{n+1} = T(x_n) \to T(x)$ and $x_{n+1} \to x$ show that $T(x) = x$. And (x_n) *does* converge, because it is Cauchy: if $n < m$, then

$$\begin{aligned}\|x_n - x_m\| &\leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \ldots + \|x_{m-1} - x_m\| \\ &< \left(k^n + k^{n+1} + \ldots\right)\|x_1 - x_0\| \\ &= k^n \|x_1 - x_0\| / (1-k) \qquad \text{with } k < 1,\end{aligned}$$

because, for example,

$$\begin{aligned}\|x_n - x_{n+1}\| &= \|T(x_{n-1}) - T(x_n)\| \\ &\leqslant k\|x_{n-1} - x_n\| \leqslant k^2\|x_{n-2} - x_{n-1}\| \\ &\leqslant \ldots \leqslant k^n\|x_1 - x_0\|.\end{aligned}$$

A typical application concerns the (local) existence of solutions to the differential equation $dy/dx = f(x, y)$ with initial conditions $y(x_0) = y_0$.

Some restrictions on f are necessary: we shall assume that both f and $\partial f/\partial y$ are continuous (and so bounded) in a neighbourhood of (x_0, y_0): say $|f| \leqslant M$ and $|\partial f/\partial y| \leqslant K$.

We can choose $\delta > 0$ sufficiently small that both $\delta < 1/K$ and all points satisfying $|x - x_0| \leqslant \delta$ and $|y - y_0| \leqslant M\delta$ are in the given neighbourhood of (x_0, y_0). To apply Banach's theorem, let Y be the closed subset of $C([x_0 - \delta, x_0 + \delta])$ consisting of those functions y for which $|y(x) - y_0| \leqslant M\delta$ whenever $|x - x_0| \leqslant \delta$ and define T by $Ty(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$. Then T maps Y to Y because

$$|Ty(x) - y_0| \leqslant M|x - x_0| \leqslant M\delta$$

and T is a contraction since

$$\begin{aligned}\left|Ty_1(x) - Ty_2(x)\right| &= \left|\int_{x_0}^{x} \{f(t, y_1(t)) - f(t, y_2(t))\}\,dt\right| \\ &\leqslant \left|\int_{x_0}^{x} K\|y_1 - y_2\|\,dt\right|, \qquad \text{using the mean value theorem in } y_1 \\ &\leqslant K\delta\|y_1 - y_2\|.\end{aligned}$$

Differentiating the defining equation, $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$, of the unique fixed point then shows that $y(x)$ is the (locally) unique solution to our differential equation. Banach's fixed point theorem initiated a still active area of research: see [14].

Another illustration of Banach's topological perspicacity lies in his transformation of the Baire Category Theorem from an isolated curiosity of real analysis to what can only be described as one of the 'Baire necessities' of functional analysis. This innocuous-looking consequence of the completeness axiom says that if a Banach space X can be written as the countable union of *closed* subsets F_n, then at least one of the F_n contains a non-empty closed ball. (If not, one can construct a *nested* sequence of closed balls $B_n = B[x_n; r_n]$ with B_n disjoint from F_n and $r_n \to 0$. By completeness (x_n) converges to x, but by construction x is not in any F_n, which contradicts the hypothesis that the F_n exhaust X: see [15].)

We give two examples of Banach's judicious use of this result: regrettably, we have to omit his most important theoretical application to the so-called open mapping and closed graph theorems: see [15 – 21]. First we discuss a version of the *Banach-Steinhaus Theorem.* A *functional* on a real or complex normed space X is a continuous linear map to $\mathbb{R}$ or $\mathbb{C}$, respectively: the set of all such functionals on X is called the dual space, X^*. The $\varepsilon - \delta$ definition of continuity at 0 easily shows that, if ϕ is a functional, then it is bounded in the sense that $\sup\{|\phi(x)| : \|x\| < 1\}$ is finite; taking this as the definition of $\|\phi\|$ then turns X^* into a Banach space.

The Banach-Steinhaus Theorem states that if (ϕ_n) is a sequence of functionals which is *pointwise* bounded in the sense that $M_x = \sup_n\{|\phi_n(x)|\}$ is finite for each x in X then the *uniform* bound $\sup_n\{\|\phi_n\|\} < \infty$ holds.

(To see this, for $N = 1, 2, 3, \ldots$, define $F_N = \{x : M_x \leqslant N\}$. Then F_N is closed and $X = \cup F_N$. By the Baire Category Theorem, $B[x_0 : r] \subseteq F_N$ for some N; then, for any $\|x\| \leqslant 1$, we have that x_0, $x_0 + rx$ are in F_N with

$$r|\phi_n(x)| = |\phi_n(x_0 + rx) - \phi_n(x_0)| \leqslant |\phi_n(x_0 + rx)| + |\phi_n(x_0)| \leqslant 2N$$

and thus $\|\phi_n\| \leqslant 2N/r$ for all n.)

As a beautiful application of this theorem we will use it to show that there is a continuous function whose (complex) Fourier series *diverges* at 0. Our underlying Banach space X will be the set of complex-valued continuous functions on $[-\pi, \pi]$ which have period 2π. For f in X, the partial sums of its complex Fourier series are $\sum_{-n}^{n} c_k e^{ikx}$ where the Fourier coefficients c_k are given by $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt}dt$. Then $\phi_n(f)$, the value of the partial sums at 0, is given by the functional

$$\phi_n(f) = \sum_{-n}^{n} c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, D_n(t)\, dt$$

$$\text{where } D_n(t) = \frac{\sin \tfrac{1}{2}(2n+1)t}{2\sin \tfrac{1}{2}t} \text{ is Dirichlet's kernel.}$$

From this, $|\phi_n(f)| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} |D_n(t)|\, dt\, \|f\|$. Let f be a continuous approximation to the step function which takes values $+1, -1$ according as $D_n(t)$ is positive or negative. We see that $\|\phi_n\| = \frac{1}{\pi}\int_{-\pi}^{\pi} |D_n(t)|\, dt \to \infty$ as $n \to \infty$ by a straightforward estimate of the integral [16, 17, 20]. Now we can play the Banach-Steinhaus Theorem backwards to deduce that there must exist an f for which the sequence $(\phi_n(f))$ is not pointwise bounded (and hence not convergent).

But Banach's most spectacular use of the Baire Category Theorem is probably in his exquisite 1931 proof that continuous nowhere differentiable functions exist [16, 22]. It is a truly remarkable proof for, as Kac and Ulam explain, it shows that '(such) functions are not just pathological creatures but constitute an overwhelming majority of all continuous functions' [23]. Even more remarkably, 'it is almost easier to prove that "most" continuous functions are nowhere differentiable than to exhibit an explicit example of one such function'. Banach focuses on the Banach space $X = C([0,1])$ and observes that if we define, for $n = 1, 2, 3, \ldots$,

$$F_n = \{f \in X : \text{for } some\ x \in [0, 1 - \tfrac{1}{n}], |f(x+h) - f(x)| \leqslant nh \text{ for } all\ 0 < h < \tfrac{1}{n}\}$$

then $\cup F_n$ certainly contains those functions in X which are also differentiable at some point of (0, 1). The strategy of the proof is to show that each F_n is closed but that no F_n contains a ball. The Baire Category Theorem then guarantees that it cannot be the case that $X = \cup F_n$, i.e. functions in $X - \cup F_n$ exist! To see that F_n is closed, suppose that the sequence $(f_k) \subseteq F_n$ with $\|f_k - f\| \to 0$ and let $x_k \in [0, 1 - \frac{1}{n}]$ correspond to f_k in the defining property of f_k as a member of F_n. Passing to a convergent subsequence if necessary we can suppose that $x_k \to x$ with $x \in [0, 1 - \frac{1}{n}]$ and we claim that x serves in the definition of F_n for f, so that f is in F_n as required. The justification of this claim is a triangle inequality argument of the type so beloved of analysts: for $0 < h < \frac{1}{n}$ we have:

$$\begin{aligned} |f(x+h) - f(x)| &\leqslant |f(x+h) - f(x_k+h)| + |f(x_k+h) - f_k(x_k+h)| \\ &\quad + |f_k(x_k+h) - f_k(x_k)| + |f_k(x_k) - f(x_k)| + |f(x_k) - f(x)| \\ &\leqslant |f(x+h) - f(x_k+h)| + \|f - f_k\| + nh + \|f - f_k\| + |f(x_k) - f(x)| \\ &\to nh \text{ as } k \to \infty \text{ since } x_k \to x,\ f \text{ is continuous and } \|f - f_k\| \to 0. \end{aligned}$$

Finally, for any f in F_n and any $\varepsilon > 0$, we show there is a g in $X - F_n$ with $\|f - g\| < \varepsilon$: it follows that F_n can contain no ball. By the Weierstrass approximation theorem, there is a polynomial p with $\|f - p\| < \frac{1}{2}\varepsilon$ and we can easily construct a piecewise-linear 'zigzag' function z with $\|z\| < \frac{1}{2}\varepsilon$ but whose alternate zigs and zags have gradients of magnitude greater than $n + \|p'\|$. Then, setting $g = p + z$ we have $\|f - g\| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$ but, for any $x \in [0, \frac{1}{n}]$, using the mean value theorem on p, we have:

$$\begin{aligned} |g(x+h) - g(x)| &\geqslant |z(x+h) - z(x)| - |p(x+h) - p(x)| \\ &> (n + \|p'\|)h - \|p'\|h = nh \end{aligned}$$

for all h sufficiently small (so that x, $x + h$ lie on the same zig or zag of z).

We next describe two grand theorems of Banach which make essential use of the Axiom of Choice: the Banach-Tarski Paradox and the Hahn-Banach Theorem. The Banach-Tarski Paradox must surely be one of the most astonishingly counterintuitive results in mathematics. Stated baldly, it says that if X and Y are *any* two bounded subsets of $\mathbb{R}^3$ with non-empty interiors then there exists a (finite) number n and partitions $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ of X and Y (into n pieces) each such that X_i and Y_i are geometrically congruent for each i.

As Stromberg puts it in his highly recommended self-contained account [24], 'this seems to be patently false ... (it) seems to be folly to claim that a billiards ball can be chopped into pieces which can then be put back together to form a life-size statue of Banach.'

How should one react to the Paradox? There is nothing fishy about the proof, although the Axiom of Choice is used in an essential way to construct the partitions. Mathematically it is *not* a paradox because the pieces are not Lebesgue measurable so the grounds for indignation are removed – it is simply illegitimate to add up the volumes of the partition pieces to get an arithmetical contradiction. Indeed, the message of the proof is arguably that congruence of arbitrary subsets is a more slippery notion than it may appear: the key to the paradoxical decomposition is the fact that the isometry group of $\mathbb{R}^3$ contains a free non-abelian subgroup G which enables first spheres and then balls to be split into paradoxical orbits arising from the action of subsets of G [25]. For some, notably Morris Kline [26], the Banach-Tarski Paradox is taken as clinching evidence that the present century's concentration on the axiomatic foundations of mathematics has led mathematics dangerously astray from its intuitive roots in the physical sciences. For most mathematicians, this strikes a rather churlish note (akin to bemoaning the fact that Van Gogh's *Sunflowers* are not suitable illustrations for a botanical textbook!), and they prefer instead to rejoice with Kasner and Newman, who write 'Surely no fairy tale, no fantasy of the Arabian Nights, no fevered dream can match this theorem of hard mathematical logic. Although (it) cannot be put to any practical use... (it) stand(s) as a magnificent challenge to imagination and as a tribute to mathematical conception.' [27] And, as a final ironical twist, according to a speculative article by John Gribbin, [28], it is not inconceivable that the Banach-Tarski Paradox has *applications* to quantum chromodynamics!

Finally, we turn to the Hahn-Banach Theorem. Hahn (1927) and Banach (1929) independently proved the full version of the theorem with real scalars: if Y is a subspace of the normed vector space X and if $\phi : Y \rightarrow \mathbb{R}$ is a bounded functional then ϕ extends to a bounded functional $\tilde{\phi}$ defined on X with ϕ (in Y^*) and $\tilde{\phi}$ (in X^*) having the *same* norm. The proof proceeds by 'induction' (extending 'one dimension at a time' to get from Y to X): because X may be infinite dimensional, transfinite induction (for example, in the form of Zorn's Lemma) is necessary.

The Hahn-Banach Theorem unified many specific results obtained by earlier investigators such as F. Riesz and Helly and with it duality theory at last came into its own. From a functional analytic perspective, this theorem guarantees that spaces of functionals – dual spaces – are rich and interesting objects of study. For example, if $0 \neq x \in X$, the functional defined on the subspace $Y = \{cx : c \in \mathbb{R}\}$ by $\phi(cx) = c$ extends to $\tilde{\phi}$ in X^* with $\|\tilde{\phi}\| = \|\phi\| = 1$ and $\phi(x) = 1$. Not only does this generate a rich supply of elements of X^*, it also means that the map $\hat{} : X \rightarrow (X^*)^*$ given by $\hat{x}(\phi) = \phi(x)$ $(x \in X, \phi \in X^*)$ is a norm-preserving isomorphism with the whole panoply of questions these observations raise. Is X^* or X^{**} a familiar Banach space? Does $X^{**} = \hat{X}$? ...

Moreover many problems naturally take a dual formulation; to take an example from Banach, [3, p. 76-77], hypotheses guaranteeing the existence of a solution (x_j) of an infinite system of linear equations

$$\sum_{j=1}^{\infty} a_{ij}x_j = c_j \qquad i = 1, 2, \ldots$$

may be obtained by regarding the row vectors $(a_{ij})_j$ as belonging to a familiar sequence space, such as l^1 , and the solution (x_j) to its dual l^∞ and using the Hahn-Banach Theorem to relate behaviour of the whole system to behaviour of the first n equations (for all n).

The Hahn-Banach Theorem is also, perhaps surprisingly, equivalent to some intuitive geometrical reformulations (see [16, 18, 19]). For example: if A and B are disjoint convex subsets of X with A open then A and B can be 'separated' by a closed hyperplane. Young provides a striking image for this result in 3 dimensions: a sheet of paper can always be inserted between an egg and a (disjoint) pin (it may touch one of them)! [29].

Later mathematicians such as Rudin in [18] have identified eccentricities in Banach's treatment of certain themes in his book *Théorie des opérations linéaires*. For example, he rigidly sticks to real scalars throughout and he is reluctant to jettison sequences for the more general notions of convergence (such as nets) that a really satisfactory treatment of the weak topology requires. But all testify to the book's significance. Halmos described it as 'addictive, but not easy going' [30]; Young concurred, 'not a book to read fast: I wrote several papers after reading the first chapter' [29], and comparatively recently Maddox in his preface to [20] opined, 'Every serious student of analysis should regard his education incomplete until he has read something of this remarkable germinal book.' Some of the very natural questions raised by Banach in his book continue to tantalise mathematicians: only last year, W. T. Gowers scored a notable triumph in settling the 'Banach hyperplane problem' by constructing an infinite dimensional Banach space not isomorphic to any of its proper subspaces [31].

It is surely a tribute to Banach's genius and vision that, even today, most introductory courses in functional analysis retrace the path that he mapped out and that their highlights are to be found in his book. Simmons in his preface to [21] articulates the attractions of such courses thus:

> 'It seems to me that a worthwhile distinction can be drawn between two types of pure mathematics. The first … centres attention on particular … theorems which are rich in meaning and history, or on juicy individual facts …
>
> The second is concerned primarily with form and structure … . Mathematics of this kind hardly ever yields great and memorable results. On the contrary, its theorems are generally small parts of a much larger whole and derive their main significance from the place they occupy in that whole.
>
> In my opinion, if a body of mathematics like this is to justify itself, it must possess aesthetic qualities akin to those of a good piece of architecture. It should have a solid foundation, its walls and beams

should be firmly and truly placed, each part should bear a meaningful relation to every other part, and its towers and pinnacles should exalt the mind.'

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Acknowledgements

The assistance of Dr Richard Crossley in connection with items [2, 6, 8, 10 and 12] is gratefully acknowledged. The excerpt from the *Scottish Book* is reproduced by kind permission of the publishers Birkhäuser. We would like to thank Mrs Françoise Ulam for lending us the photographs of Banach and of her late husband with Mazur. The stamps were generously provided by Robin Wilson.

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Sick as a parrot

Few programmes can boast a 100 per cent increase in their viewing audience since first going on air, but *Match of the Day,* which celebrates its 30 birthday this week proudly makes that claim. The [first] programme ... attracted just 50000 viewers. Tonight ... it is estimated that the audience will be in excess of five million.

From *The Radio Times* 20-26 August 1994, spotted by Howard Groves.

Space-time sun dials?

With multiple dials, several dials are set closely together around a pillar, tower or even a building to make a complete three- or even four-dimensional dial.

From *A celebration of Cornish sun dials* by Carolyn Martin, contributed by Peter Ransom.