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FLAT SETS, ℓ_p-GENERATING AND FIXING c_0 IN THE NONSEPARABLE SETTING

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Abstract

We define asymptotically p-flat and innerly asymptotically p-flat sets in Banach spaces in terms of uniform weak* Kadec–Klee asymptotic smoothness, and use these concepts to characterize weakly compactly generated (Asplund) spaces that are $c_0(\omega_1)$-generated or $\ell_p(\omega_1)$-generated, where $p \in (1, \infty)$. In particular, we show that every subspace of $c_0(\omega_1)$ is $c_0(\omega_1)$-generated and every subspace of $\ell_p(\omega_1)$ is $\ell_p(\omega_1)$-generated for every $p \in (1, \infty)$. As a byproduct of the technology of projectional resolutions of the identity we get an alternative proof of Rosenthal's theorem on fixing $c_0(\omega_1)$.

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1. Introduction

In [7], it was proved that a separable Banach space $(X, \|\cdot\|)$ is isomorphic to a subspace of c_0 if and only if it admits an equivalent norm that is C-Lipschitz weak* Kadec–Klee (for short, C-LKK*) for some $C \in (0, 1]$. The norm $\|\cdot\|$ on X is called C-LKK* if $\limsup_n \|x^* + x_n^*\| \geq \|x^*\| + C \limsup_n \|x_n^*\|$ whenever $x^* \in X^*$ and (x_n^*) is a weak*-null sequence in X^*. The norm is called LKK* if it is C-LKK* for some $C \in (0, 1]$. Clearly, the supremum norm on c_0 is 1-LKK*. A norm on a Banach space X is called *weak* Kadec–Klee* (*KK** for short) if the dual norm $\|\cdot\|$ on X has the property that, for every $x^* \in X^*$ and every weak*-null sequence (x_n^*) in X^* satisfying $\|x^* + x_n^*\| \to \|x^*\|$, we have $\|x_n^*\| \to 0$. Clearly, LKK* implies KK*.

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In [11], the following moduli of smoothness were introduced. If $(X, \|\cdot\|)$ is a Banach space, $x \in S_X$, Y is a linear subspace of X and $\tau > 0$, put

$$\overline{\rho}(\tau, x, Y) = \sup\{\|x + y\| - 1 : \ y \in Y, q\|y\| \leq \tau\},$$

then

$$\overline{\rho}(\tau, x) = \inf\{\overline{\rho}(\tau, x, Y) \mid Y \subset X, \dim(X/Y) < \infty\},$$

and finally,

$$\overline{\rho}(\tau) = \sup\{\overline{\rho}(\tau, x) \mid x \in S_X\}.$$

It turns out that the norm $\|\cdot\|$ on X is LKK* if and only if there exists $\tau_0 > 0$ such that $\overline{\rho}(\tau_0) = 0$, and it is 1-LKK* if and only if $\overline{\rho}(1) = 0$ (for details and more on the subject see [7], where a nonseparable theory is also developed). The geometric description provided by the use of the modulus $\overline{\rho}$ is more clear than the one given by the definition of the C-LKK*-norm above, and can be depicted as B_X being *asymptotically uniformly flat*. Accordingly, a separable Banach space X admits an equivalent LKK* norm, if and only if admits an equivalent norm whose unit ball is asymptotically uniformly flat, if and only if X isomorphic to a subspace of c_0; here the latter equivalence is the deep result from [7].

In this paper, we shall use some ideas from [7] to deal with the $c_0(\omega_1)$-generation and the $\ell_p(\omega_1)$-generation of Banach spaces, where $p \in (1, \infty)$ (Theorems 5 and 7), and to deal with operators from $c_0(\omega_1)$ fixing copies of $c_0(\omega_1)$ (Theorem 9). We work in the context of nonseparable weakly compactly generated Banach spaces. The restriction of the density to the first uncountable cardinal is done for the sake of simplicity. It is plausible that our results hold with milder cardinality restrictions.

In this paper, $(X, \|\cdot\|)$ denotes a Banach space, and B_X and S_X its closed unit ball and unit sphere. If M is a bounded subset of a Banach space X, we denote by $\|\cdot\|_M$ the seminorm in X^* defined by

$$\|x^*\|_M = \sup\{|\langle x, x^*\rangle| : \ x \in M\}, \qquad x^* \in X^*. \tag{1}$$

The first infinite ordinal and the first uncountable ordinal are denoted by ω_0 and ω_1 respectively. Sometimes, we identify the interval $[0, \omega_1)$ with ω_1. Throughout the paper, we assume that $\infty/\infty = 1$ and that $1/0 = \infty$. Other concepts used in this paper and not defined here can be found, for example, in [3].

The following concept evolves from the definition of C-LKK* property considered above. It will be used in characterizing weakly compactly generated Asplund spaces that are generated by $c_0(\omega_1)$ or by $\ell_p(\omega_1)$ for $p \in (1, \infty)$ (see Theorem 5).

DEFINITION 1. Let $(X, \|\cdot\|)$ be a Banach space X, let M be a nonempty subset of X, let $p \in (1, \infty]$, and put $q = p/(p-1)$. We say that M is $\|\cdot\|$*-asymptotically p-flat* if it is bounded and there exists $C > 0$ such that, for every $f \in X^*$ and every weak*-null sequence (f_n) in X^*, we have

$$\limsup_{n\to\infty} \|f + f_n\|^q \geq \|f\|^q + C \limsup_{n\to\infty} \|f_n\|_M{}^q. \tag{2}$$

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We say that M is *asymptotically p-flat* if there exists an equivalent norm $|||\cdot|||$ on X such that M is $|||\cdot|||$-asymptotically p-flat.

REMARK 2. (i) Nontrivial weak*-null sequences needed in the above definition do exist. Indeed, it is easy to check that 0 belongs to the weak* closure of the dual sphere S_{X^*}. Thus, if X is separable or, more generally, if (B_{X^*}, w^*) is a Corson (even angelic) compact space, then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in S_{X^*} such that $f_n \xrightarrow{w^*} 0$. In general nonseparable Banach spaces, the existence of such a sequence is guaranteed by the deep Josefson–Nissenzweig theorem (see, for example, [1, Ch. XII]).

(ii) A small effort yields that a bounded subset M of X is asymptotically p-flat for some $p \in (1, \infty]$ if and only if there exists $C > 0$ with the following property: whenever $\varepsilon \in (0, C^{-q})$, $f \in B_{X^*}$, and (g_n) is a sequence in S_{X^*} such that $g_n \xrightarrow{w^*} f$ and $\|f - g_n\|_M \geq \varepsilon$ for all $n \in \mathbb{N}$, then $\|f\|^q \leq 1 - C\varepsilon^q$, where $q = p/(p-1)$.

(iii) Let $(X, \|\cdot\|)$ be a Banach space. Assume that, for some $p \in (1, \infty]$, B_X is a $\|\cdot\|$-asymptotically p-flat set. Then, $(X, \|\cdot\|)$ has the KK* property.

(iv) It is easy to check that the unit ball in $c_0(\Gamma)$ is $\|\cdot\|_\infty$-asymptotically ∞-flat, and that the unit ball in $\ell_p(\Gamma)$ is $\|\cdot\|_p$-asymptotically p-flat for all $p \in (1, \infty)$, with constant $C = 1$.

(v) More generally, if the usual modulus of smoothness of $(X, \|\cdot\|)$ is of power type $p \in (1, 2]$, then B_X is $\|\cdot\|$-asymptotically p-flat. To prove this, take such p and put $q = p/(p-1)$. Then the modulus of rotundity

$$\delta_{\|\cdot\|}(\varepsilon) = \inf\{1 - \|(x^* + y^*)/2\| : x^*, y^* \in B_{X^*}, \|x^* - y^*\| \geq \varepsilon\}, \quad \varepsilon \in (0, 2],$$

of the dual norm $\|\cdot\|$ on X^* is of power type q, which means that there exists $K > 0$ such that $\delta_{\|\cdot\|}(\varepsilon) \geq K\varepsilon^q$ for all $\varepsilon \in (0, 2]$. This is a consequence of the basic relationship between both moduli due to Lindenstrauss (see, for example, [3, Lemma 9.9]). We shall verify the condition from Remark 2(ii) for $M = B_X$. So take ε, f and a sequence (g_n) as there. For every $n \in \mathbb{N}$ we have

$$K\|f - g_n\|^q \leq 1 - \left\|\frac{f + g_n}{2}\right\| \leq 1 - \left\|\frac{f + g_n}{2}\right\|^q.$$

Thus

$$\|f\|^q \leq \liminf_{n\to\infty}\left\|\frac{f + g_n}{2}\right\|^q \leq \liminf_{n\to\infty}(1 - K\|f - g_n\|^q) \leq 1 - K\varepsilon^q.$$

Now, Remark 2(ii) says that B_X is asymptotically p-flat.

(vi) If a set M in a Banach space $(X, \|\cdot\|)$ is $\|\cdot\|$-asymptotically p-flat for some $p \in (1, \infty]$, then M is also $\|\cdot\|$-asymptotically p'-flat for every $p' \in (1, p)$. This is a straightforward consequence of the fact that $\|\cdot\|_q \geq \|\cdot\|_{q'}$ whenever $1 \leq q < q'$.

(vii) Let M be a $\|\cdot\|$-compact set in X and (f_n) a weak*-null sequence in X^*. Then, $\lim_{n\to\infty}\|f_n\|_M = 0$. Hence, from the weak*-lower semicontinuity of the dual norm, we get that any norm compact set in an arbitrary Banach space is $\|\cdot\|$-asymptotically

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∞-flat. The same proof gives that, more generally, any limited set in any Banach space is asymptotically ∞-flat. Recall that a set M in a Banach space X is *limited* if $\lim_{n\to\infty} \|f_n\|_M = 0$ whenever (f_n) is a weak*-null sequence in X^*.

(viii) Lancien [10] proved that if K is a scattered compact space of finite height then the unit ball of $C(K)$ is an asymptotically ∞-flat set, by constructing an equivalent norm. Therefore, for instance, the space JL_0 of Johnson and Lindenstrauss is an example of a space the unit ball of which is asymptotically ∞-flat, though it does not contain any isometric copy of $c_0(\omega_1)$. This space is not weakly Lindelöf determined (see, for example, [3, Theorem 12.58]).

(ix) Godefroy, Kalton, and Lancien in [7, Theorem 4.4] proved that the unit ball of a weakly compactly generated Banach space X of density character at most ω_1 is an asymptotically ∞-flat set if and only if X is isomorphic to a subspace of $c_0(\omega_1)$.

We say that a Banach space X is *generated by a subset M of X* if M is linearly dense in it. X is said to be *generated by a Banach space Y* if there exists a bounded linear operator from Y into X such that $T(Y)$ is dense in X.

In [4] and [5], we studied questions on generating Banach spaces by, typically, Hilbert or superreflexive spaces via the usual moduli of uniform smoothness. Here we continue in this direction by using, in the Asplund setting, weak* uniform Kadec–Klee norms instead. This allows to get a characterization also for $p > 2$, where the former approach cannot work as the usual moduli of smoothness are at most of power type 2.

In Definition 3 below, we strengthen the definition of asymptotically p-flat set to what we call an *innerly asymptotically p-flat set*. That allows us to go beyond the framework of Asplund spaces required in Theorem 5. We shall see below (Lemma 13) that, under mild assumptions on the space in question, *every asymptotically p-flat set is an Asplund set*. This fact will then allow us to prove Theorem 7. To be precise, we introduce at this stage the following concept.

DEFINITION 3. Let $(X, \|\cdot\|)$ be a Banach space X, let M be a nonempty subset of X, let $p \in (1, \infty]$, and put $q = p/(p-1)$. We say that M is *innerly asymptotically p-flat* if it is bounded and there exists $C > 0$ such that,

$$\limsup_{n\to\infty} \|f + f_n\|_M{}^q \geq \|f\|_M{}^q + C \limsup_{n\to\infty} \|f_n\|_M{}^q \tag{3}$$

for every $f \in X^*$ and for every weak*-null sequence (f_n) in X^*.

REMARK 4. (i) Notice that, in the above definition, $C \in (0, 1]$. Also, being innerly asymptotically p-flat does not depend on a concrete equivalent norm on X. For $M = B_X$, the properties of being innerly asymptotically p-flat and $\|\cdot\|$-asymptotically p-flat coincide.

(ii) As in Remark 2(vi), if a set is innerly asymptotically p-flat for some $p \in (1, \infty]$, then it is also innerly asymptotically p'-flat for every $p' \in (1, p)$.

(iii) Again, any norm-compact, more generally, any limited subset of X is innerly asymptotically ∞-flat.

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(iv) The concept of inner asymptotic p-flatness, in contraposition with the asymptotic p-flatness, is not inherited by subsets. Indeed, fix $p \in (1, \infty]$. Let $(X, \|\cdot\|)$ be $(c_0, \|\cdot\|_\infty)$ if $p = \infty$ and $(\ell_p, \|\cdot\|_p)$ otherwise. It is easy to check that B_X is innerly asymptotically p-flat as well as $\|\cdot\|$-asymptotically p-flat. Put $N = \{e_1, e_2, \ldots\}$, where the e_i are the canonical unit vectors in X. Thus $N \subset B_X$, and so N is $\|\cdot\|$-asymptotically p-flat. Let $f_1, f_2, \ldots$ be the associated functional coefficients in X^*. Then $\|f_1 + f_n\|_N = 1 = \|f_1\|_N = \|f_n\|_N$ for all $n = 2, 3, \ldots$. Thus (3) is violated no matter how small $C \in (0, 1]$ is. It follows that N is not innerly asymptotically p-flat.

(v) It is not difficult to check that inner asymptotic p-flatness implies asymptotic p-flatness. To show this consider a p-flat subset M of X. Put

$$|||f|||^q = \|f\|^q + \|f\|_M{}^q, \qquad f \in X^*.$$

The triangle inequality for the ℓ_q-norm yields that $|||\cdot|||$ is a norm on X^*. Clearly, this is an equivalent and dual norm. Take any $f \in X^*$ and any weak*-null sequence (f_n) in X^*. Choose a subsequence (f_{n_i}) of (f_n) such that $\lim_{i\to\infty} \|f_{n_i}\|_M = \limsup_{n\to\infty} \|f_n\|_M$, and that both limits $\lim_{i\to\infty} \|f + f_{n_i}\|$ and $\lim_{i\to\infty} \|f + f_{n_i}\|_M$ exist. Then

$$\begin{aligned}
\limsup_{n\to\infty} |||f + f_n|||^q &\geq \lim_{i\to\infty} |||f + f_{n_i}|||^q \\
&= \lim_{i\to\infty} \|f + f_{n_i}\|^q + \lim_{i\to\infty} \|f + f_{n_i}\|_M{}^q \\
&\geq \|f\|^q + \Big(\|f\|_M{}^q + C \lim_{i\to\infty} \|f_{n_i}\|_M{}^q\Big) = |||f|||^q + C \lim_{i\to\infty} \|f_{n_i}\|_M{}^q \\
&= |||f|||^q + C \limsup_{n\to\infty} \|f_n\|_M{}^q.
\end{aligned}$$

Hence M is $|||\cdot|||$-asymptotically p-flat.

As a byproduct of the technology of using projectional resolutions of the identity, we get an alternative proof of Rosenthal's theorem on fixing $c_0(\omega_1)$ (see Theorem 9).

2. The results

THEOREM 5. *Let X be an Asplund space of density ω_1 and let $p \in (1, \infty)$ be given. Then the following assertions are equivalent.*

(i) X is weakly compactly generated and is generated by an asymptotically p-flat subset, or by an asymptotically ∞-flat subset.

(ii) X is generated by $\ell_p(\omega_1)$, respectively by $c_0(\omega_1)$.

COROLLARY 6. *For $p \in (1, \infty)$, every subspace of $\ell_p(\omega_1)$ is generated by $\ell_p(\omega_1)$.*

Every subspace of $c_0(\omega_1)$ is generated by $c_0(\omega_1)$ (and hence is weakly compactly generated).

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Note that the fact that subspaces of $c_0(\Gamma)$ are weakly compactly generated goes back to [9].

A Banach space is called *weakly Lindelöf determined* if its dual unit ball provided with the weak* topology continuously injects into a Σ-product of the real line.

THEOREM 7. *Let X be a general Banach space of density ω_1 and let $p \in (1, \infty)$ be given. Then the following assertions are equivalent.*

(i) *X is weakly Lindelöf determined and is generated by an innerly asymptotically p-flat subset, or by an innerly asymptotically ∞-flat subset.*

(ii) *X is generated by $\ell_p(\omega_1)$, respectively by $c_0(\omega_1)$.*

REMARK 8. (i) As a consequence of Theorems 5 and 7, we get that, if a weakly compactly generated Asplund space is generated by an asymptotically p-flat set, then it is generated by a (usually different) innerly asymptotically p-flat set; see also Remark 4(iv).

(ii) In connection with the first statement in Corollary 6, we note that a subspace of a Hilbert generated space may not be Hilbert generated, see Rosenthal's counterexample [13].

(iii) Given any $p \in (1, \infty)$, then every subspace of an $\ell_p(\Gamma)$-generated space is a subspace of a Hilbert generated space. Indeed, find a linear bounded operator $T : \ell_p(\Gamma) \to X$, with dense range. Then T^* continuously injects (B_{X^*}, w^*) into a multiple of (the uniform Eberlein compact space) (B_{ℓ_q}, w), and hence (B_{X^*}, w^*) itself is a uniform Eberlein compact space. Thus $C((B_{X^*}, w^*))$ is Hilbert generated [3, Theorem 12.17], and hence every subspace of X is a subspace of the Hilbert generated space $C((B_{X^*}, w^*))$. Of course, if $p > 2$, there is a simpler argument based on the inequality $\|\cdot\|_p \leq \|\cdot\|_2$.

The last result goes back to Rosenthal [12, Remark 1 after Theorem 3.4], [8, Ch. 7].

THEOREM 9. *Assume that a Banach space X of density ω_1 admits a linear bounded operator $T : c_0(\Gamma) \to X$ with dense range. Then there exists an uncountable subset Γ_0 of Γ such that T restricted to $c_0(\Gamma_0)$ is an isomorphism.*

Putting together Theorems 7 and 9, we immediately get the following result.

COROLLARY 10. *If a weakly compactly generated Banach space of density ω_1 is generated by an innerly asymptotically ∞-flat set, then it contains an isomorphic copy of $c_0(\omega_1)$.*

3. Proofs

A nonempty subset M of a Banach space X is called *Asplund* if it is bounded and the pseudometric space $(X^*, \|\cdot\|_N)$ is separable for every countable subset N of M (see, for example, [2, Definition 1.4.1]). The concept of a projectional resolution of the identity (PRI, for short) can be found, for instance, in [2, Definition 6.1.5]. Our arguments will be based on the following proposition.

PROPOSITION 11. *Let $(Z, \|\cdot\|)$ be a weakly Lindelöf determined (in particular weakly compactly generated) Banach space, let M and N be convex symmetric and closed subsets of B_Z, and assume that N is moreover an Asplund set. Finally, let Γ be a subset of Z that countably supports Z^*, that is, the set $\{\gamma \in \Gamma \mid \langle \gamma, z^* \rangle \neq 0\}$ is at most countable for every $z^* \in Z^*$. Then there exists a PRI $(P_\alpha \mid \omega_0 \leq \alpha \leq \omega_1)$ on $(Z, \|\cdot\|)$ such that $P_\alpha(M) \subset M$, $P_\alpha(N) \subset N$ for every $\alpha \in [\omega_0, \omega_1]$, and $\|P_\alpha^* z^* - P_\lambda^*\|_N \to 0$ as $\alpha \uparrow \lambda$ for every $z^* \in Z^*$ and for every limit ordinal $\lambda \in (\omega_0, \omega_1]$.*

A more general statement, with a proof, can be found in [6, Proposition 15].

PROOF OF THEOREM 5. Assume that (i) holds. Let $\|\cdot\|$ be an equivalent norm on X and let M be a linearly dense and $\|\cdot\|$-asymptotically p-flat subset of X. Put $q = p/(p-1)$. Simple gymnastics with M yields a new set—call it again M—which is symmetric, convex, closed, still $\|\cdot\|$-asymptotically p-flat, and such that $M \subset B_{(X,\|\cdot\|)}$. Since X is weakly compactly generated, Proposition 11 applied for $M = M$, $N = B_X$ and $\Gamma = \emptyset$ yields a PRI $(P_\alpha \mid \omega_0 \leq \alpha \leq \omega_1)$ on $(X, \|\cdot\|)$ such that $(P_\alpha^* \mid \omega_0 \leq \alpha \leq \omega_1)$ is a PRI on the dual space $(X, \|\cdot\|)^*$, and moreover $P_\alpha(M) \subset M$ for every $\alpha \in [\omega_0, \omega_1]$; recall that $P_{\omega_0} \equiv 0$. We note that $\bigcup_{\omega_0 \leq \alpha < \omega_1} P_\alpha^* X^* = X^*$. Indeed, given $f \in X^*$, we have $\|P_\alpha^* f - f\| \to 0$ as $\alpha \uparrow \omega_1$. We can find then an increasing sequence (α_n) in $[\omega_0, \omega_1)$ such that $\|P_{\alpha_n}^* f - f\| \to 0$ whenever $n \to \infty$. It follows that, putting $\alpha = \sup\{\alpha_n \mid n \in \mathbb{N}\}$ $(< \omega_1)$, then $f \in P_\alpha^* X^*$. The set M is $\|\cdot\|$-asymptotically p-flat; let C be the positive constant in (2) for this set M.

CLAIM 1. *For every $f \in X^*$, $f \neq 0$, every $\varepsilon > 0$, and every $\alpha \in [\omega_0, \omega_1)$, there is $\gamma_{f,\varepsilon,\alpha} \in (\alpha, \omega_1)$ such that*

$$\|f + g\|^q \geq (1-\varepsilon)\|f\|^q + C\|g\|_M{}^q \quad \textit{whenever } g \in \operatorname{Ker} P^*_{\gamma_{f,\varepsilon,\alpha}} \textit{ and } \|g\| < \frac{1}{\varepsilon}.$$

PROOF. Fix any $f \in X^*$, $f \neq 0$, $\varepsilon > 0$, and $\alpha \in [\omega_0, \omega_1)$. Assume that the claim does not hold for this triple. Find then $g_1 \in \operatorname{Ker} P_{\alpha+1}^*$ so that $\|g_1\| < 1/\varepsilon$ and $\|f + g_1\|^q < (1-\varepsilon)\|f\|^q + C\|g_1\|_M{}^q$. Further, find $\alpha_1 \in (\alpha+1, \omega_1)$ such that $g_1 \in P_{\alpha_1}^* X^*$. Find then $g_2 \in \operatorname{Ker} P_{\alpha_1}^*$ so that $\|g_2\| < 1/\varepsilon$ and $\|f + g_2\|^q < (1-\varepsilon)\|f\|^q + C\|g_2\|_M{}^q$. Find $\alpha_2 \in (\alpha_1, \omega_1)$ so that $g_2 \in P_{\alpha_2}^* X^*$ .... Find $g_{n+1} \in \operatorname{Ker} P_{\alpha_n}^*$ so that $\|g_{n+1}\| < 1/\varepsilon$ and $\|f + g_{n+1}\|^q < (1-\varepsilon)\|f\|^q + C\|g_{n+1}\|_M{}^q$. Find then $\alpha_{n+1} \in (\alpha_n, \omega_1)$ so that $g_{n+1} \in P_{\alpha_{n+1}}^* X^*$ .... Thus we get an infinite sequence $g_1, g_2, \ldots$ in X^* and an increasing sequence $\alpha_1 < \alpha_2 < \cdots < \omega_1$. The sequence (g_n) is weak*-null. Indeed, put $\lambda = \lim_{n\to\infty} \alpha_n$; we still have $\lambda < \omega_1$. Fix any $x \in X$. Then for every $n \in \mathbb{N}$ we get

$$\begin{aligned} |\langle x, g_{n+1} \rangle| &= |\langle x, P_\lambda^*(g_{n+1}) \rangle| = |\langle P_\lambda x, g_{n+1} \rangle| \\ &= |\langle P_\lambda x - P_{\alpha_n} x, g_{n+1} \rangle| \leq \frac{1}{\varepsilon}\|P_\lambda x - P_{\alpha_n} x\|. \end{aligned} \tag{4}$$

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Hence $\langle x, g_n\rangle \to 0$ as $n \to \infty$. Therefore, by (2), we have

$$\limsup_{n\to\infty} \|f + g_n\|^q \geq \|f\|^q + C \limsup_{n\to\infty} \|g_n\|_M{}^q$$
$$\Big(> (1-\varepsilon)\|f\|^q + C \limsup_{n\to\infty} \|g_n\|_M{}^q \geq \limsup_{n\to\infty} \|f + g_n\|^q \Big),$$

which contradicts the assumption. This completes the proof of Claim 1.

CLAIM 2. *For every $\alpha \in [\omega_0, \omega_1)$ there exists $\beta_\alpha \in (\alpha, \omega_1)$ such that*

$$\|f + g\|^q \geq \|f\|^q + C\|g\|_M{}^q \quad \textit{whenever } f \in P_\alpha^* X^* \textit{ and } g \in \operatorname{Ker} P_{\beta_\alpha}^*.$$

PROOF. Fix any $\alpha \in [\omega_0, \omega_1)$. Let S be a countable dense subset in the (separable) subspace $P_\alpha^* X^*$. Using Claim 1, put then $\beta_\alpha = \sup\{\gamma_{f,1/n,\alpha} \mid f \in S,\ n \in \mathbb{N}\}$. It is easy to check that this ordinal works.

CLAIM 3. *For every $\alpha \in [\omega_0, \omega_1)$ there exists $\tau_\alpha \in (\alpha, \omega_1)$ such that*

$$\|f + g\|^q \geq \|f\|^q + C\|g\|_M{}^q \quad \textit{whenever } f \in P_{\tau_\alpha}^* X^* \textit{ and } g \in \operatorname{Ker} P_{\tau_\alpha}^*.$$

PROOF. Fix any $\alpha \in [\omega_0, \omega_1)$. We shall construct ordinals $\alpha_1 < \alpha_2 < \cdots < \omega_1$ as follows. Put $\alpha_1 = \alpha$. Let $n \in \mathbb{N}$ and assume that α_n was already found. Using Claim 2, define $\alpha_{n+1} = \beta_{\alpha_n}$ $(>\alpha_n)$. Doing so for every $n \in \mathbb{N}$, we put $\tau_\alpha = \sup_{n\in\mathbb{N}}$ $(= \lim_{n\to\infty}\alpha_n)$; then $\tau_\alpha > \alpha$ and still $\tau_\alpha < \omega_1$. It remains to show that this τ_α works. So, take any $f \in P_{\tau_\alpha}^* X^*$ and any $g \in \operatorname{Ker} P_{\tau_\alpha}^*$. Fix any $n \in \mathbb{N}$. Then $P_{\alpha_n}^* f \in P_{\alpha_n}^* X^*$ and $P_{\beta_{\alpha_n}}^* g = P_{\alpha_{n+1}}^* g = P_{\tau_\alpha}^*(P_{\alpha_{n+1}}^* g) = P_{\alpha_{n+1}}^*(P_{\tau_\alpha}^* g) = 0$. Hence, Claim 2 yields $\|P_{\alpha_n}^* f + g\|^q \geq \|P_{\alpha_n}^* f\|^q + C\|g\|_M{}^q$. This holds for every $n \in \mathbb{N}$ and we know that $\|P_{\alpha_n}^* f - f\| \to 0$ as $n \to \infty$. Therefore $\|f + g\|^q \geq \|f\|^q + C\|g\|_M{}^q$.

CLAIM 4. *There exists an increasing long sequence $(\delta_\alpha)_{\omega_0\leq\alpha\leq\omega_1}$ in $[\omega_0, \omega_1]$, with $\delta_{\omega_0} = \omega_0$ and $\delta_{\omega_1} = \omega_1$, and such that for every $\alpha \in [\omega_0, \omega_1)$ we have*

$$\|f + g\|^q \geq \|f\|^q + C\|g\|_M{}^q \quad \textit{whenever } f \in P_{\delta_\alpha}^* X^* \textit{ and } g \in \operatorname{Ker} P_{\delta_\alpha}^*. \tag{5}$$

PROOF. Fix any $\alpha \in (\omega_0, \omega_1)$ and assume that we have already constructed ordinals δ_β for all $\beta \in [\omega_0, \alpha)$. If α has a predecessor, say $\alpha - 1$, then, using Claim 3, put $\delta_\alpha = \tau_{\delta_{\alpha-1}}$. If α is a limit ordinal, put simply $\delta_\alpha = \lim_{\beta\uparrow\alpha} \delta_\beta$.

CLAIM 5. *There exists a linear, bounded, injective and weak*-to-weak* continuous operator from X^* into $\ell_q(\mathbb{N} \times [\omega_0, \omega_1))$.*

PROOF. For each $\alpha \in [\omega_0, \omega_1)$ find a countable dense set $\{v_1^\alpha, v_2^\alpha, \ldots\}$ in $\frac{1}{2}(P_{\delta_{\alpha+1}} - P_{\delta_\alpha})(M)(\subset M)$. Define $T : X^* \to \mathbb{R}^{\mathbb{N}\times[\omega_0,\omega_1)}$ by

$$Tf(i, \alpha) = 2^{-i} f(v_i^\alpha), \quad (i, \alpha) \in \mathbb{N} \times [\omega_0, \omega_1),\ f \in X^*.$$

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Clearly, T is linear and weak*-to-pointwise continuous. T is injective because $(P_{\delta_\alpha} \mid \alpha \in [\omega_0, \omega_1])$ is clearly a PRI on X. We shall show that the range of T is a subset of the Banach space $\ell_q(\mathbb{N} \times [\omega_0, \omega_1))$ and that T is actually a bounded linear operator from X^* to the latter space. Denote by Y the linear span of the set $\bigcup_{\omega_0 \le \alpha < \omega_1} (P^*_{\delta_{\alpha+1}} - P^*_{\delta_\alpha})X^*$. Take any $f \in Y$. Then we can write f in the form $f = f_1 + f_2 + \cdots + f_k$, where $f_j \in (P^*_{\delta_{\alpha_j+1}} - P^*_{\delta_{\alpha_j}})X^*$, $j = 1, \ldots, k$, and $\alpha_1 < \alpha_2 < \cdots < \alpha_k$. Observing that $\delta_{\alpha_1} < \delta_{\alpha_2} < \cdots < \delta_{\alpha_k}$, we use (5) repeatedly, and thus we get

$$\begin{aligned}
\|f\|^q &= \Big\|\sum_{j=1}^{k} f_j\Big\|^q \ge \Big\|\sum_{j=1}^{k-1} f_j\Big\|^q + C\|f_k\|_M{}^q \\
&\ge \Big\|\sum_{j=1}^{k-2} f_j\Big\|^q + C\|f_{k-1}\|_M{}^q + C\|f_k\|_M^q \ge \cdots \ge \|f_1\|^q + C\sum_{j=2}^{k}\|f_j\|_M^q \\
&\ge \min\{1, C\}\sum_{i=1}^{\infty} 2^{-iq}\sum_{j=1}^{k}\|f_j\|_M{}^q \ge \min\{1, C\}\sum_{i=1}^{\infty} 2^{-iq}\sum_{j=1}^{k}|f_j(v_i^{\alpha_j})|^q \\
&= \min\{1, C\}\sum_{i=1}^{\infty}\sum_{\alpha\in[\omega_0,\omega_1)} 2^{-iq}|f(v_i^\alpha)|^q = \min\{1, C\}\|Tf\|_q{}^q. \qquad (6)
\end{aligned}$$

Therefore $Tf \in \ell_q(\mathbb{N} \times [\omega_0, \omega_1))$ for all $f \in Y$, and $T(Y) \subset \ell_q(\mathbb{N} \times [\omega_0, \omega_1))$.

Now, it follows easily from the properties of the P^*_α that Y is norm-dense in X^*. Notice that the restricted mapping $T\restriction_Y$ is a bounded linear operator from Y into $\ell_q(\mathbb{N} \times [\omega_0, \omega_1))$, so it has a bounded linear extension $\widetilde{T}$ to X^*, with values in $\ell_q(\mathbb{N} \times [\omega_0, \omega_1))$, and with the same norm. Since $T : X^* \to \mathbb{R}^{\mathbb{N}\times[\omega_0,\omega_1)}$ is pointwise continuous, we easily get that $T = \widetilde{T}$. Consequently $T(X^*) \subset \ell_q(\mathbb{N} \times [\omega_0, \omega_1))$.

Let u be an element of $\ell_p(\mathbb{N} \times [\omega_0, \omega_1))$ (or of $c_0(\mathbb{N} \times [\omega_0, \omega_1))$). In order to prove the weak* continuity of the functional $u \circ T : X^* \to \mathbb{R}$ defined by $u \circ T(x^*) = \langle u, Tx^*\rangle$, $x^* \in X^*$, it suffices, by the Banach–Dieudonné theorem, to check the weak* continuity of $u \circ T$ restricted to B_{X^*}. But, on the (bounded) set TB_{X^*}, the topology of pointwise convergence coincides with the weak* topology. Hence the weak*-to-pointwise continuity of T gives that $u \circ T$ is weak* continuous. It then follows that T is weak*-to-weak* continuous and Claim 5 is thus proved.

Finally, from the above, we can conclude that the adjoint operator T^* goes from $\ell_p(\mathbb{N} \times [\omega_0, \omega_1))$ (or $c_0(\mathbb{N} \times [\omega_0, \omega_1))$ into X. And since, T is injective, $T^*(\ell_p(\mathbb{N} \times [\omega_0, \omega_1))$ (or $T^*(c_0(\mathbb{N} \times [\omega_0, \omega_1))))$ is dense in X and we have completed our proof of the first half of Theorem 5, namely that (i) implies (ii).

Assume that (ii) holds. The space $\ell_p(\Gamma)$, with $1 < p < \infty$, is reflexive, and $c_0(\Gamma)$ is weakly compactly generated (it is enough to consider the set of the canonical unit vectors). The rest follows from the corresponding implication of Theorem 7 (proved below) and from Remark 4(v).

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PROOF OF COROLLARY 6. Let $p \in (1, \infty]$. Let $(X, \|\cdot\|)$ be a subspace of $\ell_p(\omega_1)$ (or $c_0(\omega_1)$). Put $q = p/(p-1)$. Let $Q : \ell_q(\omega_1) \to X^*$ be the canonical quotient mapping. The unit ball $B_{\ell_p(\omega_1)}$ is a $\|\cdot\|_p$-asymptotically p-flat set (with constant $C = 1$). We shall prove that B_X is a $\|\cdot\|$-asymptotically p-flat set in X. To this end take $x^* \in X^*$ and a weak*-null sequence (x_n^*) in X^*. Select first a subsequence $(x_{n_k}^*)$ of (x_n^*) such that $\|x_{n_k}^*\| \to \limsup_{n\to\infty} \|x_n^*\|$ as $k \to \infty$. Let (l_k^*) be a sequence in $\ell_q(\omega_1)$ such that $Ql_k^* = x^* + x_{n_k}^*$ and $\|l_k^*\| = \|x^* + x_{n_k}^*\|$ for all $k \in \mathbb{N}$. Further, the countability of the supports allows us to select a subsequence $(l_{k_j}^*)$ of (l_k^*) that is weak*-convergent to some $l^* \in \ell_q(\omega_1)$. Obviously, $Ql^* = x^*$. Then

$$
\begin{aligned}
\limsup_{n\to\infty} \|x^* + x_n^*\|^q &\geq \limsup_{j\to\infty} \|x^* + x_{n_{k_j}}^*\|^q \\
&= \limsup_{j\to\infty} \|l_{k_j}^*\|^q = \limsup_{j\to\infty} \|l^* + (l_{k_j}^* - l^*)\|^q = \|l^*\|^q + \limsup_{j\to\infty} \|l_{k_j}^* - l^*\|^q \\
&\geq \|x^*\|^q + \limsup_{j\to\infty} \|x_{n_{k_j}}^*\|^q = \|x^*\|^q + \limsup_{n\to\infty} \|x_n^*\|^q.
\end{aligned}
$$

We obtained that B_X is $\|\cdot\|$-asymptotically p-flat. It is enough now to apply Theorem 5. □

REMARK 12. The proof of Corollary 6 shows that, for $p \in (1, \infty]$, if X is a subspace of a Banach space Z such that B_Z is asymptotically p-flat and B_{Z^*} is weak* sequentially compact, then B_X is also asymptotically p-flat. As a byproduct, we get, from Theorem 5, that if Z is moreover WLD, then X is $\ell_p(\omega_1)$-generated. Let us recall that the class of Banach spaces whose dual unit ball is weak* sequentially compact is quite large. Weakly Lindelöf-determined spaces as well as weak Asplund spaces, even Gateaux differentiability spaces, are such (see [2, Theorem 2.1.2]).

The following intermediate result will be used in the proof of Theorem 7.

LEMMA 13. *Let X be a Banach space such that B_{X^*} is* weak**-sequentially compact. Then, for all $p \in (1, \infty]$, every asymptotically p-flat subset M of X is an Asplund set.*

PROOF. Let N be a countable subset of M. Then, $\mathrm{span}_{\mathbb{Q}}(N)$, the set of all linear rational combinations of elements in N, is also countable. Let $Y = \overline{\mathrm{span}}(N)$. Let $Q : X^* \to Y^*$ be the canonical quotient mapping. Given $y \in \mathrm{span}_{\mathbb{Q}}(N)$, find $\phi(y) = y^* \in S_{Y^*}$ such that $\langle y, y^*\rangle = \|y\|$. The separation theorem gives

$$\overline{\Gamma_{\mathbb{Q}}[\phi(\mathrm{span}_{\mathbb{Q}}(N))]}^{w^*} = B_{Y^*},$$

where $\Gamma_{\mathbb{Q}}[\cdot]$ denotes the absolutely rational-convex hull. We shall prove that the (countable) set $\Gamma_{\mathbb{Q}}[\phi(\mathrm{span}_{\mathbb{Q}}(N))]$ is $\|\cdot\|_N$-dense in X^*. This will conclude the proof.

To this end, choose any $x^* \in X^*$. If $x^* \in Y^\perp$, we can find, as $\|\cdot\|$-close (in particular, as $\|\cdot\|_N$-close) to x as we wish, an element which is not in $Y^\perp$. Thus, we may assume, without loss of generality, that $x^* \notin Y^\perp$ and that, for the moment being, $\|Qx^*\| = 1$. Let $y^* = Qx^*$ $(\in S_{Y^*})$. Since Y is separable, we can

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find a sequence (y_n^*) in $\Gamma_{\mathbb{Q}}[\phi(\mathrm{span}_{\mathbb{Q}}(N))]$ such that $y_n^* \xrightarrow{w^*} y^*$ as $n \to \infty$. For each element $z^* \in \Gamma_{\mathbb{Q}}[\phi(\mathrm{span}_{\mathbb{Q}}(N))]$, choose a single element $\psi(z^*)$ in B_{X^*} such that $Q(\psi(z^*)) = z^*$. Let $x_n^* = \psi(y_n^*)$ for all $n \in \mathbb{N}$. By the assumption, the sequence (x_n^*) has a subsequence, denoted again by (x_n^*), such that $x_n^* \xrightarrow{w^*} x_0^*$. Then we have

$$\limsup_{n\to\infty} \|x_n^*\|^q \geq \|x_0^*\|^q + C \limsup_{n\to\infty} \|x_0^* - x_n^*\|_M^q.$$

Obviously, $Qx_n^* = y_n^*$ $(\xrightarrow{w^*} y^*)$. Hence $Qx_0^* = y^*$, and so $\|x_0^*\| = 1$. It follows that $\limsup_n \|x_n^*\|^q = 1$ and we get $\|x_n^* - x_0^*\|_M \to 0$. In particular, $\|\psi(y_n^*) - x_0^*\|_N \to 0$. This proves the assertion for an element $x^* \in X^*$ such that $\|Qx^*\| = 1$, since the sequence (x_n^*) is in the countable set $\psi(\Gamma_{\mathbb{Q}}[\phi(\mathrm{span}_{\mathbb{Q}}(N))])$. A homogeneity argument involving rational multiples of arbitrary elements in X^* concludes the proof. □

PROOF OF THEOREM 7. Assume that (i) holds. We shall follow almost word for word the proof of the corresponding implication of Theorem 5 (already proved), with the following changes. By Lemma 13, M is an Asplund set. Then we apply Proposition 11 with $N = M$ and $\Gamma = \emptyset$ and get a PRI $(P_\alpha \mid \alpha \in [\omega_0, \omega_1])$ on $(X, \|\cdot\|)$ such that $P_\alpha(M) \subset M$ for every $\alpha \in [\omega_0, \omega_1]$, and $\|P_\lambda^* f - P_\alpha^* f\|_M \to 0$ as $\alpha \uparrow \lambda$ whenever $f \in X^*$ and $\lambda \in (\omega_0, \omega_1]$ is a limit ordinal. From this we get that still $\bigcup_{\omega_0 \leq \alpha < \omega_1} P_\alpha^* X^* = X^*$.

Claims 1, 2, 3, and 4 now read as follows.

CLAIM 1′. *For every $f \in X^*$, $f \neq 0$, every $\varepsilon > 0$, and every $\alpha \in [\omega_0, \omega_1)$, there is $\gamma_{f,\varepsilon,\alpha} \in (\alpha, \omega_1)$ such that*

$$\|f + g\|_M{}^q \geq (1 - \varepsilon)\|f\|_M{}^q + C\|g\|_M{}^q \quad \textit{whenever } g \in \operatorname{Ker} P_{\gamma_{f,\varepsilon,\alpha}}^* \textit{ and } \|g\| < \frac{1}{\varepsilon}.$$

CLAIM 2′. *For every $\alpha \in [\omega_0, \omega_1)$ there exists $\beta_\alpha \in (\alpha, \omega_1)$ such that*

$$\|f + g\|_M{}^q \geq \|f\|_M{}^q + C\|g\|_M{}^q \quad \textit{whenever } f \in P_\alpha^* X^* \textit{ and } g \in \operatorname{Ker} P_{\beta_\alpha}^*.$$

CLAIM 3′. *For every $\alpha \in [\omega_0, \omega_1)$ there exists $\tau_\alpha \in (\alpha, \omega_1)$ such that*

$$\|f + g\|^q \geq \|f\|^q + C\|g\|_M{}^q \quad \textit{whenever } f \in P_{\tau_\alpha}^* X^* \textit{ and } g \in \operatorname{Ker} P_{\tau_\alpha}^*.$$

CLAIM 4′. *There exists an increasing long sequence $(\delta_\alpha)_{\omega_0 \leq \alpha \leq \omega_1}$ in $[\omega_0, \omega_1]$, with $\delta_{\omega_0} = \omega_0$ and $\delta_{\omega_1} = \omega_1$, and such that for every $\alpha \in [\omega_0, \omega_1)$ we have*

$$\|f + g\|_M{}^q \geq \|f\|_M{}^q + C\|g\|_M{}^q \quad \textit{whenever } f \in P_{\delta_\alpha}^* X^* \textit{ and } g \in \operatorname{Ker} P_{\delta_\alpha}^*.$$

The proofs of Claims 1′, 2′, 3′ and 4′ follow the proofs of Claims 1, 2, 3, and 4 with the change that the norm $\|\cdot\|$ in X should be everywhere replaced by the seminorm $\|\cdot\|_M$. Moreover, in the proof of Claim 2′, we use the fact that once M is an Asplund

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set then the $P_\alpha^* X^*$, $\omega_0 \le \alpha < \omega_1$, are separable spaces in the metric coming from the seminorm $\|\cdot\|_M$.

A corresponding Claim 5′ (a duplicate of Claim 5) should be stated. To prove it, we mimic the proof of that one profiting from the inequality $\|\cdot\|_M \le \|\cdot\|$ and from the fact that the properties of the P_α^* guarantee that Y is dense in X^* in the metric coming from $\|\cdot\|_M$.

The rest of the proof that (i) implies (ii) is the same as in the proof of Theorem 5.

Assume that (ii) holds. Take $p \in (1, \infty)$. Assume there exists a bounded linear operator $S : \ell_p(\omega_1) \to X$, with dense range. Put $q = p/(p-1)$ and $M = S(B_{\ell_p(\omega_1)})$. Then $S^* : X^* \to \ell_q(\omega_1)$ is an injection, and hence the space X is weakly Lindelöf determined. Let $f \in X^*$ and consider a weak*-null sequence (f_n) in X^*. Then $S^* f_n \xrightarrow{w^*} 0$, and hence

$$\begin{aligned}\limsup_{n\to\infty} \|f + f_n\|_M{}^q &= \limsup_{n\to\infty} \sup\{|\langle x, f + f_n\rangle|^q : x \in S(B_{\ell_p(\omega_1)})\}\\ &= \limsup_{n\to\infty} \sup\{|\langle u, S^* f + S^* f_n\rangle|^q : u \in B_{\ell_p(\omega_1)}\}\\ &= \limsup_{n\to\infty} \|S^* f + S^* f_n\|_q{}^q \ge \|S^* f\|_q{}^q + \limsup_{n\to\infty} \|S^* f_n\|_q{}^q\\ &= \|f\|_M{}^q + \limsup_{n\to\infty} \|f_n\|_M{}^q;\end{aligned}$$

here $\|\cdot\|_q$ means the canonical norm on ℓ_q. This shows that the set M is innerly asymptotically p-flat.

The case of inner asymptotical ∞-flatness can be dealt with analogously. □

PROOF OF THEOREM 9. Let e_γ, $\gamma \in \Gamma$, denote the canonical unit vectors in $c_0(\Gamma)$. Put $\Gamma_1 = \{\gamma \in \Gamma \mid Te_\gamma \neq 0\}$. Clearly, Γ_1 is uncountable. We observe that the set $\{Te_\gamma \mid \gamma \in \Gamma_1\}$ countably supports all of X^*. Then we apply Proposition 11 with $M = N = \{0\}$ and $\Gamma = \Gamma_1$ to get a PRI $(P_\alpha \mid \omega_0 \le \alpha \le \omega_1)$ on $(X, \|\cdot\|)$ such that $P_\alpha(Te_\gamma) \in \{0, Te_\gamma\}$ for every $\alpha \in (\omega_0, \omega_1)$ and every $\gamma \in \Gamma_1$. Put

$$A = \{\alpha \in [\omega_0, \omega_1) : P_\alpha(Te_\gamma) = 0 \text{ and } P_{\alpha+1}(Te_\gamma) = Te_\gamma \text{for some } \gamma \in \Gamma_1\}.$$

For every $\alpha \in A$ then pick one $\gamma_\alpha \in \Gamma_1$ such that $P_\alpha(Te_{\gamma_\alpha}) = 0$ and $P_{\alpha+1}(Te_{\gamma_\alpha}) = Te_{\gamma_\alpha}$. Let $\Gamma_2 = \{\gamma_\alpha \mid \alpha \in A\}$. This set is uncountable, for otherwise $T(c_0(\Gamma_1))$ would be separable. A simple 'countability' argument yields another uncountable subset Γ_0 of Γ_2 and $\Delta > 0$ such that $\|Te_\gamma\| > \Delta$ for every $\gamma \in \Gamma_0$.

Take any $a \in c_0(\Gamma_0)$. Let $\{\delta_1, \delta_2, \ldots\}$ be an infinite countable subset of Γ_0 containing the support of a. For $i \in \mathbb{N}$ let a_i be the δ_i-th coordinate of a. Then $\|\sum_{i=1}^n a_i e_{\delta_i} - a\| \to 0$ as $n \to \infty$. For every $i \in \mathbb{N}$ find $\alpha_i \in [\omega_0, \omega_1)$ such that $P_{\alpha_i+1}(Te_{\delta_i}) = Te_{\delta_i}$ and $P_{\alpha_i}(Te_{\delta_i}) = 0$. Observe that $\alpha_i \neq \alpha_j$ whenever $i, j \in \mathbb{N}$ are distinct. Then the 'orthogonality' of the projections $P_{\alpha_i+1} - P_{\alpha_i}$, $i \in \mathbb{N}$, yields that for every fixed $n, j \in \mathbb{N}$, with $n > j$, we have

$$\Big\|\sum_{i=1}^n a_i Te_{\delta_i}\Big\| \ge \frac{1}{2}\Big\|(P_{\alpha_j+1} - P_{\alpha_j})\Big(\sum_{i=1}^n a_i Te_{\delta_i}\Big)\Big\| = \frac{1}{2}\|a_j Te_{\delta_j}\|\Big(\ge \frac{1}{2}|a_j|\Delta\Big).$$

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Hence

$$\begin{aligned}\|Ta\| &= \lim_{n\to\infty}\left\|T\left(\sum_{i=1}^{n} a_i e_{\delta_i}\right)\right\| = \lim_{n\to\infty}\left\|\sum_{i=1}^{n} a_i T e_{\delta_i}\right\| \\ &\geq \frac{\Delta}{2}\max\{|a_j|,\ j\in\mathbb{N}\} = \frac{\Delta}{2}\max\{|a_\gamma| : \gamma\in\Gamma_0\} = \frac{\Delta}{2}\|a\|.\end{aligned}$$

This proves that T is an isomorphism from $c_0(\Gamma_0)$ into X. □

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