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# A REMARK ON RATIONAL CUBOIDS

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It is a well known unsolved problem whether there exists a *perfect rational cuboid*, a rectangular parallelepiped whose edges, face diagonals and body diagonal all have integer lengths. (For a recent survey see Leech [4].) This would require simultaneous solution in integers of the four equations

$$x_2^2+x_3^2=y_1^2,\quad x_3^2+x_1^2=y_2^2,\quad x_1^2+x_2^2=y_3^2,\quad x_1^2+x_2^2+x_3^2=z^2.$$

An early parametric solution of the first three of these equations (a *rational cuboid*) was given by Saunderson [5], but is usually associated with the name of Euler; it has

(1) $$x_1=4abc,\qquad x_2=a(4b^2-c^2),\qquad x_3=b(4a^2-c^2),$$

where $a, b, c$ satisfy $a^2+b^2=c^2$. Spohn [6] showed that such a cuboid cannot be perfect.

A construction, known to Euler, for deriving one rational cuboid from another, consists of replacing $x_1, x_2, x_3$ by their products in pairs $x_2x_3, x_3x_1, x_1x_2$. From the Euler cuboid (1), after removing common factors, we obtain

(2) $$x_1=(4a^2-c^2)(4b^2-c^2),\qquad x_2=4bc(4a^2-c^2),\qquad x_3=4ac(4b^2-c^2).$$

Spohn [7] was unable to complete a proof that such a cuboid cannot be perfect; complete proofs have been given by Chein [1] and Lagrange [3].

This note is to remark that both impossibilities are immediate consequences of the nonexistence of integers $m, n$ $(n\neq 0)$ satisfying both $m^2+n^2=p^2$ and $m^2+5n^2=q^2$. This last impossibility was probably known to Euler, but seems to have been first formally established by Collins [2].

For the Euler cuboid (1) we have $x_2^2+x_3^2=c^6$, and so

$$\begin{aligned}x_1^2+x_2^2+x_3^2&=c^2(16a^2b^2+c^4)\\&=c^2((a^2-b^2)^2+5(2ab)^2).\end{aligned}$$

This last factor cannot be square, as $m=a^2-b^2$, $n=2ab$ satisfy $m^2+n^2=(a^2+b^2)^2$ and cannot also satisfy $m^2+5n^2=q^2$.

For the derived cuboid (2) we have $x_1=16a^2b^2-3c^4$ and $x_2^2+x_3^2=16c^8$,

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and so

$$x_1^2+x_2^2+x_3^2=256a^4b^4-96a^2b^2c^4+25c^8$$
$$=(16a^2b^2-5c^4)^2+(8abc^2)^2.$$

This expression cannot be square, as $m=16a^2b^2-5c^4$, $n=8abc^2$ satisfy $m^2+5n^2=(16a^2b^2+5c^4)^2$ and cannot also satisfy $m^2+n^2=p^2$.

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