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CENTRAL LIMIT THEOREM FOR A BIRTH–GROWTH MODEL WITH POISSON ARRIVALS AND RANDOM GROWTH SPEED

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Abstract

We consider Gaussian approximation in a variant of the classical Johnson–Mehl birth–growth model with random growth speed. Seeds appear randomly in $\mathbb{R}^d$ at random times and start growing instantaneously in all directions with a random speed. The locations, birth times, and growth speeds of the seeds are given by a Poisson process. Under suitable conditions on the random growth speed, the time distribution, and a weight function $h:\mathbb{R}^d \times [0, \infty) \to [0, \infty)$, we prove a Gaussian convergence of the sum of the weights at the exposed points, which are those seeds in the model that are not covered at the time of their birth. Such models have previously been considered, albeit with fixed growth speed. Moreover, using recent results on stabilization regions, we provide non-asymptotic bounds on the distance between the normalized sum of weights and a standard Gaussian random variable in the Wasserstein and Kolmogorov metrics.

Keywords: Spatial birth growth; inhomogeneous Poisson process; Johnson–Mehl tessellation; stabilization; growth frontier; exposed seeds

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1. Introduction

In the spatial Johnson–Mehl growth model, seeds arrive at random times t_i, $i \in \mathbb{N}$, at random locations x_i, $i \in \mathbb{N}$, in $\mathbb{R}^d$, according to a Poisson process $(x_i, t_i)_{i\in\mathbb{N}}$ on $\mathbb{R}^d \times \mathbb{R}_+$, where $\mathbb{R}_+ := [0, \infty)$. Once a seed is born at time t, it begins to form a cell by growing radially in all directions at a constant speed $v \geq 0$, so that by time t' it occupies the ball of radius $v(t' - t)$. The parts of the space claimed by the seeds form the so-called Johnson–Mehl tessellation; see [7] and [16]. This is a generalization of the classical Voronoi tessellation, which is obtained if all births occur simultaneously at time zero.

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The study of such birth–growth processes started with the work of Kolmogorov [11] to model crystal growth in two dimensions. Since then, this model has seen applications in various contexts, such as phase transition kinetics, polymers, ecological systems, and DNA replications, to name a few; see [4, 7, 16] and references therein. A central limit theorem for the Johnson–Mehl model with inhomogeneous arrivals of the seeds was obtained in [5].

Variants of the classical spatial birth–growth model can be found, sometimes as a particular case of other models, in many papers. Among them, we mention [2] and [17], where the birth–growth model appears as a particular case of a random sequential packing model, and [20], which studied a variant of the model with non-uniform deterministic growth patterns. The main tools rely on the concept of stabilization by considering regions where the appearance of new seeds influences the functional of interest.

In this paper, we consider a generalization of the Johnson–Mehl model by introducing random growth speeds for the seeds. This gives rise to many interesting features in the model, most importantly, long-range interactions if the speed can take arbitrarily large values with positive probability. Therefore, the model with random speed is no longer stabilizing in the classical sense of [13] and [18], since distant points may influence the growth pattern if their speeds are sufficiently high. It should be noted that, even in the constant-speed setting, we substantially improve and extend limit theorems obtained in [5].

We consider a birth–growth model, determined by a Poisson process η in $\mathbb{X} := \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $\mu := \lambda \otimes \theta \otimes \nu$, where λ is the Lebesgue measure on $\mathbb{R}^d$, θ is a non-null locally finite measure on $\mathbb{R}_+$, and ν is a probability distribution on $\mathbb{R}_+$ with $\nu(\{0\}) < 1$. Each point $\boldsymbol{x}$ of this point process η has three components (x, t_x, v_x), where $v_x \in \mathbb{R}_+$ denotes the random speed of a seed born at location $x \in \mathbb{R}^d$ and whose growth commences at time $t_x \in \mathbb{R}_+$. In a given point configuration, a point $\boldsymbol{x} := (x, t_x, v_x)$ is said to be *exposed* if there is no other point (y, t_y, v_y) in the configuration with $t_y < t_x$ and $\|x - y\| \le v_y(t_x - t_y)$, where $\|\cdot\|$ denotes the Euclidean norm. Notice that the event that a point $(x, t_x, v_x) \in \eta$ is exposed depends only on the point configuration in the region

$$L_{x,t_x} := \big\{(y, t_y, v_y) \in \mathbb{X} : \|x - y\| \le v_y(t_x - t_y)\big\}. \tag{1.1}$$

Namely, $\boldsymbol{x}$ is exposed if and only if η has no points (apart from $\boldsymbol{x}$) in L_{x,t_x}.

The *growth frontier* of the model can be defined as the random field

$$\min_{(x,t_x,v_x)\in\eta} \left(t_x + \frac{1}{v_x}\|y - x\|\right), \quad y \in \mathbb{R}^d. \tag{1.2}$$

This is an example of an extremal shot-noise process; see [10]. Its value at a point $y \in \mathbb{R}^d$ corresponds to a seed from η whose growth region covers y first. It should be noted here that this covering seed need not be an exposed one. In other words, because of random speeds, it may happen that the cell grown from a non-exposed seed shades a subsequent seed which would be exposed otherwise. This excludes possible applications of our model with random growth speed to crystallisation, where a more natural model would be to not allow a non-exposed seed to affect any future seeds. But this creates a causal chain of influences that seems quite difficult to study with the currently known methods of stabilization for Gaussian approximation.

Nonetheless, models such as ours are natural in telecommunication applications, with the speed playing the role of the weight or strength of a particular transmission node, where the growth frontier defined above can be used as a variant of the additive signal-to-interference model from [1, Chapter 5]. Furthermore, similar models can be applied in the ecological or epidemiological context, where a non-visible event influences appearances of others. Suppose we have a barren land and a drone/machine is planting seeds from a mixture of plant species at random times and random locations for reforestation. Each seed, after falling on the ground, starts growing a bush around it at a random speed depending on its species. If a new seed falls on a part of the ground that is already covered in bushes, it is still allowed to form its own bush; i.e., there is no exclusion. Now the number of exposed points in our model above translates to the number of seeds that start a bush on a then barren piece of land, rather than starting on a piece of ground already covered in bushes. This, in some sense, can explain the efficiency of the reforestation process, i.e., what fraction of the seeds were planted on barren land, in contrast to being planted on already existing bushes.

Given a measurable weight function $h: \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$, the main object of interest in this paper is the sum of h over the space–time coordinates (x, t_x) of the exposed points in η. These can be defined as those points (y, t_y) where the growth frontier defined at (1.2) has a local minimum (see Section 2 for a precise definition). Our aim is to provide sufficient conditions for Gaussian convergence of such sums. A standard approach for proving Gaussian convergence for such statistics relies on stabilization theory [2, 8, 17, 20]. While in the stabilization literature one commonly assumes that the so-called stabilization region is a ball around a given reference point, the region L_{x,t_x} is unbounded and it seems that it is not expressible as a ball around $\boldsymbol{x}$ in some different metric. Moreover, our stabilization region is set to be empty if $\boldsymbol{x}$ is not exposed.

The main challenge when working with random unbounded speeds of growth is that there are possibly very long-range interactions between seeds. This makes the use of balls as stabilization regions vastly suboptimal and necessitates the use of regions of a more general shape. In particular, we only assume that the random growth speed in our model has finite moment of order $7d$ (see the assumption (C) in Section 2), and this allows for some power-tailed distributions for the speed.

The recent work [3] introduced a new notion of *region-stabilization* which allows for more general regions than balls and, building on the seminal work [14], provides bounds on the rate of Gaussian convergence for certain sums of region-stabilizing score functions. We will utilize this to derive bounds on the Wasserstein and Kolmogorov distances, defined below, between a suitably normalized sum of weights and the standard Gaussian distribution. For real-valued random variables X and Y, the *Wasserstein distance* between their distributions is given by

$$d_{\mathrm{W}}(X, Y) := \sup_{f \in \mathrm{Lip}_1} |\mathbb{E} f(X) - \mathbb{E} f(Y)|,$$

where Lip_1 denotes the class of all Lipschitz functions $f: \mathbb{R} \to \mathbb{R}$ with Lipschitz constant at most one. The *Kolmogorov distance* between the distributions is given by

$$d_{\mathrm{K}}(X, Y) := \sup_{t \in \mathbb{R}} |\mathbb{P}\{X \le t\} - \mathbb{P}\{Y \le t\}|.$$

The rest of the paper is organized as follows. In Section 2, we describe the model and state our main results. In Section 3, we prove a result providing necessary upper and lower bounds for the variance of our statistic of interest. Section 4 presents the proofs of our quantitative bounds.

2. Model and main results

Recall that we work in the space $\mathbb{X} := \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$, $d \in \mathbb{N}$, with the Borel σ-algebra. The points from $\mathbb{X}$ are written as $\boldsymbol{x} := (x, t_x, v_x)$, so that $\boldsymbol{x}$ designates a seed born in position x at time t_x, which then grows radially in all directions with speed v_x. For $\boldsymbol{x} \in \mathbb{X}$, the set

$$G_{\boldsymbol{x}} = G_{x,t_x,v_x} := \left\{(y, t_y) \in \mathbb{R}^d \times \mathbb{R}_+ : t_y \ge t_x, \|y - x\| \le v_x(t_y - t_x)\right\}$$

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is the growth region of the seed $\boldsymbol{x}$. Denote by $\mathbf{N}$ the family of σ-finite counting measures $\mathcal{M}$ on $\mathbb{X}$ equipped with the smallest σ-algebra $\mathscr{N}$ such that the maps $\mathcal{M} \mapsto \mathcal{M}(A)$ are measurable for all Borel A. We write $\boldsymbol{x} \in \mathcal{M}$ if $\mathcal{M}(\{\boldsymbol{x}\}) \geq 1$. For $\mathcal{M} \in \mathbf{N}$, a point $\boldsymbol{x} \in \mathcal{M}$ is said to be *exposed* in $\mathcal{M}$ if it does not belong to the growth region of any other point $\boldsymbol{y} \in \mathcal{M}$, $\boldsymbol{y} \neq \boldsymbol{x}$. Note that the property of being exposed is not influenced by the speed component of $\boldsymbol{x}$.

The *influence set* $L_{\boldsymbol{x}} = L_{x,t_x}$, $\boldsymbol{x} \in \mathbb{X}$, defined at (1.1), is exactly the set of points that were born before time t_x and which at time t_x occupy a region that covers the location x, thereby shading it. Note that $\boldsymbol{y} \in L_{\boldsymbol{x}}$ if and only if $\boldsymbol{x} \in G_{\boldsymbol{y}}$. Clearly, a point $\boldsymbol{x} \in \mathcal{M}$ is exposed in $\mathcal{M}$ if and only if $\mathcal{M}(L_{\boldsymbol{x}} \setminus \{x\}) = 0$. We write $(y, t_y, v_y) \preceq (x, t_x)$ or $\boldsymbol{y} \preceq \boldsymbol{x}$ if $\boldsymbol{y} \in L_{x,t_x}$ (recall that the speed component of $\boldsymbol{x}$ is irrelevant in such a relation), and so $\boldsymbol{x}$ is not an exposed point with respect to $\delta_{\boldsymbol{y}}$, where $\delta_{\boldsymbol{y}}$ denotes the Dirac measure at $\boldsymbol{y}$.

For $\mathcal{M} \in \mathbf{N}$ and $\boldsymbol{x} \in \mathcal{M}$, define

$$H_{\boldsymbol{x}}(\mathcal{M}) \equiv H_{x,t_x}(\mathcal{M}) := \mathbb{1}\{\boldsymbol{x} \text{ is exposed in } \mathcal{M}\} = \mathbb{1}_{\mathcal{M}(L_{x,t_x} \setminus \{\boldsymbol{x}\})=0}.$$

A generic way to construct an additive functional on the exposed points is to consider the sum of weights of these points, where each exposed point $\boldsymbol{x}$ contributes a weight $h(\boldsymbol{x})$ for some measurable $h : \mathbb{X} \to \mathbb{R}_+$. In the following we consider weight functions $h(\boldsymbol{x})$ which are products of two measurable functions $h_1 : \mathbb{R}^d \to \mathbb{R}_+$ and $h_2 : \mathbb{R}_+ \to \mathbb{R}_+$ of the locations and birth times, respectively, of the exposed points. In particular, we let $h_1(x) = \mathbb{1}_W(x) = \mathbb{1}\{x \in W\}$ for a window $W \subset \mathbb{R}^d$, and $h_2(t) = \mathbb{1}\{t \leq a\}$ for $a \in (0, \infty)$. Then

$$F(\mathcal{M}) := \int_{\mathbb{X}} h_1(x) h_2(t_x) H_{\boldsymbol{x}}(\mathcal{M}) \mathcal{M}(\mathrm{d}\boldsymbol{x}) = \sum_{\boldsymbol{x} \in \mathcal{M}} \mathbb{1}_{x \in W} \mathbb{1}_{t_x \leq a} H_{\boldsymbol{x}}(\mathcal{M}) \tag{2.1}$$

is the number of exposed points from $\mathcal{M}$ located in W and born before time a. Note here that when we add a new point $\boldsymbol{y} = (y, t_y, v_y) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ to a configuration $\mathcal{M} \in \mathbf{N}$ not containing it, the change in the value of F is not a function of only $\boldsymbol{y}$ and some local neighborhood of it, but rather depends on points in the configuration that might be very far away. Indeed, for $\boldsymbol{y} \notin \mathcal{M}$ we have

$$F(\mathcal{M} + \delta_{\boldsymbol{y}}) - F(\mathcal{M}) = \mathbb{1}_{y \in W} \mathbb{1}_{t_y \leq a} H_{\boldsymbol{y}}(\mathcal{M} + \delta_{\boldsymbol{y}}) - \sum_{\boldsymbol{x} \in \mathcal{M}} \mathbb{1}_{x \in W} \mathbb{1}_{t_x \leq a} \mathbb{1}_{\boldsymbol{x} \in L_{(y,t_y)}};$$

that is, F may increase by one when $\boldsymbol{y}$ is exposed in $\mathcal{M} + \delta_{\boldsymbol{y}}$, while simultaneously, any point $\boldsymbol{x} \in \mathcal{M}$ which was previously exposed in $\mathcal{M}$ may not be so anymore after the addition of $\boldsymbol{y}$, if it happens to fall in the influence set $L_{(y,t_y)}$ of $\boldsymbol{y}$. This necessitates the use of region-stabilization.

Recall that η is a *Poisson process* in $\mathbb{X}$ with intensity measure μ, being the product of the Lebesgue measure λ on $\mathbb{R}^d$, a non-null locally finite measure θ on $\mathbb{R}_+$, and a probability measure ν on $\mathbb{R}_+$ with $\nu(\{0\}) < 1$. Note that η is a simple random counting measure. The main goal of this paper is to find sufficient conditions for a Gaussian convergence of $F \equiv F(\eta)$ as defined at (2.1). The functional $F(\eta)$ is a region-stabilizing functional, in the sense of [3], and can be represented as $F(\eta) = \sum_{\boldsymbol{x} \in \eta} \xi(\boldsymbol{x}, \eta)$, where the score function ξ is given by

$$\xi(\boldsymbol{x}, \mathcal{M}) := \mathbb{1}_{x \in W} \mathbb{1}_{t_x \leq a} H_{\boldsymbol{x}}(\mathcal{M}), \quad \boldsymbol{x} \in \mathcal{M}, \tag{2.2}$$

with the region of stabilization being L_{x,t_x} when $\boldsymbol{x}$ is an exposed point (see Section 4 for more details). As a convention, let $\xi(\boldsymbol{x}, \mathcal{M}) = 0$ if $\mathcal{M} = 0$ or if $x \notin \mathcal{M}$. Theorem 2.1 in [3] yields ready-to-use bounds on the Wasserstein and Kolmogorov distances between F, suitably

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normalized, and a standard Gaussian random variable N upon validating Equation (2.1) and the conditions (A1) and (A2) therein. We consistently follow the notation of [3].

Now we are ready to state our main results. First, we list the necessary assumptions on our model. In the sequel, we drop the λ in Lebesgue integrals and simply write dx instead of $\lambda(\mathrm{d}x)$. Our assumptions are as follows:

(A) The window W is compact convex with non-empty interior.

(B) For all $x > 0$,

$$\int_0^\infty e^{-x\Lambda(t)}\,\theta(\mathrm{d}t) < \infty,$$

where

$$\Lambda(t) := \omega_d \int_0^t (t-s)^d \theta(\mathrm{d}s) \tag{2.3}$$

and ω_d is the volume of the d-dimensional unit Euclidean ball.

(C) The moment of ν of order $7d$ is finite, i.e., $\nu_{7d} < \infty$, where

$$\nu_u := \int_0^\infty v^u \nu(\mathrm{d}v), \quad u \geq 0.$$

Note that the function $\Lambda(t)$ given at (2.3) is, up to a constant, the measure of the influence set of any point $\boldsymbol{x} \in \mathbb{X}$ with time component $t_x = t$ (the measure of the influence set does not depend on the location and speed components of $\boldsymbol{x}$). Indeed, the μ-content of L_{x,t_x} is given by

$$\begin{aligned}\mu(L_{x,t_x}) &= \int_0^\infty \int_0^{t_x} \int_{\mathbb{R}^d} \mathbb{1}_{y \in B_{v_y(t_x - t_y)}(x)}\,\mathrm{d}y\,\theta(\mathrm{d}t_y)\nu(\mathrm{d}v_y)\\ &= \int_0^\infty \nu(\mathrm{d}v_y) \int_0^{t_x} \omega_d v_y^d (t_x - t_y)^d \theta(\mathrm{d}t_y) = \nu_d \Lambda(t_x),\end{aligned}$$

where $B_r(x)$ denotes the closed d-dimensional Euclidean ball of radius r centered at $x \in \mathbb{R}^d$. In particular, if θ is the Lebesgue measure on $\mathbb{R}_+$, then $\Lambda(t) = \omega_d\, t^{d+1}/(d+1)$.

The following theorem is our first main result. We denote by $(V_j(W))_{0\leq j\leq d}$ the intrinsic volumes of W (see [19, Section 4.1]), and let

$$V(W) := \max_{0\leq j\leq d} V_j(W). \tag{2.4}$$

Theorem 2.1. *Let η be a Poisson process on $\mathbb{X}$ with intensity measure μ as above, such that the assumptions (A)–(C) hold. Then, for $F := F(\eta)$ as in (2.1) with $a \in (0, \infty)$,*

$$d_{\mathrm{W}}\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq C\left[\frac{\sqrt{V(W)}}{\operatorname{Var} F} + \frac{V(W)}{(\operatorname{Var} F)^{3/2}}\right]$$

and

$$d_{\mathrm{K}}\left(\frac{F - \mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq C\left[\frac{\sqrt{V(W)}}{\operatorname{Var} F} + \frac{V(W)}{(\operatorname{Var} F)^{3/2}} + \frac{V(W)^{5/4} + V(W)^{3/2}}{(\operatorname{Var} F)^2}\right]$$

for a constant $C \in (0, \infty)$ which depends on a, d, the first $7d$ moments of ν, and θ.

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To derive a quantitative central limit theorem from Theorem 2.1, a lower bound on the variance is needed. The following proposition provides general lower and upper bounds on the variance, which are then specialized for measures on $\mathbb{R}_+$ given by

$$\theta(\mathrm{d}t) := t^{\tau}\,\mathrm{d}t, \qquad \tau \in (-1, \infty). \tag{2.5}$$

In the following, $t_1 \wedge t_2$ denotes $\min\{t_1, t_2\}$ for $t_1, t_2 \in \mathbb{R}$. For $a \in (0, \infty)$ and $\tau > -1$, define the function

$$l_{a,\tau}(x) := \gamma\left(\frac{\tau+1}{d+\tau+1}, a^{d+\tau+1}x\right) x^{-(\tau+1)/(d+\tau+1)}, \qquad x > 0, \tag{2.6}$$

where $\gamma(p, z) := \int_0^z t^{p-1}e^{-t}\mathrm{d}t$ is the lower incomplete gamma function.

Proposition 2.1. *Let the assumptions (A)–(C) be in force. For a Poisson process η with intensity measure μ as above and $F := F(\eta)$ as in (2.1),*

$$\frac{\mathrm{Var}(F)}{\lambda(W)} \geq \left[\int_0^a w(t)\theta(\mathrm{d}t) - 2\omega_d \nu_d \int_0^a \int_0^t (t-s)^d w(s)w(t)\theta(\mathrm{d}s)\theta(\mathrm{d}t)\right] \tag{2.7}$$

and

$$\frac{\mathrm{Var}(F)}{\lambda(W)} \leq \Bigg[2\int_0^a w(t)^{1/2}\theta(\mathrm{d}t) + \omega_d^2 \nu_{2d} \int_{[0,a]^2}\int_0^{t_1\wedge t_2} (t_1-s)^d(t_2-s)^d w(t_1)^{1/2}w(t_2)^{1/2}\theta(\mathrm{d}s)\theta^2(\mathrm{d}(t_1,t_2))\Bigg], \tag{2.8}$$

where

$$w(t) := e^{-\nu_d\Lambda(t)} = \mathbb{E}\left[H_{0,t}(\eta)\right]. \tag{2.9}$$

If θ is given by (2.5), then

$$C_1(d-1-\tau) < C_1' \leq \frac{\mathrm{Var}(F)}{\lambda(W)l_{a,\tau}(\nu_d)} \leq C_2(1+\nu_{2d}\nu_d^{-2}) \tag{2.10}$$

for constants C_1, $C_1{}'$, C_2 depending on the dimension d and τ, and $C_1, C_2 > 0$.

We remark here that the lower bound in (2.10) is useful only when $\tau \leq d-1$. We believe that a positive lower bound still exists when $\tau > d-1$, even though our arguments in general do not apply for such τ.

In the case of a deterministic speed v, Proposition 2.1 provides an explicit condition on θ ensuring that the variance scales like the volume of the observation window in the classical Johnson–Mehl growth model. The problem of finding such a condition, explicitly formulated in [6, p. 754], arose in [5], where asymptotic normality for the number of exposed seeds in a region, as the volume of the region approaches infinity, is obtained under the assumption that the variance scales properly. This was by then only shown numerically for the case when θ is the Lebesgue measure and $d = 1, 2, 3, 4$. Subsequent papers [17, 20] derived the variance scaling for θ being the Lebesgue measure and some generalizations of it, but in a slightly different formulation of the model, in which seeds that do not appear in the observation window are automatically rejected and cannot influence the growth pattern in the region W.

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It should be noted that it might also be possible to use [12, Theorem 1.2] to obtain a quantitative central limit theorem and variance asymptotics for statistics of the exposed points in a domain W which is the union of unit cubes around a subset of points in $\mathbb{Z}^d$. For this, one would need to check Assumption 1.1 from the cited paper, which ensures non-degeneracy of the variance, and a moment condition in the form of Equation (1.10) therein. It seems to us that checking Assumption 1.1 may be a challenging task and would involve further assumptions on the model, such as the one we also need in our Proposition 2.1. Controls on the long-range interactions would also be necessary to check [12, Equation (1.10)]. Thus, while this might indeed yield results similar to ours, the goal of the present work is to highlight the application of region-stabilization in this context, which in general is of a different nature from the methods in [12]. For example, the approach in [12] does not apply for Pareto-minimal points in a hypercube considered in [3], since there is no polynomial decay in long-range interactions, while region-stabilization yields optimal rates for the Gaussian convergence.

The bounds in Theorem 2.1 can be specified under two different scenarios. When considering a sequence of weight functions, under suitable conditions Theorem 2.1 provides a quantitative central limit theorem for the corresponding functionals $(F_n)_{n\in\mathbb{N}}$. Keeping all other quantities fixed with respect to n, consider the sequence of non-negative location-weight functions on $\mathbb{R}^d$ given by $h_{1,n} = \mathbb{1}_{n^{1/d}W}$ for a fixed convex body $W \subset \mathbb{R}^d$ satisfying (A). In view of Proposition 2.1, this provides the following quantitative central limit theorem.

Theorem 2.2. *Let the assumptions (A)–(C) be in force. For $n \in \mathbb{N}$ and η as in Theorem 2.1, let $F_n := F_n(\eta)$, where F_n is defined as in (2.1) with a independent of n and $h_1 = h_{1,n} = \mathbb{1}_{n^{1/d}W}$. Assume that θ and ν satisfy*

$$\int_0^a w(t)\theta(\mathrm{d}t) - 2\omega_d\nu_d \int_0^a \int_0^t (t-s)^d w(s)w(t)\theta(\mathrm{d}s)\theta(\mathrm{d}t) > 0\,, \tag{2.11}$$

where w(t) is given at (2.9). Then there exists a constant $C \in (0, \infty)$, depending on a, d, the first 7d moments of ν, θ, and W, such that

$$\max\left\{d_{\mathrm{W}}\left(\frac{F_n - \mathbb{E}F_n}{\sqrt{\operatorname{Var} F_n}}, N\right), d_{\mathrm{K}}\left(\frac{F_n - \mathbb{E}F_n}{\sqrt{\operatorname{Var} F_n}}, N\right)\right\} \le Cn^{-1/2}$$

for all $n \in \mathbb{N}$. In particular, (2.11) is satisfied for θ given at (2.5) with $\tau \in (-1, d-1]$.

Furthermore, the bound on the Kolmogorov distance is of optimal order; i.e., when (2.11) holds, there exists a constant $0 < C' \le C$ depending only on a, d, the first 2d moments of ν, θ, and W, such that

$$d_{\mathrm{K}}\left(\frac{F_n - \mathbb{E}F_n}{\sqrt{\operatorname{Var} F_n}}, N\right) \ge C'n^{-1/2}.$$

When (2.11) is satisfied, Theorem 2.2 yields a central limit theorem for the number of exposed seeds born before time $a \in (0, \infty)$, with rate of convergence of order $n^{-1/2}$. This extends the central limit theorem for the number of exposed seeds from [5] in several directions: the model is generalized to random growth speeds, there is no constraint of any kind on the shape of the window W except convexity, and a logarithmic factor is removed from the rate of convergence.

In a different scenario, if θ has a power-law density (2.5) with $\tau \in (-1, d-1]$, it is possible to explicitly specify the dependence of the bound in Theorem 2.1 on the moments of ν, as

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stated in the following result. Note that for the above choice of θ, the assumption (B) is trivially satisfied. Define

$$V_\nu(W) := \sum_{i=0}^{d} V_{d-i}(W)\nu_{d+i},$$

which is the sum of the intrinsic volumes of W weighted by the moments of the speed.

Theorem 2.3. *Let the assumptions (A) and (C) be in force. For θ given at (2.5) with $\tau \in (-1, d-1]$, consider $F = F(\eta)$, where η is as in Theorem 2.1 and F is defined as in (2.1) with $a \in (0, \infty)$. Then there exists a constant $C \in (0, \infty)$, depending only on d and τ, such that*

$$d_{\mathrm{W}}\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq C(1+a^d)\left(1+\nu_{7d}\nu_d^{-7}\right)^2\left[\frac{\nu_d^{-\frac{1}{2}\left(\frac{\tau+1}{d+\tau+1}+1\right)}\sqrt{V_\nu(W)}}{l_{a,\tau}(\nu_d)\lambda(W)} + \frac{\nu_d^{-\frac{\tau+1}{d+\tau+1}-1}V_\nu(W)}{l_{a,\tau}(\nu_d)^{3/2}\lambda(W)^{3/2}}\right],$$

and

$$d_{\mathrm{K}}\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \leq C(1+a^d)^{3/2}\left(1+\nu_{7d}\nu_d^{-7}\right)^2\left[\frac{\nu_d^{-\frac{1}{2}\left(\frac{\tau+1}{d+\tau+1}+1\right)}\sqrt{V_\nu(W)}}{l_{a,\tau}(\nu_d)\lambda(W)} + \frac{\nu_d^{-\frac{\tau+1}{d+\tau+1}-1}V_\nu(W)}{l_{a,\tau}(\nu_d)^{3/2}\lambda(W)^{3/2}} + \frac{\nu_d^{-\frac{5}{4}\left(\frac{\tau+1}{d+\tau+1}+1\right)}V_\nu(W)^{5/4} + \nu_d^{-\frac{3}{2}\left(\frac{\tau+1}{d+\tau+1}+1\right)}V_\nu(W)^{3/2}}{l_{a,\tau}(\nu_d)^2\lambda(W)^2}\right],$$

where $l_{a,\tau}$ is defined at (2.6).

Note that our results for the number of exposed points can also be interpreted as quantitative central limit theorems for the number of local minima of the growth frontier, which is of independent interest. As an application of Theorem 2.3, we consider the case when the intensity of the underlying point process grows to infinity. The quantitative central limit theorem for this case is contained in the following result.

Corollary 2.1. *Let the assumptions (A) and (C) be in force. Consider $F(\eta_s)$ defined at (2.1) with $a \in (0, \infty)$, evaluated at the Poisson process η_s with intensity $s\lambda \otimes \theta \otimes \nu$ for $s \geq 1$ and θ given at (2.5) with $\tau \in (-1, d-1]$. Then there exists a finite constant $C \in (0, \infty)$ depending only on W, d, a, τ, ν_d, and ν_{7d}, such that, for all $s \geq 1$,*

$$\max\left\{d_{\mathrm{W}}\left(\frac{F(\eta_s)-\mathbb{E}F(\eta_s)}{\sqrt{\operatorname{Var} F(\eta_s)}}, N\right), d_{\mathrm{K}}\left(\frac{F(\eta_s)-\mathbb{E}F(\eta_s)}{\sqrt{\operatorname{Var} F(\eta_s)}}, N\right)\right\} \leq Cs^{-\frac{d}{2(d+\tau+1)}}.$$

Furthermore, the bound on the Kolmogorov distance is of optimal order.

3. Variance estimation

In this section, we estimate the mean and variance of the statistic F, thus providing a proof of Proposition 2.1. Recall the weight function $h(\boldsymbol{x}) := h_1(x)h_2(t_x)$, where $h_1(x) = \mathbb{1}\{x \in W\}$ and

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$h_2(t) = \mathbb{1}\{t \le a\}$. Notice that by the Mecke formula, the mean of F is given by

$$\begin{aligned}\mathbb{E}F(\eta) &= \int_{\mathbb{X}} h(\boldsymbol{x})\mathbb{E}H_{\boldsymbol{x}}(\eta + \delta_{\boldsymbol{x}})\mu(\mathrm{d}\boldsymbol{x}) \\ &= \int_{\mathbb{R}^d} h_1(x)\mathrm{d}x \int_0^\infty h_2(t)w(t)\theta(\mathrm{d}t) = \lambda(W)\int_0^a w(t)\theta(\mathrm{d}t),\end{aligned}$$

where $w(t)$ is defined at (2.9). In many instances, we will use the simple inequality

$$2ab \le a^2 + b^2, \quad a, b \in \mathbb{R}_+. \tag{3.1}$$

Also notice that for $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \lambda\big(B_{r_1}(0) \cap B_{r_2}(x)\big)\mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{1}_{y\in B_{r_1}(0)} \int_{\mathbb{R}^d} \mathbb{1}_{y\in B_{r_2}(x)}\mathrm{d}x\mathrm{d}y = \omega_d^2 r_1^d r_2^d\,. \tag{3.2}$$

The multivariate Mecke formula (see, e.g., [15, Theorem 4.4]) yields that

$$\begin{aligned}\mathrm{Var}(F) = \int_{\mathbb{X}} h(\boldsymbol{x})^2\mathbb{E}H_{\boldsymbol{x}}(\eta + \delta_{\boldsymbol{x}})\mu(\mathrm{d}\boldsymbol{x}) - \Big(\int_{\mathbb{X}} h(\boldsymbol{x})\mathbb{E}H_{\boldsymbol{x}}(\eta + \delta_{\boldsymbol{x}})\mu(\mathrm{d}\boldsymbol{x})\Big)^2 \\ + \int_D h(\boldsymbol{x})h(\boldsymbol{y})\mathbb{E}\big[H_{\boldsymbol{x}}(\eta + \delta_{\boldsymbol{y}} + \delta_{\boldsymbol{x}})H_{\boldsymbol{y}}(\eta + \delta_{\boldsymbol{x}} + \delta_{\boldsymbol{y}})\big]\mu^2(\mathrm{d}(\boldsymbol{x}, \boldsymbol{y})),\end{aligned}$$

where the double integration is over the region $D \subset \mathbb{X}$ where the points $\boldsymbol{x}$ and $\boldsymbol{y}$ are incomparable ($\boldsymbol{x} \not\preceq \boldsymbol{y}$ and $\boldsymbol{y} \not\preceq \boldsymbol{x}$), i.e.,

$$D := \big\{(\boldsymbol{x}, \boldsymbol{y}) : \|x - y\| > \max\{v_x(t_y - t_x), v_y(t_x - t_y)\}\big\}.$$

It is possible to get rid of one of the Dirac measures in the inner integral, since on D the points are incomparable. Thus, using the translation-invariance of $\mathbb{E}H_{\boldsymbol{x}}(\eta)$, we have

$$\mathrm{Var}(F) = \lambda(W)\int_0^a w(t)\theta(\mathrm{d}t) - I_0 + I_1, \tag{3.3}$$

where

$$I_0 := 2\int_{\mathbb{X}^2} \mathbb{1}_{\boldsymbol{y}\preceq\boldsymbol{x}} h_1(x)h_1(y)h_2(t_x)h_2(t_y)w(t_x)w(t_y)\mu^2(\mathrm{d}(\boldsymbol{x}, \boldsymbol{y})),$$

and

$$I_1 := \int_D h_1(x)h_1(y)h_2(t_x)h_2(t_y)\Big[\mathbb{E}\big[H_{\boldsymbol{x}}(\eta + \delta_{\boldsymbol{x}})H_{\boldsymbol{y}}(\eta + \delta_{\boldsymbol{y}})\big] - w(t_x)w(t_y)\Big]\mu^2(\mathrm{d}(\boldsymbol{x}, \boldsymbol{y})).$$

Finally, we will use the following simple inequality for the incomplete gamma function:

$$\min\{1, b^x\}\gamma(x, y) \le \gamma(x, by) \le \max\{1, b^x\}\gamma(x, y), \tag{3.4}$$

which holds for all $x \in \mathbb{R}_+$ and $b, y > 0$.

Proof of Proposition 2.1. First, notice that the term I_1 in (3.3) is non-negative, since

$$\mathbb{E}[H_{\boldsymbol{x}}(\eta)H_{\boldsymbol{y}}(\eta)] = e^{-\mu(L_{\boldsymbol{x}}\cup L_{\boldsymbol{y}})} \ge e^{-\mu(L_{\boldsymbol{x}})}e^{-\mu(L_{\boldsymbol{y}})} = w(t_x)w(t_y).$$

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Furthermore, (3.1) yields that

$$I_0 \le \int_{\mathbb{X}} h_1(x)^2 h_2(t_x) w(t_x) \left[\int_{\mathbb{X}} \mathbb{1}_{\boldsymbol{y} \preceq \boldsymbol{x}} h_2(t_y) w(t_y) \mu(\mathrm{d}\boldsymbol{y}) \right] \mu(\mathrm{d}\boldsymbol{x}) \\ + \int_{\mathbb{X}} h_1(y)^2 h_2(t_y) w(t_y) \left[\int_{\mathbb{X}} \mathbb{1}_{\boldsymbol{y} \preceq \boldsymbol{x}} h_2(t_x) w(t_x) \mu(\mathrm{d}\boldsymbol{x}) \right] \mu(\mathrm{d}\boldsymbol{y}).$$

Since $\boldsymbol{y} \preceq \boldsymbol{x}$ is equivalent to $\|y - x\| \le v_y(t_x - t_y)$, the first summand on the right-hand side above can be simplified as

$$\int_{\mathbb{X}} h_1(x)^2 h_2(t_x) w(t_x) \left[\int_{\mathbb{X}} \mathbb{1}_{\boldsymbol{y} \preceq \boldsymbol{x}} h_2(t_y) w(t_y) \mu(\mathrm{d}\boldsymbol{y}) \right] \mu(\mathrm{d}\boldsymbol{x}) \\ = \int_{\mathbb{R}^d} \int_0^\infty h_1(x)^2 h_2(t_x) w(t_x) \theta(\mathrm{d}t_x) \mathrm{d}x \int_0^\infty \int_0^{t_x} \omega_d v_y^d (t_x - t_y)^d h_2(t_y) w(t_y) \theta(\mathrm{d}t_y) \nu(\mathrm{d}v_y) \\ = \omega_d \nu_d \lambda(W) \int_0^a \int_0^t (t-s)^d w(s) w(t) \theta(\mathrm{d}s) \theta(\mathrm{d}t).$$

The second summand in the bound on I_0, upon interchanging integrals for the second step, turns into

$$\int_{\mathbb{X}} h_1(y)^2 h_2(t_y) w(t_y) \left[\int_{\mathbb{X}} \mathbb{1}_{\boldsymbol{y} \preceq \boldsymbol{x}} h_2(t_x) w(t_x) \mu(\mathrm{d}\boldsymbol{x}) \right] \mu(\mathrm{d}\boldsymbol{y}) \\ = \int_{\mathbb{R}^d} \int_0^\infty h_1(y)^2 h_2(t_y) w(t_y) \theta(\mathrm{d}t_y) \mathrm{d}y \int_0^\infty \int_{t_y}^\infty \omega_d v_y^d (t_x - t_y)^d h_2(t_x) w(t_x) \theta(\mathrm{d}t_x) \nu(\mathrm{d}v_y) \\ = \omega_d \nu_d \lambda(W) \int_0^a \int_0^t (t-s)^d w(s) w(t) \theta(\mathrm{d}s) \theta(\mathrm{d}t).$$

Combining, by (3.3) we obtain (2.7).

To prove (2.8), note that by the Poincaré inequality (see [15, Section 18.3]),

$$\mathrm{Var}(F) \le \int_{\mathbb{X}} \mathbb{E}\big(F(\eta + \delta_{\boldsymbol{x}}) - F(\eta)\big)^2 \mu(\mathrm{d}\boldsymbol{x}).$$

Observe that η is simple, and for $x \notin \eta$,

$$F(\eta + \delta_{\boldsymbol{x}}) - F(\eta) = h(\boldsymbol{x}) H_{\boldsymbol{x}}(\eta + \delta_{\boldsymbol{x}}) - \sum_{\boldsymbol{y} \in \eta} h(\boldsymbol{y}) H_{\boldsymbol{y}}(\eta) \mathbb{1}_{\boldsymbol{y} \succeq \boldsymbol{x}}\,.$$

The inequality

$$-\sum_{\boldsymbol{y} \in \eta} h(\boldsymbol{x}) h(\boldsymbol{y}) H_{\boldsymbol{x}}(\eta + \delta_{\boldsymbol{x}}) H_{\boldsymbol{y}}(\eta) \mathbb{1}_{\boldsymbol{y} \succeq \boldsymbol{x}} \le 0$$

in the first step and the Mecke formula in the second step yield that

$$\int_{\mathbb{X}} \mathbb{E}\big|F(\eta + \delta_{\boldsymbol{x}}) - F(\eta)\big|^2 \mu(\mathrm{d}\boldsymbol{x}) \\ \le \int_{\mathbb{X}} \mathbb{E}\big[h(\boldsymbol{x})^2 H_{\boldsymbol{x}}(\eta + \delta_{\boldsymbol{x}})\big] \mu(\mathrm{d}\boldsymbol{x}) + \int_{\mathbb{X}} \mathbb{E}\Big[\sum_{\boldsymbol{y}, \boldsymbol{z} \in \eta} \mathbb{1}_{\boldsymbol{y} \succeq \boldsymbol{x}} \mathbb{1}_{\boldsymbol{z} \succeq \boldsymbol{x}} h(\boldsymbol{y}) h(\boldsymbol{z}) H_{\boldsymbol{y}}(\eta) H_{\boldsymbol{z}}(\eta)\Big] \mu(\mathrm{d}\boldsymbol{x}) \\ = \int_{\mathbb{X}} h(\boldsymbol{x})^2 w(t_x) \mu(\mathrm{d}\boldsymbol{x}) + \int_{\mathbb{X}^2} \mathbb{1}_{\boldsymbol{y} \succeq \boldsymbol{x}} h(\boldsymbol{y})^2 w(t_y) \mu^2(\mathrm{d}(\boldsymbol{x}, \boldsymbol{y})) \\ + \int_{\mathbb{X}} \int_{D_{\boldsymbol{x}}} h(\boldsymbol{y}) h(\boldsymbol{z}) e^{-\mu(L_{\boldsymbol{y}} \cup L_{\boldsymbol{z}})} \mu^2(\mathrm{d}(\boldsymbol{y}, \boldsymbol{z})) \mu(\mathrm{d}\boldsymbol{x}), \tag{3.5}$$

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where

$$D_{\boldsymbol{x}} := \left\{(\boldsymbol{y}, \boldsymbol{z}) \in \mathbb{X}^2 : \boldsymbol{y} \succeq \boldsymbol{x}, \boldsymbol{z} \succeq \boldsymbol{x}, \boldsymbol{y} \not\succeq \boldsymbol{z}, \boldsymbol{z} \not\succeq \boldsymbol{y}\right\}.$$

Using that $xe^{-x/2} \le 1$ for $x \in \mathbb{R}_+$, observe that

$$\begin{aligned}\int_{\mathbb{X}^2} \mathbb{1}_{\boldsymbol{y}\succeq \boldsymbol{x}} h(\boldsymbol{y})^2 w(t_y)\mu^2(\mathrm{d}(\boldsymbol{x}, \boldsymbol{y})) &= \int_{\mathbb{X}} h(\boldsymbol{y})^2 w(t_y)\mu(L_{\boldsymbol{y}})\mu(\mathrm{d}\boldsymbol{y}) \\ &\le \int_{\mathbb{X}} h(\boldsymbol{y})^2 w(t_y)^{1/2}\mu(\mathrm{d}\boldsymbol{y}). \end{aligned} \tag{3.6}$$

Next, using that $\mu(L_{\boldsymbol{y}} \cup L_{\boldsymbol{z}}) \ge (\mu(L_{\boldsymbol{y}}) + \mu(L_{\boldsymbol{z}}))/2$ and that $D_{\boldsymbol{x}} \subseteq \{\boldsymbol{y}, \boldsymbol{z} \succeq \boldsymbol{x}\}$ for the first inequality, and (3.1) for the second one, we have

$$\begin{aligned}&\int_{\mathbb{X}} \int_{D_{\boldsymbol{x}}} h(\boldsymbol{y})h(\boldsymbol{z})e^{-\mu(L_{\boldsymbol{y}}\cup L_{\boldsymbol{z}})}\mu^2(\mathrm{d}(\boldsymbol{y}, \boldsymbol{z}))\mu(\mathrm{d}\boldsymbol{x}) \\ &\le \int_{\mathbb{X}} \int_{\mathbb{X}^2} \mathbb{1}_{\boldsymbol{y},\boldsymbol{z}\succeq \boldsymbol{x}} h(\boldsymbol{y})h(\boldsymbol{z})w(t_y)^{1/2}w(t_z)^{1/2}\mu^2(\mathrm{d}(\boldsymbol{y}, \boldsymbol{z}))\mu(\mathrm{d}\boldsymbol{x}) \\ &\le \int_{[0,a]^2} w(t_y)^{1/2}w(t_z)^{1/2} \int_{\mathbb{R}^{2d}} h_1(z)^2 \left(\int_{\mathbb{X}} \mathbb{1}_{\boldsymbol{x}\preceq \boldsymbol{y},\boldsymbol{z}}\mu(\mathrm{d}\boldsymbol{x})\right) \mathrm{d}(y, z)\theta^2(\mathrm{d}(t_y, t_z)). \end{aligned} \tag{3.7}$$

By (3.2),

$$\begin{aligned}&\int_{\mathbb{R}^d} \int_{\mathbb{X}} \mathbb{1}_{\boldsymbol{x}\preceq \boldsymbol{y},\boldsymbol{z}}\mu(\mathrm{d}\boldsymbol{x})\mathrm{d}y \\ &= \int_0^{t_y\wedge t_z} \int_0^{\infty} \nu(\mathrm{d}v_x)\theta(\mathrm{d}t_x) \int_{\mathbb{R}^d} \lambda\big(B_{v_x(t_y-t_x)}(y) \cap B_{v_x(t_z-t_x)}(z)\big)\mathrm{d}y \\ &= \omega_d^2 \nu_{2d} \int_0^{t_y\wedge t_z} (t_y - t_x)^d (t_z - t_x)^d \theta(\mathrm{d}t_x).\end{aligned}$$

Plugging in (3.7), we obtain

$$\begin{aligned}&\int_{\mathbb{X}} \int_{D_{\boldsymbol{x}}} h(\boldsymbol{y})h(\boldsymbol{z})e^{-\mu(L_{\boldsymbol{y}}\cup L_{\boldsymbol{z}})}\mu^2(\mathrm{d}(\boldsymbol{y}, \boldsymbol{z}))\mu(\mathrm{d}\boldsymbol{x}) \\ &\qquad \le \omega_d^2\nu_{2d}\lambda(W) \int_{[0,a]^2} \int_0^{t_1\wedge t_2} (t_1 - s)^d (t_2 - s)^d w(t_1)^{1/2}w(t_2)^{1/2}\theta(\mathrm{d}s)\theta^2(\mathrm{d}(t_1, t_2)).\end{aligned}$$

This in combination with (3.5) and (3.6) proves (2.8).

Now we move on to prove (2.10). We first confirm the lower bound. Fix $\tau \in (-1, d-1]$, as otherwise the bound is trivial, and $a \in (0, \infty)$. Then

$$\Lambda(t) = \omega_d \int_0^t (t-s)^d s^\tau \mathrm{d}s = \omega_d t^{d+\tau+1} B(d+1, \tau+1) = B\,\omega_d t^{d+\tau+1},$$

where $B := B(d+1, \tau+1)$ is a value of the beta function. Hence, we have $w(t) = \exp\{-B\,\omega_d\nu_d t^{d+\tau+1}\}$. Plugging in, we obtain

$$\begin{aligned}\frac{\mathrm{Var}(F)}{\lambda(W)} &\ge \int_0^a e^{-B\,\omega_d\nu_d t^{d+\tau+1}}\theta(\mathrm{d}t) \\ &\qquad - 2\omega_d\nu_d \int_0^a \int_0^t (t-s)^d e^{-B\,\omega_d\nu_d(s^{d+\tau+1}+t^{d+\tau+1})}\theta(\mathrm{d}s)\theta(\mathrm{d}t) \\ &= \left(\frac{1}{B\,\omega_d\nu_d}\right)^{\frac{\tau+1}{d+\tau+1}} \left[\int_0^b e^{-t^{d+\tau+1}} t^\tau \mathrm{d}t - \frac{2}{B}\int_0^b \int_0^t (t-s)^d e^{-(s^{d+\tau+1}+t^{d+\tau+1})} t^\tau s^\tau \mathrm{d}s\mathrm{d}t\right], \end{aligned} \tag{3.8}$$

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where $b := a(B\,\omega_d \nu_d)^{1/(d+\tau+1)}$. Writing $s = tu$ for some $u \in [0, 1]$, we have

$$\begin{aligned}&\frac{2}{B}\int_0^b\int_0^t (t-s)^d e^{-(s^{d+\tau+1}+t^{d+\tau+1})} t^\tau s^\tau \,\mathrm{d}s\mathrm{d}t\\ &\le \frac{2}{B}\int_0^b t^{d+2\tau+1}\int_0^1 (1-u)^d u^\tau e^{-t^{d+\tau+1}(u^{d+\tau+1}+1)}\mathrm{d}u\mathrm{d}t < 2\int_0^b t^{d+2\tau+1} e^{-t^{d+\tau+1}}\mathrm{d}t.\end{aligned}$$

By substituting $t^{d+\tau+1} = z$, it is easy to check that for any $\rho > -1$,

$$\int_0^b e^{-t^{d+\tau+1}} t^\rho \,\mathrm{d}t = \frac{1}{d+\tau+1}\gamma\left(\frac{\rho+1}{d+\tau+1}, b^{d+\tau+1}\right),$$

where γ is the lower incomplete gamma function. In particular, using that $x\gamma(x, y) > \gamma(x + 1, y)$ for $x, y > 0$, we have

$$\begin{aligned}\int_0^b e^{-t^{d+\tau+1}} t^{d+2\tau+1}\mathrm{d}t &= \frac{1}{d+\tau+1}\gamma\left(1+\frac{\tau+1}{d+\tau+1}, b^{d+\tau+1}\right)\\ &< \frac{\tau+1}{(d+\tau+1)^2}\gamma\left(\frac{\tau+1}{d+\tau+1}, b^{d+\tau+1}\right).\end{aligned}$$

Thus, since $\tau \in (-1, d-1]$,

$$\begin{aligned}&\int_0^b e^{-t^{d+\tau+1}} t^\tau \mathrm{d}t - \frac{2}{B}\int_0^b\int_0^t (t-s)^d e^{-(s^{d+\tau+1}+t^{d+\tau+1})} t^\tau s^\tau \,\mathrm{d}s\mathrm{d}t\\ &\qquad > \gamma\left(\frac{\tau+1}{d+\tau+1}, b^{d+\tau+1}\right)\frac{1}{d+\tau+1}\left[1-\frac{2(\tau+1)}{d+\tau+1}\right] \ge 0.\end{aligned}$$

By (3.8) and (3.4), we obtain the lower bound in (2.10).

For the upper bound in (2.10), for θ as in (2.5), arguing as above we have

$$\int_0^a w(t)^{1/2}\theta(\mathrm{d}t) = \int_0^a e^{-B\,\omega_d\nu_d t^{d+\tau+1}/2}\theta(\mathrm{d}t) = \frac{(2/B\,\omega_d\nu_d)^{\frac{\tau+1}{d+\tau+1}}}{d+\tau+1}\gamma\left(\frac{\tau+1}{d+\tau+1}, b^{d+\tau+1}\right).$$

Finally, substituting $s' = (B\,\omega_d\nu_d)^{\frac{1}{d+\tau+1}} s$ and similarly for t_1 and t_2, it is straightforward to see that

$$\begin{aligned}&\nu_{2d}\int_{[0,a)^2}\int_0^{t_1\wedge t_1} (t_1-s)^d (t_2-s)^d w(t_1)^{1/2} w(t_2)^{1/2}\theta(\mathrm{d}s)\theta^2(\mathrm{d}(t_1,t_2))\\ &\qquad \le C\nu_{2d}\nu_d^{-2}\nu_d^{-\frac{\tau+1}{d+\tau+1}}\left(\int_{\mathbb{R}_+} t^{d+\tau} e^{-t^{d+\tau+1}/4}\mathrm{d}t\right)^2\int_0^b s'^\tau e^{-s'^{d+\tau+1}/2}\mathrm{d}s'\\ &\qquad \le C'\nu_{2d}\nu_d^{-2}\nu_d^{-\frac{\tau+1}{d+\tau+1}}\gamma\left(\frac{\tau+1}{d+\tau+1}, \frac{b^{d+\tau+1}}{2}\right)\end{aligned}$$

for some constants C, C' depending only on d and τ. The upper bound in (2.10) now follows from (2.8) upon using the above computation and (3.4). □

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4. Proofs of the theorems

In this section, we derive our main results using [3, Theorem 2.1]. While we do not restate this theorem here, referring the reader to [3, Section 2], it is important to note that the Poisson process considered in [3, Theorem 2.1] has the intensity measure $s\mathbb{Q}$ obtained by scaling a fixed measure $\mathbb{Q}$ on $\mathbb{X}$ with s. Nonetheless, the main result is non-asymptotic, and while in the current paper we consider a Poisson process with fixed intensity measure μ (without a scaling parameter), we can still use [3, Theorem 2.1] with $s = 1$ and the measure $\mathbb{Q}$ replaced by μ. While still following the notation from [3], we drop the subscript s for ease of notation.

Recall that for $\mathcal{M} \in \mathbf{N}$, the score function $\xi(\boldsymbol{x}, \mathcal{M})$ is defined at (2.2). It is straightforward to check that if $\xi(\boldsymbol{x}, \mathcal{M}_1) = \xi(\boldsymbol{x}, \mathcal{M}_2)$ for some $\mathcal{M}_1, \mathcal{M}_2 \in \mathbf{N}$ with $0 \neq \mathcal{M}_1 \leq \mathcal{M}_2$ (meaning that $\mathcal{M}_2 - \mathcal{M}_1$ is a nonnegative measure) and $\boldsymbol{x} \in \mathcal{M}_1$, then $\xi(\boldsymbol{x}, \mathcal{M}_1) = \xi(\boldsymbol{x}, \mathcal{M})$ for all $\mathcal{M} \in \mathbf{N}$ such that $\mathcal{M}_1 \leq \mathcal{M} \leq \mathcal{M}_2$, so that [3, Equation (2.1)] holds. Next we check the assumptions (A1) and (A2) in [3].

For $\mathcal{M} \in \mathbf{N}$ and $x \in \mathcal{M}$, define the stabilization region

$$R(\boldsymbol{x}, \mathcal{M}) := \begin{cases} L_{x,t_x} & \text{if } \boldsymbol{x} \text{ is exposed in } \mathcal{M}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Notice that

$$\{\mathcal{M} \in \mathbf{N} \colon \boldsymbol{y} \in R(\boldsymbol{x}, \mathcal{M} + \delta_{\boldsymbol{x}})\} \in \mathscr{N} \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{X},$$

and that

$$\mathbb{P}\left\{\boldsymbol{y} \in R(\boldsymbol{x}, \eta + \delta_{\boldsymbol{x}})\right\} = \mathbb{1}_{\boldsymbol{y} \preceq \boldsymbol{x}} e^{-\mu(L_{x,t_x})} = \mathbb{1}_{\boldsymbol{y} \preceq \boldsymbol{x}} w(t_x)$$

and

$$\mathbb{P}\{\{\boldsymbol{y}, \boldsymbol{z}\} \subseteq R(\boldsymbol{x}, \eta + \delta_{\boldsymbol{x}})\} = \mathbb{1}_{\boldsymbol{y} \preceq \boldsymbol{x}} \mathbb{1}_{\boldsymbol{z} \preceq \boldsymbol{x}} e^{-\mu(L_{x,t_x})} = \mathbb{1}_{\boldsymbol{y} \preceq \boldsymbol{x}} \mathbb{1}_{\boldsymbol{z} \preceq \boldsymbol{x}} w(t_x)$$

are measurable functions of $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{X}^2$ and $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathbb{X}^3$ respectively, with $w(t)$ defined at (2.9). It is not hard to see that R is monotonically decreasing in the second argument, and that for all $\mathcal{M} \in \mathbf{N}$ and $\boldsymbol{x} \in \mathcal{M}$, $\mathcal{M}(R(x, \mathcal{M})) \geq 1$ implies that $\boldsymbol{x}$ is exposed, so that $(\mathcal{M} + \delta_{\boldsymbol{y}})(R(x, \mathcal{M} + \delta_{\boldsymbol{y}})) \geq 1$ for all $\boldsymbol{y} \notin R(\boldsymbol{x}, \mathcal{M})$. Moreover, the function R satisfies

$$\xi\left(\boldsymbol{x}, \mathcal{M}\right) = \xi\left(\boldsymbol{x}, \mathcal{M}_{R(\boldsymbol{x},\mathcal{M})}\right), \quad \mathcal{M} \in \mathbf{N},\ \boldsymbol{x} \in \mathcal{M}\,,$$

where $\mathcal{M}_{R(\boldsymbol{x},\mathcal{M})}$ denotes the restriction of the measure $\mathcal{M}$ to the region $R(\boldsymbol{x}, \mathcal{M})$. It is important to note here that this holds even when $\boldsymbol{x}$ is not exposed in $\mathcal{M}$, since in this case, the left-hand side is 0 where the right-hand side is 0 by our convention that $\xi(\boldsymbol{x}, 0) = 0$. Hence, the assumptions (A1.1)–(A1.4) in [3] are satisfied. Furthermore, notice that for any $p \in (0, 1]$, for all $\mathcal{M} \in \mathbf{N}$ with $\mathcal{M}(\mathbb{X}) \leq 7$, we have

$$\mathbb{E}\left[\xi(\boldsymbol{x}, \eta + \delta_{\boldsymbol{x}} + \mathcal{M})^{4+p}\right] \leq \mathbb{1}_{x \in W} \mathbb{1}_{t_x \leq a} w(t_x),$$

confirming the condition (A2) in [3] with $M_p(\boldsymbol{x}) := \mathbb{1}\{x \in W, t_x \leq a\}$. For definiteness, we take $p = 1$ and define

$$\widetilde{M}(\boldsymbol{x}) := \max\{M_1(\boldsymbol{x})^2, M_1(\boldsymbol{x})^4\} = \mathbb{1}_{x \in W} \mathbb{1}_{t_x \leq a}.$$

Finally, define

$$r(\boldsymbol{x}, \boldsymbol{y}) := \begin{cases} \nu_d \Lambda(t_x) & \text{if } \boldsymbol{y} \preceq \boldsymbol{x}, \\ \infty & \text{if } \boldsymbol{y} \npreceq \boldsymbol{x}, \end{cases}$$

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so that

$$\mathbb{P}\left\{\boldsymbol{y} \in R(\boldsymbol{x}, \eta+\delta_{\boldsymbol{x}})\right\} = \mathbb{1}_{\boldsymbol{y}\preceq \boldsymbol{x}} w(t_x) = e^{-r(\boldsymbol{x},\boldsymbol{y})}, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{X},$$

which corresponds to [3, Equation (2.4)]. Now that we have checked all the necessary conditions, we can invoke [3, Theorem 2.1]. Let $\zeta := \frac{p}{40+10p} = 1/50$, and define functions of $\boldsymbol{y} \in \mathbb{X}$ by

$$g(\boldsymbol{y}) := \int_{\mathbb{X}} e^{-\zeta r(\boldsymbol{x},\boldsymbol{y})}\, \mu(\mathrm{d}\boldsymbol{x}), \tag{4.1}$$

$$h(\boldsymbol{y}) := \int_{\mathbb{X}} \mathbb{1}_{x\in W} \mathbb{1}_{t_x \le a} e^{-\zeta r(\boldsymbol{x},\boldsymbol{y})}\, \mu(\mathrm{d}\boldsymbol{x}), \tag{4.2}$$

$$G(\boldsymbol{y}) := \mathbb{1}_{y\in W} \mathbb{1}_{t_x\le a} + \max\{h(\boldsymbol{y})^{4/9}, h(\boldsymbol{y})^{8/9}\}\big(1+g(\boldsymbol{y})^4\big). \tag{4.3}$$

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{X}$, let

$$q(\boldsymbol{x},\boldsymbol{y}) := \int_{\mathbb{X}} \mathbb{P}\Big\{\{\boldsymbol{x},\boldsymbol{y}\} \subseteq R\big(z, \eta+\delta_z\big)\Big\}\, \mu(\mathrm{d}z) = \int_{\boldsymbol{x}\preceq z, \boldsymbol{y}\preceq z} w(t_z)\, \mu(\mathrm{d}z). \tag{4.4}$$

For $\alpha > 0$, let

$$f_\alpha(\boldsymbol{y}) := f_\alpha^{(1)}(\boldsymbol{y}) + f_\alpha^{(2)}(\boldsymbol{y}) + f_\alpha^{(3)}(\boldsymbol{y}), \quad \boldsymbol{y} \in \mathbb{X},$$

where, for $\boldsymbol{y} \in \mathbb{X}$,

$$\begin{aligned} f_\alpha^{(1)}(\boldsymbol{y}) &:= \int_{\mathbb{X}} G(\boldsymbol{x}) e^{-\alpha r(\boldsymbol{x},\boldsymbol{y})}\, \mu(\mathrm{d}\boldsymbol{x}) = \int_{\boldsymbol{y}\preceq\boldsymbol{x}} G(\boldsymbol{x}) w(t_x)^\alpha \mu(\mathrm{d}\boldsymbol{x}), \\ f_\alpha^{(2)}(\boldsymbol{y}) &:= \int_{\mathbb{X}} G(\boldsymbol{x}) e^{-\alpha r(\boldsymbol{y},\boldsymbol{x})}\, \mu(\mathrm{d}\boldsymbol{x}) = w(t_y)^\alpha \int_{\boldsymbol{x}\preceq\boldsymbol{y}} G(\boldsymbol{x}) \mu(\mathrm{d}\boldsymbol{x}), \\ f_\alpha^{(3)}(\boldsymbol{y}) &:= \int_{\mathbb{X}} G(\boldsymbol{x}) q(\boldsymbol{x},\boldsymbol{y})^\alpha\, \mu(\mathrm{d}\boldsymbol{x}). \end{aligned} \tag{4.5}$$

Finally, let

$$\kappa(\boldsymbol{x}) := \mathbb{P}\left\{\xi(\boldsymbol{x}, \eta+\delta_{\boldsymbol{x}}) \neq 0\right\} = \mathbb{1}_{x\in W} \mathbb{1}_{t_x\le a} w(t_x), \quad \boldsymbol{x} \in \mathbb{X}.$$

For an integrable function $f : \mathbb{X} \to \mathbb{R}$, denote $\mu f := \int_{\mathbb{X}} f(\boldsymbol{x}) \mu(\mathrm{d}\boldsymbol{x})$. With $\beta := \frac{p}{32+4p} = 1/36$, [3, Theorem 2.1] yields that $F = F(\eta)$ as in (2.1) satisfies

$$d_{\mathrm{W}}\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \le C\left[\frac{\sqrt{\mu f_\beta^2}}{\operatorname{Var} F} + \frac{\mu((\kappa+g)^{2\beta} G)}{(\operatorname{Var} F)^{3/2}}\right] \tag{4.6}$$

and

$$\begin{aligned} d_{\mathrm{K}}\left(\frac{F-\mathbb{E}F}{\sqrt{\operatorname{Var} F}}, N\right) \le C\Bigg[&\frac{\sqrt{\mu f_\beta^2} + \sqrt{\mu f_{2\beta}}}{\operatorname{Var} F} + \frac{\sqrt{\mu((\kappa+g)^{2\beta} G)}}{\operatorname{Var} F} \\ &+ \frac{\mu((\kappa+g)^{2\beta} G)}{(\operatorname{Var} F)^{3/2}} + \frac{(\mu((\kappa+g)^{2\beta} G))^{5/4} + (\mu((\kappa+g)^{2\beta} G))^{3/2}}{(\operatorname{Var} F)^2}\Bigg], \end{aligned} \tag{4.7}$$

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where N is a standard normal random variable and $C \in (0, \infty)$ is a constant.

In the rest of this section, we estimate the summands on the right-hand side of the above two bounds to obtain our main results. While the bounds above are admittedly quite difficult to interpret, they essentially involve integrals of functions which are products involving an exponential part and a polynomial part. Because of the faster decay of the exponential part, the integrals grow at a rate that is at most some small enough power of the variance of F, and this yields the presumably optimal rates of convergence in Theorem 2.2. We start with a simple lemma.

Lemma 4.1. *For all $x \in \mathbb{R}_+$ and $y > 0$,*

$$Q(x, y) := \int_0^\infty t^x \, e^{-y\Lambda(t)} \, \theta(\mathrm{d}t) = \int_0^\infty t^x \, w(t)^{y/\nu_d} \, \theta(\mathrm{d}t) < \infty. \tag{4.8}$$

Proof. Assume that $\theta([0, c]) > 0$ for some $c \in (0, \infty)$, since otherwise the result holds trivially. Notice that

$$\int_0^{2c} t^x \, e^{-y\Lambda(t)} \theta(\mathrm{d}t) \le (2c)^x \int_0^\infty e^{-y\Lambda(t)} \theta(\mathrm{d}t) < \infty$$

by the assumption (B). Hence, it suffices to show the finiteness of the integral over $[2c, \infty)$. The inequality $w^{x/d} e^{-w/2} \le C$ for some finite constant $C > 0$ yields that

$$\int_{2c}^\infty t^x \, e^{-y\Lambda(t)} \theta(\mathrm{d}t) \le \frac{C}{y^{x/d}} \int_{2c}^\infty \frac{t^x}{\Lambda(t)^{x/d}} \, e^{-y\Lambda(t)/2} \theta(\mathrm{d}t).$$

For $t \ge 2c$,

$$\Lambda(t) = \int_0^t (t - s)^d \theta(\mathrm{d}s) \ge \int_0^{t/2} (t - s)^d \theta(\mathrm{d}s) \ge (t/2)^d \theta([0, t/2]) \ge 2^{-d} t^d \theta([0, c]).$$

Thus,

$$\int_{2c}^\infty t^x \, e^{-y\Lambda(t)} \theta(\mathrm{d}t) \le \frac{C2^x}{(y\theta([0, c]))^{x/d}} \int_{2c}^\infty e^{-y\Lambda(t)/2} \theta(\mathrm{d}t) < \infty$$

by the assumption (B), yielding the result. □

To compute the bounds in (4.6) and (4.7), we need to bound $\mu f_{2\beta}$ and μf_β^2, with $\beta = 1/36$. Nonetheless, we provide bounds on μf_α and μf_α^2 for any $\alpha > 0$. By Jensen's inequality, it suffices to bound $\mu f_\alpha^{(i)}$ and $\mu (f_\alpha^{(i)})^2$ for $i = 1, 2, 3$. This is the objective of the following three lemmas.

For g defined at (4.1),

$$\begin{aligned} g(\mathbf{y}) &= \int_{\mathbb{X}} \mathbb{1}_{\mathbf{y} \preceq \mathbf{x}} w(t_x)^\zeta \, \mu(\mathrm{d}\mathbf{x}) = \int_{t_y}^\infty \int_{\mathbb{R}^d} \mathbb{1}_{x \in B_{\nu_y(t_x - t_y)}(y)} w(t_x)^\zeta \, \mathrm{d}x \theta(\mathrm{d}t_x) \\ &= \omega_d \nu_y^d \int_{t_y}^\infty (t_x - t_y)^d w(t_x)^\zeta \, \theta(\mathrm{d}t_x) \le \omega_d \nu_y^d \int_0^\infty t_x^d w(t_x)^\zeta \, \theta(\mathrm{d}t_x) = \omega_d Q(d, \zeta \nu_d) \, \nu_y^d, \end{aligned}$$

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where Q is defined at (4.8). Similarly, for h as in (4.2) with $a \in (0, \infty)$, we have

$$\begin{aligned} h(\mathbf{y}) &= \int_{\mathbb{X}} \mathbb{1}_{\mathbf{y} \preceq \mathbf{x}} w(t_x)^\zeta \mathbb{1}_{x \in W} \mathbb{1}_{t_x \le a} \mu(\mathrm{d}\mathbf{x}) \\ &= \mathbb{1}_{t_y \le a} \int_{t_y}^{a} \left(\int_{\mathbb{R}^d} \mathbb{1}_{x \in B_{v_y(t_x - t_y)}(y)} \mathbb{1}_{x \in W} \, \mathrm{d}x \right) w(t_x)^\zeta \, \theta(\mathrm{d}t_x) \\ &\le \mathbb{1}_{y \in W + B_{v_y(a-t_y)}(0)} \int_{t_y}^{\infty} \int_{\mathbb{R}^d} \mathbb{1}_{x \in B_{v_y(t_x - t_y)}(y)} w(t_x)^\zeta \, \mathrm{d}x \theta(\mathrm{d}t_x) \\ &\le \mathbb{1}_{y \in W + B_{v_y(a-t_y)}(0)} \int_{t_y}^{\infty} \omega_d v_y^d t_x^d w(t_x)^\zeta \, \theta(\mathrm{d}t_x) \\ &\le \mathbb{1}_{y \in W + B_{v_y a}(0)} \omega_d Q(d, \zeta \nu_d) \, v_y^d. \end{aligned}$$

Therefore, the function G defined at (4.3) for $a \in (0, \infty)$ is bounded by

$$\begin{aligned} G(\mathbf{y}) &\le \mathbb{1}_{y \in W} + \mathbb{1}_{y \in W + B_{v_y a}(0)} (1 + \omega_d Q(d, \zeta \nu_d) \, v_y^d)(1 + \omega_d^4 Q(d, \zeta \nu_d)^4 \, v_y^{4d}) \\ &\le 6\omega_d^5 \mathbb{1}_{y \in W + B_{v_y a}(0)} p(v_y) \, , \end{aligned} \tag{4.9}$$

with

$$p(v_y) := 1 + Q(d, \zeta \nu_d)^5 v_y^{5d}.$$

Define

$$M_u := \int_0^\infty v^u p(v) \nu(\mathrm{d}v), \qquad u \in \mathbb{R}_+.$$

In particular,

$$M_0 := \int_0^\infty p(v) \nu(\mathrm{d}v) = 1 + Q(d, \zeta \nu_d)^5 \nu_{5d},$$

and

$$M := M_0 + M_d = \int_0^\infty (1 + v^d) p(v) \nu(\mathrm{d}v) = 1 + \nu_d + Q(d, \zeta \nu_d)^5 (\nu_{5d} + \nu_{6d}).$$

Recall $V(W)$ defined at (2.4), and let $\omega = \max_{0 \le j \le d} \omega_j$. The Steiner formula (see [19, Section 4.1]) yields that

$$\begin{aligned} \int_{\mathbb{R}_+} \lambda(W + B_{v_x a}(0)) p(v_x) \nu(\mathrm{d}v_x) &= \sum_{i=0}^{d} \int_{\mathbb{R}_+} \omega_i v_x^i a^i V_{d-i}(W) \, p(v_x) \nu(\mathrm{d}v_x) \\ &\le \omega(1 + a^d) \sum_{i=0}^{d} V_{d-i}(W) M_i \qquad (4.10) \\ &\le c_d (1 + a^d) M \, V(W), \qquad (4.11) \end{aligned}$$

with $c_d = (d+1)\omega$, where in the final step we have used the simple inequality $v_x^a \le 1 + v_x^b$ for any $0 \le a \le b < \infty$. We will use this fact many times in the sequel without mentioning it explicitly.

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We will also often use the fact that for an increasing function f of the speed v, since p is also increasing, by positive association, we have

$$\int_{\mathbb{R}_+} f(v)p(v)\nu(\mathrm{d}v) \geq M_0 \int_{\mathbb{R}_+} f(v)\nu(\mathrm{d}v).$$

Lemma 4.2. *For $a \in (0, \infty)$, $\alpha > 0$, and $f_\alpha^{(1)}$ defined at (4.5),*

$$\int_{\mathbb{X}} f_\alpha^{(1)}(\boldsymbol{y})\mu(\mathrm{d}\boldsymbol{y}) \leq C_1\, V(W) \;\text{ and }\; \int_{\mathbb{X}} f_\alpha^{(1)}(\boldsymbol{y})^2\mu(\mathrm{d}\boldsymbol{y}) \leq C_2\, V(W),$$

where

$$\begin{aligned} C_1 &:= C(1+a^d)M\frac{Q(0, \alpha\nu_d/2)}{\alpha}, \\ C_2 &:= C(1+a^d)\, M_0 M \nu_{2d} Q(d, \alpha\nu_d/2)^2 Q(0, \alpha\nu_d), \end{aligned}$$

for a constant $C \in (0, \infty)$ depending only on d.

Proof. Using (4.9), we can write

$$\begin{aligned} \int_{\mathbb{X}} f_\alpha^{(1)}(\boldsymbol{y})\mu(\mathrm{d}\boldsymbol{y}) &= \int_{\mathbb{X}}\int_{\boldsymbol{y}\preceq\boldsymbol{x}} G(\boldsymbol{x})w(t_x)^\alpha \mu(\mathrm{d}\boldsymbol{x})\mu(\mathrm{d}\boldsymbol{y}) \\ &\leq 6\omega_d^5\nu_d \int_{\mathbb{X}} \Lambda(t_x)\mathbb{1}_{x\in W+B_{v_x a}(0)} p(v_x)w(t_x)^\alpha \mu(\mathrm{d}\boldsymbol{x}) \;=: 6\omega_d^5\nu_d\, I_1, \end{aligned}$$

whence, using (4.11) and the fact that $xe^{-x/2} \leq 1$ for $x \in \mathbb{R}_+$, we obtain

$$\begin{aligned} I_1 :&= \int_{\mathbb{R}_+^2} \lambda(W + B_{v_x a}(0))p(v_x)\Lambda(t_x)w(t_x)^\alpha\theta(\mathrm{d}t_x)\nu(\mathrm{d}v_x) \\ &\leq c_d(1+a^d)M\, V(W)\int_{\mathbb{R}_+} \Lambda(t_x)w(t_x)^\alpha\theta(\mathrm{d}t_x) \\ &\leq c_d(1+a^d)M\frac{Q(0, \alpha\nu_d/2)}{\alpha\nu_d}\, V(W), \end{aligned}$$

proving the first assertion.

For the second assertion, first, by (3.2), for any $t_1, t_2 \in \mathbb{R}_+$ we have

$$\begin{aligned} \int_{\mathbb{R}^d} \mu(L_{0,t_1}\cap L_{x,t_2})\mathrm{d}x &= \int_0^{t_1\wedge t_2}\theta(\mathrm{d}s)\int_0^\infty \nu(\mathrm{d}v)\int_{\mathbb{R}^d}\lambda(B_{v(t_1-s)}(0)\cap B_{v(t_2-s)}(x))\mathrm{d}x \\ &= \omega_d^2\int_0^\infty v^{2d}\nu(\mathrm{d}v)\int_0^{t_1\wedge t_2}(t_1-s)^d(t_2-s)^d\theta(\mathrm{d}s) \\ &= \omega_d^2\nu_{2d}\int_0^{t_1\wedge t_2}(t_1-s)^d(t_2-s)^d\theta(\mathrm{d}s) \;=: \ell(t_1, t_2), \end{aligned} \tag{4.12}$$

which is symmetric in t_1 and t_2. Thus, changing the order of the integrals in the second step and using (4.9) for the final step, we get

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$$\begin{aligned}\int_{\mathbb{X}} f_\alpha^{(1)}(\boldsymbol{y})^2\mu(\mathrm{d}\boldsymbol{y}) &= \int_{\mathbb{X}}\int_{\boldsymbol{y}\preceq\boldsymbol{x}_1}\int_{\boldsymbol{y}\preceq\boldsymbol{x}_2} G(\boldsymbol{x}_1)w(t_{x_1})^\alpha G(\boldsymbol{x}_2)w(t_{x_2})^\alpha\mu(\mathrm{d}\boldsymbol{x}_1)\mu(\mathrm{d}\boldsymbol{x}_2)\mu(\mathrm{d}\boldsymbol{y})\\ &= \int_{\mathbb{X}}\int_{\mathbb{X}}\left(\int_{\boldsymbol{y}\preceq\boldsymbol{x}_1,\boldsymbol{y}\preceq\boldsymbol{x}_2}\mu(\mathrm{d}\boldsymbol{y})\right)G(\boldsymbol{x}_1)G(\boldsymbol{x}_2)\big(w(t_{x_1})w(t_{x_2})\big)^\alpha\mu(\mathrm{d}\boldsymbol{x}_1)\mu(\mathrm{d}\boldsymbol{x}_2)\\ &= \int_{\mathbb{X}}\int_{\mathbb{X}}\mu(L_{x_1,t_{x_1}}\cap L_{x_2,t_{x_2}})G(\boldsymbol{x}_1)G(\boldsymbol{x}_2)\big(w(t_{x_1})w(t_{x_2})\big)^\alpha\mu(\mathrm{d}\boldsymbol{x}_1)\mu(\mathrm{d}\boldsymbol{x}_2)\\ &\le 36\omega_d^{10}M_0I_2, \end{aligned}\tag{4.13}$$

where

$$I_2 := \int_{\mathbb{R}_+^3}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\mu(L_{0,t_{x_1}}\cap L_{x_2-x_1,t_{x_2}})\mathrm{d}x_2\right)\mathbb{1}_{x_1\in W+B_{v_{x_1}a}(0)}\mathrm{d}x_1\right) \times p(v_{x_1})\big(w(t_{x_1})w(t_{x_2})\big)^\alpha\theta^2(\mathrm{d}(t_{x_1},t_{x_2})\nu(\mathrm{d}v_{x_1}).$$

By (4.11) and (4.12), we have

$$\begin{aligned}I_2 &= \int_{\mathbb{R}_+^3}\ell(t_{x_1},t_{x_2})\lambda(W+B_{v_{x_1}a}(0))p(v_{x_1})\big(w(t_{x_1})w(t_{x_2})\big)^\alpha\theta^2(\mathrm{d}(t_{x_1},t_{x_2})\nu(\mathrm{d}v_{x_1})\\ &\le c_d(1+a^d)M\,V(W)\int_{\mathbb{R}_+^2}\ell(t_{x_1},t_{x_2})\big(w(t_{x_1})w(t_{x_2})\big)^\alpha\theta^2(\mathrm{d}(t_{x_1},t_{x_2})).\end{aligned}$$

Using that w is a decreasing function, the result now follows from (4.13) and (4.12) by noticing that

$$\begin{aligned}&\int_{\mathbb{R}_+^2}\ell(t_{x_1},t_{x_2})\big(w(t_{x_1})w(t_{x_2})\big)^\alpha\theta^2(\mathrm{d}(t_{x_1},t_{x_2}))\\ &= \omega_d^2\nu_{2d}\int_{\mathbb{R}_+^2}\int_0^{t_{x_1}\wedge t_{x_2}}(t_{x_1}-s)^d(t_{x_2}-s)^d\big(w(t_{x_1})w(t_{x_2})\big)^\alpha\theta(\mathrm{d}s)\theta^2(\mathrm{d}(t_{x_1},t_{x_2}))\\ &= \omega_d^2\nu_{2d}\int_0^\infty\left(\int_s^\infty(t-s)^dw(t)^\alpha\theta(\mathrm{d}t)\right)^2\theta(\mathrm{d}s)\\ &\le \omega_d^2\nu_{2d}\int_0^\infty\left(\int_0^\infty t^dw(t)^{\alpha/2}\theta(\mathrm{d}t)\right)^2w(s)^\alpha\theta(\mathrm{d}s) = \omega_d^2\nu_{2d}Q(d,\alpha\nu_d/2)^2Q(0,\alpha\nu_d).\end{aligned}$$

□

Arguing as in (4.11), we also have

$$\int_{\mathbb{R}_+}\lambda(W+B_{v_xa}(0))v_x^dp(v_x)\nu(\mathrm{d}v_x) \le \omega(1+a^d)\sum_{i=0}^{d}V_{d-i}(W)M_{d+i}\tag{4.14}$$

$$\le c_d(1+a^d)M'\,V(W),\tag{4.15}$$

with

$$M' := \int_0^\infty(1+v^d)v^dp(v)\nu(\mathrm{d}v) = \nu_d+\nu_{2d}+Q(d,\zeta\nu_d)^5(\nu_{6d}+\nu_{7d}).$$

Note that by positive association, we have $\nu_dM\le M'$.

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Lemma 4.3. *For $a \in (0, \infty)$, $\alpha > 0$, and $f_\alpha^{(2)}$ defined at (4.5),*

$$\int_{\mathbb{X}} f_\alpha^{(2)}(\boldsymbol{y})\mu(\mathrm{d}\boldsymbol{y}) \le C_1\, V(W) \quad \text{and} \quad \int_{\mathbb{X}} f_\alpha^{(2)}(\boldsymbol{y})^2\mu(\mathrm{d}\boldsymbol{y}) \le C_2\, V(W)$$

for

$$C_1 := C(1+a^d)M'\, Q(0, \alpha\nu_d/2)Q(d, \alpha\nu_d/2)\,,$$
$$C_2 := C(1+a^d)\, M_d M'\, Q(0, \alpha\nu_d/3)^2 Q(2d, \alpha\nu_d/3),$$

for a constant $C \in (0, \infty)$ depending only on d.

Proof. By the definition of $f_\alpha^{(2)}$, (4.9), and (4.15), we obtain

$$\begin{aligned}
&\int_{\mathbb{X}} f_\alpha^{(2)}(\boldsymbol{y})\mu(\mathrm{d}\boldsymbol{y}) \\
&\le 6\omega_d^5 \int_{\mathbb{X}} \left(\int_{\boldsymbol{x}\preceq\boldsymbol{y}} w(t_y)^\alpha \mu(\mathrm{d}\boldsymbol{y})\right) \mathbb{1}_{x\in W+B_{\nu_x a}(0)} p(\nu_x)\mu(\mathrm{d}\boldsymbol{x}) \\
&= 6\omega_d^6 \int_0^\infty \int_{t_x}^\infty w(t_y)^\alpha (t_y-t_x)^d \int_0^\infty \lambda(W+B_{\nu_x a}(0))\nu_x^d p(\nu_x)\nu(\mathrm{d}\nu_x)\theta(\mathrm{d}t_y)\theta(\mathrm{d}t_x) \\
&\le 6\omega_d^6 c_d(1+a^d)M'\, V(W) \int_0^\infty w(t_x)^{\alpha/2}\theta(\mathrm{d}t_x) \int_0^\infty t_y^d w(t_y)^{\alpha/2}\theta(\mathrm{d}t_y) \\
&\le 6\omega_d^6 c_d(1+a^d)M'\, Q(0, \alpha\nu_d/2)Q(d, \alpha\nu_d/2)\, V(W),
\end{aligned}$$

where in the penultimate step we have used that w is decreasing. This proves the first assertion.

For the second assertion, using (4.9), we have

$$\begin{aligned}
\int_{\mathbb{X}} f_\alpha^{(2)}(\boldsymbol{y})^2\mu(\mathrm{d}\boldsymbol{y}) &= \int_{\mathbb{X}} w(t_y)^{2\alpha} \left(\int_{\boldsymbol{x}_1\preceq\boldsymbol{y}} G(\boldsymbol{x}_1)\mu(\mathrm{d}\boldsymbol{x}_1) \int_{\boldsymbol{x}_2\preceq\boldsymbol{y}} G(\boldsymbol{x}_2)\mu(\mathrm{d}\boldsymbol{x}_2)\right)\mu(\mathrm{d}\boldsymbol{y}) \\
&= \int_{\mathbb{X}}\int_{\mathbb{X}} \left(\int_{\boldsymbol{x}_1\preceq\boldsymbol{y},\boldsymbol{x}_2\preceq\boldsymbol{y}} w(t_y)^{2\alpha}\mu(\mathrm{d}\boldsymbol{y})\right) G(\boldsymbol{x}_1)G(\boldsymbol{x}_2)\mu(\mathrm{d}\boldsymbol{x}_1)\mu(\mathrm{d}\boldsymbol{x}_2) \\
&\le 36\omega_d^{10} \int_{\mathbb{X}}\int_{\mathbb{X}} p(\nu_{x_1})p(\nu_{x_2}) \\
&\quad \times \mathbb{1}_{x_1\in W+B_{\nu_{x_1}a}(0)} \left(\int_{\boldsymbol{x}_1\preceq\boldsymbol{y},\boldsymbol{x}_2\preceq\boldsymbol{y}} w(t_y)^{2\alpha}\mu(\mathrm{d}\boldsymbol{y})\right)\mu(\mathrm{d}\boldsymbol{x}_1)\mu(\mathrm{d}\boldsymbol{x}_2). \qquad (4.16)
\end{aligned}$$

For fixed $\boldsymbol{x}_1$, t_{x_2}, and ν_{x_2}, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\boldsymbol{x}_1\preceq\boldsymbol{y},\boldsymbol{x}_2\preceq\boldsymbol{y}} w(t_y)^{2\alpha}\mu(\mathrm{d}\boldsymbol{y})\mathrm{d}x_2 \\
&= \int_{t_{x_1}\vee t_{x_2}}^\infty w(t_y)^{2\alpha} \left(\int_{\mathbb{R}^d} \lambda\big(B_{\nu_{x_1}(t_y-t_{x_1})}(0) \cap B_{\nu_{x_2}(t_y-t_{x_2})}(x)\big)\mathrm{d}x\right)\theta(\mathrm{d}t_y) \\
&= \omega_d^2 \nu_{x_1}^d \nu_{x_2}^d \int_{t_{x_1}\vee t_{x_2}}^\infty (t_y-t_{x_1})^d(t_y-t_{x_2})^d w(t_y)^{2\alpha}\theta(\mathrm{d}t_y)\,.
\end{aligned}$$

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Arguing similarly as for $\mu(f_\alpha^{(2)})$ above, we obtain from (4.16) and (4.15) that

$$\begin{aligned}\int_{\mathbb{X}} f_\alpha^{(2)}(\boldsymbol{y})^2\mu(\mathrm{d}\boldsymbol{y}) &\le 36\omega_d^{12}\, M_d \int_0^\infty \lambda\big(W + B_{v_{x_1}a}(0)\big) v_{x_1}^d p(v_{x_1})\nu(\mathrm{d}v_{x_1})\\ &\quad \times \int_{\mathbb{R}_+^2} \left(\int_{t_{x_1}\vee t_{x_2}}^\infty (t_y - t_{x_1})^d (t_y - t_{x_2})^d w(t_y)^\alpha \theta(\mathrm{d}t_y)\right)\theta^2(\mathrm{d}(t_{x_1}, t_{x_2}))\\ &\le 36\omega_d^{12} c_d(1+a^d)\, M_d M' V(W) \int_0^\infty w(t_{x_1})^{\alpha/3}\theta(\mathrm{d}t_{x_1})\\ &\quad \times \int_0^\infty w(t_{x_2})^{\alpha/3}\theta(\mathrm{d}t_{x_2}) \int_0^\infty t_y^{2d} w(t_y)^{\alpha/3}\theta(\mathrm{d}t_y)\\ &\le 36\omega_d^{12} c_d(1+a^d)\, M_d M'\, Q(0, \alpha\nu_d/3)^2 Q(2d, \alpha\nu_d/3)\, V(W). \end{aligned}$$ □

Before proceeding to bound the integrals of $f^{(3)}$, notice that, since θ is a non-null measure,

$$\begin{aligned} M'_\alpha = M'_\alpha(\nu_d) &:= \int_0^\infty t^{d-1} e^{-\frac{\alpha\nu_d}{3}\Lambda(t)}\mathrm{d}t = \int_0^\infty t^{d-1} e^{-\frac{\alpha\omega_d\nu_d}{3}\int_0^t (t-s)^d\theta(\mathrm{d}s)}\mathrm{d}t\\ &\le \int_0^\infty t^{d-1} e^{-\frac{\alpha\omega_d\nu_d}{3}\int_0^{t/2}(t/2)^d\theta(\mathrm{d}s)}\mathrm{d}t = \int_0^\infty t^{d-1} e^{-\frac{\alpha\omega_d\nu_d}{3}\theta([0,t/2))(t/2)^d}\mathrm{d}t < \infty\,. \end{aligned} \tag{4.17}$$

Lemma 4.4. *For $a \in (0, \infty)$, $\alpha \in (0, 1]$, and $f_\alpha^{(3)}$ defined at (4.5),*

$$\int_{\mathbb{X}} f_\alpha^{(3)}(\boldsymbol{y})\mu(\mathrm{d}\boldsymbol{y}) \le C_1\, V(W) \quad \text{and} \quad \int_{\mathbb{X}} f_\alpha^{(3)}(\boldsymbol{y})^2\mu(\mathrm{d}\boldsymbol{y}) \le C_2\, V(W),$$

where

$$\begin{aligned} C_1 &:= C\,(1+a^d)M'Q(0, \alpha\nu_d/3)^2\Big[M'_\alpha + Q(2d, \alpha\nu_d/3)\nu_d\Big],\\ C_2 &:= C\,(1+a^d)M_dM'(1+\nu_{2d}\nu_d^{-2})Q(0,\alpha\nu_d/3)^3\\ &\qquad \times \Big({M'_\alpha}^2 + M'_\alpha Q(2d, \alpha\nu_d/3)\nu_d + Q(2d, \alpha\nu_d/3)^2\nu_{2d}\Big), \end{aligned}$$

for a constant $C \in (0, \infty)$ depending only on d.

Proof. Note that $\boldsymbol{x}, \boldsymbol{y} \preceq \boldsymbol{z}$ implies

$$|x-y| \le |x-z| + |y-z| \le t_z(v_x + v_y).$$

For q defined at (4.4), we have

$$q(\boldsymbol{x},\boldsymbol{y}) \le e^{-\nu_d\Lambda(r_0)} \int_{r_0}^\infty \lambda\big(B_{v_x(t_z-t_x)}(0)\cap B_{v_y(t_z-t_y)}(y-x)\big)e^{-\nu_d(\Lambda(t_z)-\Lambda(r_0))}\theta(\mathrm{d}t_z)\,,$$

where

$$r_0 = r_0(\boldsymbol{x},\boldsymbol{y}) := \frac{|x-y|}{v_x+v_y} \vee t_x \vee t_y\,.$$

Therefore,

$$\begin{aligned} q(\boldsymbol{x},\boldsymbol{y})^\alpha &\le e^{-\alpha\nu_d\Lambda(r_0)}\\ &\qquad\times\left(1 + \int_{r_0}^\infty \lambda\big(B_{v_x(t_z-t_x)}(0)\cap B_{v_y(t_z-t_y)}(y-x)\big)e^{-\nu_d(\Lambda(t_z)-\Lambda(r_0))}\theta(\mathrm{d}t_z)\right)\\ &\le e^{-\alpha\nu_d\Lambda(r_0)} + \int_{r_0}^\infty \lambda\big(B_{v_x(t_z-t_x)}(0)\cap B_{v_y(t_z-t_y)}(y-x)\big)e^{-\alpha\nu_d\Lambda(t_z)}\theta(\mathrm{d}t_z)\,. \end{aligned} \tag{4.18}$$

Then, with $f_\alpha^{(3)}$ defined at (4.5),

$$\int_{\mathbb{X}} f_\alpha^{(3)}(\mathbf{y})\mu(\mathbf{y}) \le \int_{\mathbb{X}^2} G(\mathbf{x}) e^{-\alpha v_d \Lambda(r_0)} \mu^2(\mathrm{d}(\mathbf{x}, \mathbf{y}))$$
$$+ \int_{\mathbb{X}^2} G(\mathbf{x}) \int_{r_0}^{\infty} \lambda\big(B_{v_x(t_z - t_x)}(0) \cap B_{v_y(t_z - t_y)}(y - x)\big) e^{-\alpha v_d \Lambda(t_z)} \theta(\mathrm{d}t_z) \mu^2(\mathrm{d}(\mathbf{x}, \mathbf{y})). \quad (4.19)$$

Since Λ is increasing,

$$\exp\{-\alpha v_d \Lambda(r_0(\mathbf{x}, \mathbf{y}))\} \le \exp\left\{-\frac{\alpha v_d}{3}\left[\Lambda\left(\frac{|x-y|}{v_x + v_y}\right) + \Lambda(t_x) + \Lambda(t_y)\right]\right\}, \quad (4.20)$$

and, by a change of variable and passing to polar coordinates, we obtain

$$\int_{\mathbb{R}^d} e^{-\frac{\alpha v_d}{3}\Lambda\left(\frac{|x|}{v_x + v_y}\right)} \mathrm{d}x \le d\omega_d (v_x + v_y)^d \int_0^\infty \rho^{d-1} e^{-\frac{\alpha v_d}{3}\Lambda(\rho)} \mathrm{d}\rho = d\omega_d (v_x + v_y)^d M'_\alpha. \quad (4.21)$$

Thus, using (4.9), (4.20), and (4.21), we can bound the first summand on the right-hand side of (4.19) as

$$\int_{\mathbb{X}^2} G(\mathbf{x}) e^{-\alpha v_d \Lambda(r_0)} \mu^2(\mathrm{d}(\mathbf{x}, \mathbf{y})) \le 6\omega_d^5 \int_0^\infty e^{-\frac{\alpha v_d}{3}\Lambda(t_x)} \mathrm{d}t_x \int_0^\infty e^{-\frac{\alpha v_d}{3}\Lambda(t_y)} \mathrm{d}t_y$$
$$\times \int_{\mathbb{R}^d} \mathbb{1}_{x \in W + B_{v_x a}(0)} \mathrm{d}x \iint_{\mathbb{R}_+^2 \times \mathbb{R}^d} p(v_x) e^{-\frac{\alpha v_d}{3}\Lambda\left(\frac{|x-y|}{v_x + v_y}\right)} \mathrm{d}y\, \nu^2(\mathrm{d}(v_x, v_y))$$
$$\le 6\omega_d^6 d Q(0, \alpha v_d/3)^2 M'_\alpha \int_{\mathbb{R}_+^2} \lambda\big(W + B_{v_x a}(0)\big) p(v_x)(v_x + v_y)^d \nu^2(\mathrm{d}(v_x, v_y))$$
$$\le 2^{d+3}\omega_d^6 d c_d (1 + a^d) M' M'_\alpha Q(0, \alpha v_d/3)^2\, V(W),$$

where for the final step we have used Jensen's inequality, (4.11) and (4.15), and the fact that $v_d M \le M'$. Arguing similarly for the second summand in (4.19), using (4.9) and the fact that $r_0 \ge t_x \vee t_y$ in the first step, (3.2) in the second step, and (4.15) in the final step, we obtain

$$\int_{\mathbb{X}^2} G(\mathbf{x}) \int_{r_0}^{\infty} \lambda\big(B_{v_x(t_z - t_x)}(0) \cap B_{v_y(t_z - t_y)}(y - x)\big) e^{-\alpha v_d \Lambda(t_z)} \theta(\mathrm{d}t_z) \mu^2(\mathrm{d}(\mathbf{x}, \mathbf{y}))$$
$$\le 6\omega_d^5 \int_{\mathbb{R}_+^2} \int_{\mathbb{R}_+^2} \lambda\big(W + B_{v_x a}(0)\big) p(v_x) \int_{t_x \vee t_y}^{\infty} w(t_z)^\alpha$$
$$\times \left(\int_{\mathbb{R}^d} \lambda\big(B_{v_x(t_z - t_x)}(0) \cap B_{v_y(t_z - t_y)}(y)\big) \mathrm{d}y\right) \theta(\mathrm{d}t_z) \theta^2(\mathrm{d}(t_x, t_y))\, \nu^2(\mathrm{d}(v_x, v_y))$$
$$\le 6\omega_d^7 \int_{\mathbb{R}_+^2} \lambda\big(W + B_{v_x a}(0)\big) p(v_x) v_x^d v_y^d\, \nu^2(\mathrm{d}(v_x, v_y))$$
$$\times \int_{\mathbb{R}_+^3} t_z^{2d} w(t_z)^{\alpha/3} w(t_x)^{\alpha/3} w(t_y)^{\alpha/3} \theta^3(\mathrm{d}(t_z, t_x, t_y))$$
$$\le 6\omega_d^7 c_d\, (1 + a^d)\, Q(0, \alpha v_d/3)^2 Q(2d, \alpha v_d/3) v_d M'\, V(W).$$

This concludes the proof of the first assertion.

Next, we prove the second assertion. For ease of notation, we drop obvious subscripts and write $\boldsymbol{y} = (y, s, v)$, $\boldsymbol{x}_1 = (x_1, t_1, u_1)$, and $\boldsymbol{x}_2 = (x_2, t_2, u_2)$. Using (4.18), write

$$\int_{\mathbb{X}} f_\alpha^{(3)}(\boldsymbol{y})^2 \mu(\boldsymbol{y}) = \int_{\mathbb{X}^3} G(\boldsymbol{x}_1)G(\boldsymbol{x}_2)q(\boldsymbol{x}_1, \boldsymbol{y})^\alpha q(\boldsymbol{x}_2, \boldsymbol{y})^\alpha \mu^3(\mathrm{d}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}))$$
$$\leq \int_{\mathbb{X}^3} G(\boldsymbol{x}_1)G(\boldsymbol{x}_2)(\mathfrak{I}_1 + 2\mathfrak{I}_2 + \mathfrak{I}_3)\mu^3(\mathrm{d}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y})), \tag{4.22}$$

with

$$\mathfrak{I}_1 = \mathfrak{I}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) := \exp\Big\{ -\alpha\nu_d\big[\Lambda(r_0(\boldsymbol{x}_1, \boldsymbol{y})) + \Lambda(r_0(\boldsymbol{x}_2, \boldsymbol{y}))\big]\Big\},$$
$$\mathfrak{I}_2 = \mathfrak{I}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) := e^{-\alpha\nu_d\Lambda(r_0(\boldsymbol{x}_1, \boldsymbol{y}))} \times \int_{s\vee t_2}^\infty \lambda(B_{u_2(r-t_2)}(0) \cap B_{v(r-s)}(y - x_2))e^{-\alpha\nu_d\Lambda(r)}\theta(\mathrm{d}r),$$
$$\mathfrak{I}_3 = \mathfrak{I}_3(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}) := \int_{s\vee t_2}^\infty \lambda\big(B_{u_2(r-t_2)}(0) \cap B_{v(r-s)}(y - x_2)\big)e^{-\alpha\nu_d\Lambda(r)}\theta(\mathrm{d}r) \times \int_{s\vee t_1}^\infty \lambda\big(B_{u_1(\rho-t_1)}(0) \cap B_{v(\rho-s)}(y - x_1)\big)e^{-\alpha\nu_d\Lambda(\rho)}\theta(\mathrm{d}\rho).$$

By (4.21),

$$\iint_{\mathbb{R}^{2d}} \exp\left\{ -\frac{\alpha\nu_d}{3}\left[\Lambda\left(\frac{|y|}{u_1 + v}\right) + \Lambda\left(\frac{|x - y|}{u_2 + v}\right)\right]\right\} \mathrm{d}x\mathrm{d}y$$
$$\leq \int_{\mathbb{R}^d} \exp\left\{ -\frac{\alpha\nu_d}{3}\Lambda\left(\frac{|y|}{u_1 + v}\right)\right\}\mathrm{d}y \int_{\mathbb{R}^d} \exp\left\{ -\frac{\alpha\nu_d}{3}\Lambda\left(\frac{|x|}{u_2 + v}\right)\right\}\mathrm{d}x$$
$$\leq d^2\omega_d^2(u_1 + v)^d(u_2 + v)^d {M'_\alpha}^2.$$

Hence, using (4.9) and (4.20) for the first step, we have

$$\int_{\mathbb{X}^3} G(\boldsymbol{x}_1)G(\boldsymbol{x}_2)\mathfrak{I}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y})\ \mu^3(\mathrm{d}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}))$$
$$\leq 36\omega_d^{10} \int_{\mathbb{R}^d} \mathbb{1}_{x_1\in W + B_{u_1 a}(0)}\mathrm{d}x_1 \int_{\mathbb{R}_+^3} e^{-\frac{\alpha\nu_d}{3}[\Lambda(t_1)+\Lambda(t_2)+2\Lambda(s)]}\theta^3(\mathrm{d}(t_1, t_2, s))$$
$$\times \iint_{\mathbb{R}_+^3\times(\mathbb{R}^d)^2} p(u_1)p(u_2)e^{-\frac{\alpha\nu_d}{3}\left[\Lambda\left(\frac{|x_1-y|}{u_1+v}\right)+\Lambda\left(\frac{|x_2-y|}{u_2+v}\right)\right]}\,\mathrm{d}y\,\mathrm{d}x_2\ \nu^3(\mathrm{d}(u_1, u_2, v))$$
$$\leq 36\omega_d^{12}d^2\ {M'_\alpha}^2 Q(0, \alpha\nu_d/3)^3$$
$$\times \int_{\mathbb{R}_+^3} \lambda\big(W + B_{u_1 a}(0)\big)\ (u_1 + v)^d(u_2 + v)^d p(u_1)p(u_2)\nu^3(\mathrm{d}(u_1, u_2, v))$$
$$\leq c_1\ {M'_\alpha}^2 Q(0, \alpha\nu_d/3)^3(1 + a^d)(1 + \nu_{2d}\nu_d^{-2})M_d M'\ V(W)$$

for some constant $c_1 \in (0, \infty)$ depending only on d. Here we have used the monotonicity of Q with respect to its second argument in the penultimate step, and in the final step we have used Jensen's inequality and (4.15) along with the fact that

$$\int_{\mathbb{R}_+^2} (1 + \nu_d^{-1}v^d)(u_2^d + v^d)p(u_2)\nu^2(\mathrm{d}(u_2, v)) \leq C(1 + \nu_d^{-2}\nu_{2d})M_d$$

for some constant $C \in (0, \infty)$ depending only on d.

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Next we bound the second summand in (4.22). Using (3.2) in the second step, the monotonicity of Λ and (4.20) in the third step, and (4.21) in the final step, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}} \mathfrak{I}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y})\mathrm{d}x_2\mathrm{d}y \\
&= \int_{\mathbb{R}^d} e^{-\alpha\nu_d\Lambda(r_0(\boldsymbol{x}_1,\boldsymbol{y}))}\mathrm{d}y \int_{s\vee t_2}^{\infty}\int_{\mathbb{R}^d} \lambda\big(B_{u_2(r-t_2)}(0)\cap B_{v(r-s)}(y-x_2)\big)\mathrm{d}x_2\, e^{-\alpha\nu_d\Lambda(r)}\theta(\mathrm{d}r) \\
&= \omega_d^2 u_2^d v^d \int_{\mathbb{R}^d} e^{-\alpha\nu_d\Lambda(r_0(\boldsymbol{x}_1,\boldsymbol{y}))}\mathrm{d}y \int_{s\vee t_2}^{\infty} (r-t_2)^d(r-s)^d\, e^{-\alpha\nu_d\Lambda(r)}\theta(\mathrm{d}r) \\
&\le \omega_d^2 u_2^d v^d \exp\left\{-\frac{\alpha\nu_d}{3}\left[\Lambda(t_1)+\Lambda(t_2)+2\Lambda(s)\right]\right\} \\
&\qquad\qquad \times \int_0^{\infty} r^{2d}\, e^{-\alpha\nu_d\Lambda(r)/3}\theta(\mathrm{d}r)\int_{\mathbb{R}^d}\exp\left\{-\frac{\alpha\nu_d}{3}\Lambda\left(\frac{|x_1-y|}{u_1+v}\right)\right\}\mathrm{d}y \\
&= d\omega_d^3 M'_\alpha Q(2d,\alpha\nu_d/3)u_2^d v^d(u_1+v)^d \exp\left\{-\frac{\alpha\nu_d}{3}\left[\Lambda(t_1)+\Lambda(t_2)+2\Lambda(s)\right]\right\}.
\end{aligned}$$

Therefore, arguing similarly as before, we obtain

$$\begin{aligned}
&\int_{\mathbb{X}^3} G(\boldsymbol{x}_1)G(\boldsymbol{x}_2)\mathfrak{I}_2(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{y})\,\mu^3(\mathrm{d}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{y})) \\
&\le 36\omega_d^{10}\int_{\mathbb{R}^d}\mathbb{1}_{x_1\in W+B_{u_1a}(0)}\mathrm{d}x_1\iint_{\mathbb{R}_+^6} p(u_1)p(u_2) \\
&\qquad\qquad \times\left(\iint_{\mathbb{R}^{2d}}\mathfrak{I}_2(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{y})\mathrm{d}x_2\,\mathrm{d}y\right)\theta^3(\mathrm{d}(t_1,t_2,s))\nu^3(\mathrm{d}(u_1,u_2,v)) \\
&\le 36\omega_d^{13} d\, M'_\alpha Q(2d,\alpha\nu_d/3)\int_{\mathbb{R}_+^3} e^{-\frac{\alpha\nu_d}{3}[\Lambda(t_1)+\Lambda(t_2)+2\Lambda(s)]}\theta^3(\mathrm{d}(t_1,t_2,s)) \\
&\qquad\qquad \times\int_{\mathbb{R}_+^3}\lambda\big(W+B_{u_1a}(0)\big)u_2^d v^d(u_1+v)^d p(u_1)p(u_2)\nu^3(\mathrm{d}(u_1,u_2,v)) \\
&\le c_2\, M'_\alpha Q(2d,\alpha\nu_d/3)Q(0,\alpha\nu_d/3)^3(1+a^d)(1+\nu_{2d}\nu_d^{-2})\nu_d M_d M'\,V(W)
\end{aligned}$$

for some constant $c_2\in(0,\infty)$ depending only on d, where for the final step we have used

$$\int_{\mathbb{R}_+^2}(1+\nu_d^{-1}v^d)u_2^d v^d p(u_2)\nu^2(\mathrm{d}(u_2,v))\le C'(1+\nu_d^{-2}\nu_{2d})\nu_d M_d$$

for some constant $C'\in(0,\infty)$ depending only on d.

Finally, we bound the third summand in (4.22). Arguing as above,

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}}\mathfrak{I}_3(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{y})\,\mathrm{d}x_2\,\mathrm{d}y \\
&= \int_{s\vee t_1}^{\infty}\left(\int_{\mathbb{R}^d}\lambda\big(B_{u_1(\rho-t_1)}(0)\cap B_{v(\rho-s)}(y-x_1)\big)\mathrm{d}y\right)e^{-\alpha\nu_d\Lambda(\rho)}\theta(\mathrm{d}\rho) \\
&\qquad\qquad \times\int_{s\vee t_2}^{\infty}\left(\int_{\mathbb{R}^d}\lambda\big(B_{u_2(r-t_2)}(0)\cap B_{v(r-s)}(y-x_2)\big)\mathrm{d}x_2\right)e^{-\alpha\nu_d\Lambda(r)}\theta(\mathrm{d}r) \\
&= \omega_d^4 u_1^d u_2^d v^{2d}\int_{s\vee t_1}^{\infty}(\rho-t_1)^d(\rho-s)^d e^{-\alpha\nu_d\Lambda(\rho)}\theta(\mathrm{d}\rho)\int_{s\vee t_2}^{\infty}(r-t_1)^d(r-s)^d e^{-\alpha\nu_d\Lambda(r)}\theta(\mathrm{d}r)
\end{aligned}$$

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$$\leq \omega_d^4 u_1^d u_2^d v^{2d} \left(\int_0^\infty r^{2d}\, e^{-\alpha\nu_d\Lambda(r)/3}\theta(\mathrm{d}r) \right)^2 \exp\left\{ -\frac{\alpha}{3}\nu_d\big[\Lambda(t_1)+\Lambda(t_2)+2\Lambda(s)\big] \right\}$$
$$\leq \omega_d^4\, Q(2d, \alpha\nu_d/3)^2 u_1^d u_2^d v^{2d} \exp\left\{ -\frac{\alpha}{3}\nu_d\big[\Lambda(t_1)+\Lambda(t_2)+2\Lambda(s)\big] \right\}.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{X}^3} G(\boldsymbol{x}_1)G(\boldsymbol{x}_2)\mathfrak{I}_3(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{y})\;\mu^3(\mathrm{d}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{y}))\\
&\leq 36\omega_d^{10} \int_{\mathbb{R}^d} \mathbb{1}_{x_1\in W+B_{u_1a}(0)}\mathrm{d}x_1 \iint_{\mathbb{R}_+^6} p(u_1)p(u_2)\\
&\qquad\times\left(\iint_{\mathbb{R}^{2d}} \mathfrak{I}_3(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{y})\mathrm{d}x_2\,\mathrm{d}y\right)\theta^3(\mathrm{d}(t_1,t_2,s))\nu^3(\mathrm{d}(u_1,u_2,v))\\
&\leq 36\omega_d^{14}\, Q(2d,\alpha\nu_d/3)^2 \int_{\mathbb{R}_+^3} \lambda\big(W+B_{u_1a}(0)\big)p(u_1)p(u_2)u_1^d u_2^d v^{2d}\nu^3(\mathrm{d}(u_1,u_2,v))\\
&\qquad\times\int_{\mathbb{R}_+^3} e^{-\frac{\alpha\nu_d}{3}[\Lambda(t_1)+\Lambda(t_2)+2\Lambda(s)]}\theta^3(\mathrm{d}(t_1,t_2,s))\\
&\leq c_3\, Q(2d,\alpha\nu_d/3)^2 Q(0,\alpha\nu_d/3)^3(1+a^d)\nu_{2d}M_dM'\,V(W),
\end{aligned}$$

for some constant $c_3\in(0,\infty)$ depending only on d. Combining the bounds for the summands on the right-hand side of (4.22) yields the desired conclusion. □

To compute the bounds in (4.6) and (4.7), we now only need to bound $\mu((\kappa+g)^{2\beta}G)$.

Lemma 4.5. *For $a\in(0,\infty)$ and $\alpha\in(0,1]$,*

$$\mu((\kappa+g)^{\alpha}G)\leq C_1\,V(W),$$

where

$$C_1 := C\,(1+a^d)Q(0,\alpha\zeta\nu_d/2)[M+(M+M')Q(d,\zeta\nu_d/2)^{\alpha}]$$

for a constant $C\in(0,\infty)$ depending only on d.

Proof. Define the function

$$\psi(t) := \int_t^\infty (s-t)^d e^{-\zeta\nu_d\Lambda(s)}\theta(\mathrm{d}s),$$

so that $g(\boldsymbol{x})=\omega_d v_x^d\psi(t_x)$. By subadditivity, it suffices to separately bound

$$\int_{\mathbb{X}}\kappa^{\alpha}(\boldsymbol{x})G(\boldsymbol{x})\mu(\mathrm{d}\boldsymbol{x}) \quad\text{and}\quad \int_{\mathbb{X}} g(\boldsymbol{x})^{\alpha}G(\boldsymbol{x})\mu(\mathrm{d}\boldsymbol{x}).$$

By (4.9) and (4.11),

$$\begin{aligned}
\int_{\mathbb{X}}\kappa^{\alpha}(\boldsymbol{x})G(\boldsymbol{x})\mu(\mathrm{d}\boldsymbol{x}) &\leq 6\omega_d^5\int_{\mathbb{X}}\mathbb{1}_{x\in W+B_{v_xa}(0)}p(v_x)e^{-\alpha\nu_d\Lambda(t_x)}\,\mathrm{d}x\,\theta(\mathrm{d}t_x)\,\nu(\mathrm{d}v_x)\\
&\leq 6\omega_d^5 c_d(1+a^d)Q(0,\alpha\zeta\nu_d/2)M\,V(W).
\end{aligned}$$

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For the second integral, using (4.15) write

$$\int g(\boldsymbol{x})^\alpha G(\boldsymbol{x})\mu(\mathrm{d}\boldsymbol{x}) \le 6\omega_d^{5+\alpha}\int_0^\infty\int_0^\infty \psi(t_x)^\alpha \lambda(W+B_{v_x a}(0))v_x^{\alpha d}p(v_x)\nu(\mathrm{d}v_x)\theta(\mathrm{d}t_x)$$
$$\le 6\omega_d^6 c_d(1+a^d)(M+M')\,V(W)\int_0^\infty \psi(t_x)^\alpha\theta(\mathrm{d}t_x).$$

Note that

$$\int_0^\infty \psi(t)^\alpha\theta(\mathrm{d}t) = \int_0^\infty \left(\int_t^\infty (s-t)^d e^{-\zeta\nu_d\Lambda(s)}\theta(\mathrm{d}s)\right)^\alpha \theta(\mathrm{d}t)$$
$$\le \int_0^\infty e^{-\alpha\zeta\nu_d\Lambda(t)/2}\theta(\mathrm{d}t)\left(\int_0^\infty s^d e^{-\zeta\nu_d\Lambda(s)/2}\theta(\mathrm{d}s)\right)^\alpha$$
$$= Q(0,\alpha\zeta\nu_d/2)Q(d,\zeta\nu_d/2)^\alpha\,,$$

where we have used the monotonicity of Λ in the second step. Combining this with the above bounds yields the result. □

Proofs of Theorems 2.1 and 2.2. Theorem 2.1 follows from (4.6) and (4.7) upon using Lemmas 4.2, 4.3, 4.4, and 4.5 and including the factors involving the moments of the speed in the constants.

The upper bound in Theorem 2.2 follows by combining Theorem 2.1 and Proposition 2.1, upon noting that $V(n^{1/d}W) \le nV(W)$ for $n \in \mathbb{N}$.

The optimality of the bound in Theorem 2.2 in the Kolmogorov distance follows by a general argument employed in the proof of [9, Theorem 1.1, Equation (1.6)], which shows that the Kolmogorov distance between any integer-valued random variable, suitably normalized, and a standard normal random variable is always lower-bounded by a universal constant times the inverse of the standard deviation; see [9, Section 6] for further details. The variance upper bound in (2.10) now yields the result. □

Proof of Theorem 2.3. Let θ be as given at (2.5). Then, as in the proof of Proposition 2.1,

$$\Lambda(t) = B\,\omega_d t^{d+\tau+1},$$

where $B := B(d+1,\tau+1)$. By (4.8), for $x \in \mathbb{R}_+$ and $y > 0$,

$$Q(x,y) = \int_0^\infty t^{x+\tau}e^{-y\omega_d B\,t^{d+\tau+1}}\mathrm{d}t = \frac{(y\omega_d B)^{-\frac{x+\tau+1}{d+\tau+1}}}{d+\tau+1}\Gamma\left(\frac{x+\tau+1}{d+\tau+1}\right) = C_1 y^{-\frac{x+\tau+1}{d+\tau+1}} \tag{4.23}$$

for some constant $C_1 \in (0,\infty)$ depending only on x, τ, and d. Then, using the inequality $\nu_\delta\nu_{7d-\delta} \le \nu_{7d}$ for any $\delta \in [0,7d]$, we have that for any $u \in [0,2d]$,

$$M_u = \int_{\mathbb{R}_+} v^u p(v)\nu(\mathrm{d}v) = \nu_u + Q(d,\zeta\nu_d)^5\nu_{5d+u}$$
$$\le C_2\nu_u(1+\nu_{5d+u}\nu_u^{-1}\nu_d^{-5}) \le C_2\nu_u\big(1+\nu_{7d}\nu_d^{-7}\big)$$

for $C_2 \in (0,\infty)$ depending only on τ and d, where in the last step we have used positive association and the Cauchy–Schwartz inequality to obtain

$$\nu_{7d}\nu_d^{-7} \ge \nu_{5d+u}\nu_u^{-1}\nu_d^{-5}\nu_{2d-u}\nu_u\nu_d^{-2} \ge \nu_{5d+u}\nu_u^{-1}\nu_d^{-5}.$$

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In particular,

$$M_0 \le C_2\big(1+\nu_{7d}\nu_d^{-7}\big) \quad \text{and} \quad M \le C_2(1+\nu_d)\big(1+\nu_{7d}\nu_d^{-7}\big).$$

Similarly, by (4.17),

$$M'_\alpha := \frac{1}{d+\tau+1}\Gamma\left(\frac{d}{d+\tau+1}\right)(\alpha B\omega_d\nu_d/3)^{-\frac{d}{d+\tau+1}} = C_3\nu_d^{-\frac{d}{d+\tau+1}}$$

for some constant $C_3 \in (0, \infty)$ depending only on α, τ, and d. Also, by (4.23), for $b > 0$,

$$Q(x, by) = b^{-\frac{x+\tau+1}{d+\tau+1}} Q(x, y).$$

Recall the parameters $p = 1$, $\beta = 1/36$, and $\zeta = 1/50$. We will need a slightly refined version of Lemmas 4.2–4.4 that uses (4.10) and (4.14) instead of (4.11) and (4.15), respectively. Arguing exactly as in Lemmas 4.2–4.4, this yields

$$\int_{\mathbb{X}} f_\alpha^{(1)}(\mathbf{y})\mu(\mathrm{d}\mathbf{y}) \le C(1+a^d)\frac{Q(0, \alpha\nu_d/2)}{\alpha}\sum_{i=0}^{d} V_{d-i}(W)M_i,$$

$$\int_{\mathbb{X}} f_\alpha^{(2)}(\mathbf{y})\mu(\mathrm{d}\mathbf{y}) \le C(1+a^d)\, Q(0, \alpha\nu_d/2)Q(d, \alpha\nu_d/2)\sum_{i=0}^{d} V_{d-i}(W)M_{d+i},$$

$$\int_{\mathbb{X}} f_\alpha^{(3)}(\mathbf{y})\mu(\mathrm{d}\mathbf{y}) \le C(1+a^d)Q(0, \alpha\nu_d/3)^2\Big[M'_\alpha + Q(2d, \alpha\nu_d/3)\nu_d\Big]\sum_{i=0}^{d} V_{d-i}(W)M_{d+i},$$

$$\mu((\kappa+g)^\alpha G) \le C(1+a^d)Q(0, \alpha\zeta\nu_d/2) \times\left[\sum_{i=0}^{d} V_{d-i}(W)M_i + Q(d, \zeta\nu_d/2)^\alpha \sum_{i=0}^{d} V_{d-i}(W)M_{\alpha d+i}\right],$$

and

$$\int_{\mathbb{X}} f_\alpha^{(1)}(\mathbf{y})^2\mu(\mathrm{d}\mathbf{y}) \le C(1+a^d)\, M_0\nu_{2d}Q(d, \alpha\nu_d/2)^2 Q(0, \alpha\nu_d)\sum_{i=0}^{d} V_{d-i}(W)M_i,$$

$$\int_{\mathbb{X}} f_\alpha^{(2)}(\mathbf{y})^2\mu(\mathrm{d}\mathbf{y}) \le C(1+a^d)\, M_d\, Q(0, \alpha\nu_d/3)^2 Q(2d, \alpha\nu_d/3)\sum_{i=0}^{d} V_{d-i}(W)M_{d+i},$$

$$\int_{\mathbb{X}} f_\alpha^{(3)}(\mathbf{y})^2\mu(\mathrm{d}\mathbf{y}) \le C(1+a^d)M_d(1+\nu_{2d}\nu_d^{-2})Q(0, \alpha\nu_d/3)^3 \times\Big({M'_\alpha}^2 + M'_\alpha Q(2d, \alpha\nu_d/3)\nu_d + Q(2d, \alpha\nu_d/3)^2\nu_{2d}\Big)\sum_{i=0}^{d} V_{d-i}(W)M_{d+i},$$

where $C \in (0, \infty)$ is a constant depending only on d.

These modified bounds in combination with the above estimates, along with the fact that $\nu_i \le \nu_d^{-1}\nu_{d+i}$, yield that there exists a constant C depending only on d and τ such that for $i \in \{1, 2, 3\}$,

$$\int_{\mathbb{X}} f_{2\beta}^{(i)}(\mathbf{y})\mu(\mathrm{d}\mathbf{y}) \le C(1+a^d)\nu_d^{-\frac{\tau+1}{d+\tau+1}-1}\big(1+\nu_{7d}\nu_d^{-7}\big)\sum_{i=0}^{d} V_{d-i}(W)\nu_{d+i}. \tag{4.24}$$

Also, note that by Hölder's inequality and positive association, for $i = 0, \ldots, d$, we have

$$\nu_d^{-\alpha}\nu_{\alpha d+i} \le \nu_d^{-\alpha}\nu_i^{1-\alpha}\nu_{d+i}^{\alpha} \le \nu_d^{-1}\nu_{d+i}.$$

Combining this with the estimates above yields that there exists a constant C depending only on d and τ such that

$$\mu((\kappa+g)^{2\beta}G) \le C(1+a^d)\nu_d^{-\frac{\tau+1}{d+\tau+1}-1}\big(1+\nu_{7d}\nu_d^{-7}\big)\sum_{i=0}^{d} V_{d-i}(W)\nu_{d+i}. \tag{4.25}$$

Arguing similarly, we also obtain that there exists a constant C depending only on d and τ such that

$$\int_{\mathbb{X}} f_\beta^{(1)}(\mathbf{y})^2\mu(\mathrm{d}\mathbf{y}) \le C(1+a^d)\,\nu_d^{-\frac{\tau+1}{d+\tau+1}-3}\nu_{2d}\big(1+\nu_{7d}\nu_d^{-7}\big)^2\sum_{i=0}^{d} V_{d-i}(W)\nu_{d+i}\,,$$

$$\int_{\mathbb{X}} f_\beta^{(2)}(\mathbf{y})^2\mu(\mathrm{d}\mathbf{y}) \le C(1+a^d)\,\nu_d^{-\frac{\tau+1}{d+\tau+1}-1}\big(1+\nu_{7d}\nu_d^{-7}\big)^2\sum_{i=0}^{d} V_{d-i}(W)\nu_{d+i}\,,$$

$$\int_{\mathbb{X}} f_\beta^{(3)}(\mathbf{y})^2\mu(\mathrm{d}\mathbf{y}) \le C(1+a^d)\,\nu_d^{-\frac{\tau+1}{d+\tau+1}-1}\big(1+\nu_{7d}\nu_d^{-7}\big)^4\sum_{i=0}^{d} V_{d-i}(W)\nu_{d+i}.$$

Thus, there exists a constant C depending only on d and τ such that for $i \in \{1, 2, 3\}$,

$$\int_{\mathbb{X}} f_\beta^{(i)}(\mathbf{y})^2\mu(\mathrm{d}\mathbf{y}) \le C(1+a^d)\,\nu_d^{-\frac{\tau+1}{d+\tau+1}-1}\big(1+\nu_{7d}\nu_d^{-7}\big)^4\sum_{i=0}^{d} V_{d-i}(W)\nu_{d+i}. \tag{4.26}$$

Plugging (4.24), (4.25), and (4.26) into (4.6) and (4.7) and using Proposition 2.1 to lower-bound the variance yields the desired bounds. □

Proof of Corollary 2.1. Define the Poisson process $\eta^{(s)}$ with intensity measure $\mu^{(s)} := \lambda \otimes \theta \otimes \nu^{(s)}$, where $\nu^{(s)}(A) := \nu(s^{-1/d}A)$ for all Borel sets A. It is straightforward to see that the set of locations of exposed points of η_s has the same distribution as that of those of $\eta^{(s)}$, multiplied by $s^{-1/d}$, i.e., the set $\{x : \boldsymbol{x} \in \eta_s \text{ is exposed}\}$ coincides in distribution with $\{s^{-1/d}x : \boldsymbol{x} \in \eta^{(s)} \text{ isexposed}\}$. Hence, the functional $F(\eta_s)$ has the same distribution as $F_s(\eta^{(s)})$, where F_s is defined as in (2.1) for the weight function

$$h(\boldsymbol{x}) = h_{1,s}(x)h_2(t_x) = \mathbb{1}_{x\in W_s}\mathbb{1}_{t_x<a}\,,$$

with $W_s := s^{1/d}W$. It is easy to check that for $k \in \mathbb{N}$, the kth moment of $\nu^{(s)}$ is given by $\nu_k^{(s)} = s^{k/d}\nu_k$ and $\lambda(W_s) = s\lambda(W)$. We also have

$$V_{\nu^{(s)}}(W_s) = \sum_{k=0}^{d} V_{d-i}(s^{1/d}W)\nu_{d+i}^{(s)} = \sum_{k=0}^{d} s^{\frac{d-i}{d}} V_{d-i}(W)s^{\frac{d+i}{d}}\nu_{d+i} = s^2 V_\nu(W).$$

Finally, noticing that

$$\begin{aligned} l_{a,\tau}(\nu_d^{(s)}) &= \gamma\left(\frac{\tau+1}{d+\tau+1}, a^{d+\tau+1}s\nu_d\right)(s\nu_d)^{-\frac{\tau+1}{d+\tau+1}} \\ &\ge \gamma\left(\frac{\tau+1}{d+\tau+1}, a^{d+\tau+1}\nu_d\right)(s\nu_d)^{-\frac{\tau+1}{d+\tau+1}} \end{aligned}$$

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for $s \geq 1$, we deduce the result directly from Theorem 2.3. The optimality of the Kolmogorov bound follows from arguing as in the proof of Theorem 2.2. □

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