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TRUNCATED MICROSUPPORT AND HYPERBOLIC INEQUALITIES

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Abstract. We prove that the k-truncated microsupport of the specialization of a complex of sheaves F along a submanifold is contained in the normal cone to the conormal bundle along the k-truncated microsupport of F. In the complex case, applying our estimates to $F = R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$, where $\mathcal{M}$ is a coherent $\mathcal{D}$-module, we obtain new estimates for the truncated microsupport of real analytic and hyperfunction solutions. When $\mathcal{M}$ is regular along Y we also obtain estimates for the truncated microsupport of the holomorphic solutions of the induced system along Y as well as for the nearby-cycle sheaf of $\mathcal{M}$ when Y is a hypersurface.

§1. Introduction and statement of the main results

Let X be a real manifold and let F denote an object of the derived category of abelian sheaves on X. The microsupport of F, denoted by $SS(F)$, was introduced by M. Kashiwara and P. Schapira ([13]; [14]), as a subset of the cotangent bundle $\pi : T^*X \to X$ describing the directions of non propagation for F. The truncated microsupport of a given degree k (or k-truncated microsupport), $SS_k(F)$, defined by the same authors, is only concerned by degrees of cohomology up to the order k and allows us to consider some phenomenon of propagation in specific degrees. Such notion is particularly useful in the framework of the theory of linear partial differential equations. More precisely, when F is the complex of holomorphic solutions of a coherent module $\mathcal{M}$ over the sheaf $\mathcal{D}_X$ of holomorphic differential operators on a complex manifold X, $SS_k(F)$ is completely determined as a subset of the characteristic variety $\mathrm{Char}(\mathcal{M})$, which itself coincides with

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$SS(F)$. In the characteristic case, interesting propagation results (cf. [2], [19], [11]) may be obtained with the truncated microsupport. The truncated microsupport and its functorial properties were studied in [11] and [12].

It is now natural to study the behaviour of $SS_k(F)$ under specialization along a submanifold. That is the main purpose of this work, having in scope the application to $\mathcal{D}$-modules, specially to holomorphic solutions of induced systems and to real analytic solutions.

Let $\mathbf{k}$ be a field. Let $D^b(\mathbf{k}_X)$ denote the bounded derived category of complexes of sheaves of $\mathbf{k}$-vector spaces.

Let M be a submanifold of X. We shall identify $T_{T^*_M X}(T^*X)$, $T^*(T_M X)$ and $T^*(T^*_M X)$ thanks to the Hamiltonian isomorphism. Unless otherwise specified, we shall follow the notations in [13]. In particular, for $F \in D^b(\mathbf{k}_X)$, $\nu_M(F)$ denotes the specialization of F along M, an object of $D^b(\mathbf{k}_{T_M X})$ and $C_{T^*_M X}(SS_k(F))$ denotes the normal cone to $SS_k(F)$ along $T^*_M X$. For a morphism $f : Y \to X$ we shall use $f^\#$, a correspondence which associates conic subsets of T^*Y to conic subsets of T^*X as well as the operation $\widehat{+}$ which associates to pairs of conic closed subsets of T^*X conic closed subsets of T^*X.

The main result of this work is the following:

THEOREM 1.1. *Let M be a closed submanifold of X and let $F \in D^b(\mathbf{k}_X)$. Then:*

$$SS_k(\nu_M(F)) \subset C_{T^*_M X}(SS_k(F)).$$

The main difficulty in its proof is that the use of distinguished triangles is not always convenient because of the shift they introduce. To overcome it, we needed to deduce a number of further functorial properties. Namely, as an essential step of the proof of this theorem, we obtain the following estimate:

THEOREM 1.2. *Let Y and X be real manifolds, let $f\colon Y \to X$ be a morphism and let $F \in D^b(\mathbf{k}_X)$. Then*

$$SS_k(f^{-1}F) \subset f^\#(SS_k(F)).$$

Let us denote by f_d and f_π the canonical morphisms (f_d was noted by ${}^t f'$ in [13]):

$$f_\pi : Y \times_X T^*X \to T^*X \text{ and } f_d : Y \times_X T^*X \to T^*Y.$$

Regarding f as the composition of a smooth map with a closed embedding, the proof of Theorem 1.2 relies in two steps. The first is to apply Proposition 4.4 of [11] which proves the estimate when f is smooth. The second is Proposition 6.1, where we obtain the estimate

$$SS_k(F|_M) \subset j_d j_\pi^{-1}(SS_k(F) \widehat{+} T^*_M X),$$

when $j : M \to X$ is a closed embedding.

Remark that, in that case, $j^\#(SS_k(F)) = j_d j_\pi^{-1}(SS_k(F) \widehat{+} T^*_M X)$.

In particular, when f is non characteristic with respect to F, we get

$$SS_k(f^{-1}F) \subset f_d f_\pi^{-1}(SS_k(F)).$$

Namely, when M is non characteristic with respect to F, in other words,

$$SS(F) \cap T^*_M X \subset T^*_X X,$$

we have $SS_k(F) \widehat{+} T^*_M X = SS_k(F) + T^*_M X$ and

$$j_d j_\pi^{-1}(SS_k(F) + T^*_M X) = j_d j_\pi^{-1}(SS_k(F)).$$

Let now Y be a complex closed smooth hypersurface of a complex analytic manifold X and assume that Y is defined as the zero locus of a holomorphic function f. Let ψ_Y denote the functor of nearby cycles associated to Y. Recall that Y may be regarded as a submanifold Y' of $T_Y X$ by a canonical section s given by s such that $\psi_Y(F) \simeq s^{-1}\nu_Y(F)$.

Then, Theorem 1.1 entails:

Corollary 1.3. *Let $F \in D^b(\mathbf{k}_X)$. Then*

$$SS_k(\psi_Y(F)) \subset s_d s_\pi^{-1}(C_{T^*_Y X}(SS_k(F)) \widehat{+} T^*_{Y'}(T_Y X)).$$

Let us point out that one interesting application of Proposition 6.1 is the new estimate for the k-truncated microsupport of the tensor product (see Proposition 6.7).

We end this paper with the application of our results to the complex $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ of holomorphic solutions of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ on a complex manifold X(see Section 6.2). Let $\mathcal{D}_X$ be the sheaf of linear partial differential operators of finite order and $\mathcal{O}_X$ the sheaf of holomorphic functions. Let Y be a complex submanifold of X and j be the embedding of Y in X. We shall denote by $\mathcal{M}_Y$ the induced system, an object of the derived category of left $\mathcal{D}_Y$-modules. Recall that, when $\mathcal{M}$ is regular in the sense of [10], $\mathcal{M}_Y$ has coherent cohomology. Let $\tau\colon T_YX \to Y$ be the projection. Still under the assumption that $\mathcal{M}$ is regular along Y, one defines a coherent $\mathcal{D}_{T_YX}$-module $\nu_Y(\mathcal{M})$, the specialization of $\mathcal{M}$ along Y, satisfying a natural isomorphism

$$\nu_Y(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)) \simeq R\mathcal{H}om_{\mathcal{D}_{T_YX}}(\nu_Y(\mathcal{M}),\mathcal{O}_{T_YX}).$$

Moreover, if Y has codimension 1, one defines the nearby-cycle module $\psi_Y(\mathcal{M})$, a coherent $\mathcal{D}_Y$-module, satisfying a natural isomorphism

$$\psi_Y(F) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\psi_Y(\mathcal{M}),\mathcal{O}_Y).$$

We refer to [8] for the details on these isomorphisms.

Set $V = SS(F) = \mathrm{Char}(\mathcal{M})$ and denote by $V = \bigsqcup_\alpha V_\alpha$ the (local) decomposition of V in its irreducible components. Let Y_α be the irreducible complex analytic subset $\pi(V_\alpha)$ of X. The notion of orthogonality between a submanifold Y of X and an involutive subvariety V of T^*X will be recalled at Section 6.2. We recall in Lemma 6.8 that if Y is orthogonal to V and V is irreducible, then $V' = j_d(j_\pi^{-1}(V))$ is irreducible and $\pi(V)$ has the same codimension of $\pi'(V')$. Here, $\pi' : T^*Y \to Y$ denotes the projection.

As a consequence of Theorem 1.1 together with the results of [8] we obtain:

Theorem 1.4. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then:*

$$SS_k(R\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}\mathcal{M},\nu_Y(\mathcal{O}_X))) \subset C_{T_Y^*X}(SS_k(F)).$$

If, moreover, $\mathcal{M}$ is regular along Y in the sense of [10] *we have:*

$$SS_k(R\mathcal{H}om_{\mathcal{D}_{T_YX}}(\nu_Y(\mathcal{M}),\mathcal{O}_{T_YX})) \subset C_{T_Y^*X}(SS_k(F)).$$

From the preceding theorem, the results of [8] and Corollary 1.3 we obtain:

Corollary 1.5. *Assume that $\mathcal{M}$ is regular along Y in the sense of* [10]. *Then*

$$SS_k(R\mathcal{H}om_{\mathcal{D}_Y}(\psi_Y(\mathcal{M}),\mathcal{O}_Y)) \subset s_d s_\pi^{-1}(C_{T_Y^*X}(SS_k(F))\widehat{+}T^*_{Y'}(T_YX)).$$

Furthermore, Proposition 6.1 together with the results of [8] and Theorem 6.7 of [11] entails:

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THEOREM 1.6. *Assume that $\mathcal{M}$ is regular along Y in the sense of* [10]. *Then:*

$$SS_k(R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)) \subset j_d j_\pi^{-1}(SS_k(F)\widehat{+}T_Y^*X).$$

If, moreover, Y is non characteristic for $\mathcal{M}$, we have

$$SS_k(R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)) \subset j_d j_\pi^{-1}(SS_k(F)).$$

If Y is orthogonal to each V_α such that codim $\pi(V_\alpha) \leq k$, the preceding inclusion becomes an equality, for every $i \leq k$:

$$SS_i(R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)) = j_d j_\pi^{-1}(SS_i(F)).$$

Recall that M. Kashiwara has proven in [9] that, when Y is non characteristic for $\mathcal{M}$,

$$SS(R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)) = j_d j_\pi^{-1}(SS(F)).$$

The condition of orthogonality is required in Theorem 1.6 in order to have the analogous equality up to a given degree k.

Let us now assume that the complex manifold X is the complexified of a real analytic manifold M. Denote by $\mathcal{A}_M$ the sheaf of real analytic functions on M and by j the embedding of M in X.

Another important application of Theorem 1.2 is:

PROPOSITION 1.7. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then we have the estimate:*

$$SS_k(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)) \subset j_d j_\pi^{-1}(SS_k(F)\widehat{+}T_M^*X).$$

Let $\mathcal{B}_M$ denote the sheaf of Sato's hyperfuntions on M. As an immediate consequence of Proposition 1.7 together with Theorem 6.7 of [11] we get:

COROLLARY 1.8. *Let $\mathcal{M}$ be an coherent $\mathcal{D}_X$-module. Assume that*

$$SS_k(F) \cap T_M^*X \subset T_X^*X.$$

Then,

$$SS_k(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)) = SS_k(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)) \subset j_d j_\pi^{-1}(SS_k(F))$$
$$\subset j_d j_\pi^{-1}\Big(\Big(\bigcup_{codim Y_\alpha < k} V_\alpha\Big) \cup \Big(\bigcup_{codim Y_\alpha = k} T_{Y_\alpha}^*X\Big)\Big).$$

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We shall illustrate this corollary with an example (see Example 6.10) of a propagation phenomenon for real analytic solutions of a class of non elliptic differential operators, which, as far as we know, is new.

When $\mathcal{M}$ is elliptic, in other words,

$$SS(\mathcal{M}) \cap T^*_M X \subset T^*_X X,$$

we get the estimate:

$$\text{For any k, } SS_k(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)) = SS_k(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M))$$

$$\subset j_d j_\pi^{-1}((\bigcup_{codim Y_\alpha < k} V_\alpha) \cup (\bigcup_{codim Y_\alpha) = k} T^*_{Y_\alpha} X)).$$

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§2. Notations

We will mainly follow the notations in [13].

Let X be a real manifold. We denote by $\tau : TX \to X$ the tangent bundle to X and by $\pi : T^*X \to X$ the cotangent bundle. We identify X with the zero section of T^*X. Given a smooth submanifold Y of X, $T_Y X$ denotes the normal bundle to Y and $T^*_Y X$ the conormal bundle. Given a submanifold Y of X and a subset S of X we denote by $C_Y(S)$ the normal cone to S along Y, a closed conic subset of $T_Y X$.

Let $f : X \to Y$ be a morphism of manifolds. We denote by

$$f_\pi : X \times_Y T^*Y \to T^*Y \text{ and } f_d : X \times_Y T^*Y \to T^*X$$

the associated morphisms.

Given a subset A of T^*X, we denote by A^a the image of A by the antipodal map

$$a : (x; \xi) \mapsto (x; -\xi).$$

The closure of A is denoted by $\overline{A}$. For a cone $\gamma \subset TX$, the polar cone γ° to γ is the convex cone in T^*X defined by

$$\gamma^\circ = \{(x; \xi) \in TX; x \in \pi(\gamma) \text{ and } \langle v, \xi \rangle \geq 0 \text{ for any } (x; v) \in \gamma\}.$$

Given conic subsets A and B of T^*X, the operations $A+B$ and $A\widehat{+}B$ are defined in [13] and will be recalled in Section 3.

Given an open subset Ω of X, as in [13], we denote by $N^*(\Omega)$ the conormal cone to Ω.

When X is an open subset of a real finite-dimensional vector space E and γ is a closed convex cone (with vertex at 0) in E, we denote by X_γ the open set X endowed with the induced γ-topology of E.

Let $\mathbf{k}$ be a field. We denote by $D(\mathbf{k}_X)$ the derived category of complexes of sheaves of $\mathbf{k}$-vector spaces on X and by $D^b(\mathbf{k}_X)$ the full subcategory of $D(\mathbf{k}_X)$ consisting of complexes with bounded cohomologies.

For $k \in \mathbb{Z}$, we denote by $D^{\geq k}(\mathbf{k}_X)$ (resp. $D^{\leq k}(\mathbf{k}_X)$) the full additive subcategory of $D^b(\mathbf{k}_X)$ consisting of objects F satisfying $H^j(F) = 0$, for any $j < k$ (resp. $H^j(F) = 0$, for any $j > k$). The category $D^{\geq k+1}(\mathbf{k}_X)$ is denoted by $D^{>k}(\mathbf{k}_X)$.

Given an object F of $D^b(\mathbf{k}_X)$ and a submanifold M of X, $\nu_M(F)$ denotes the specialization of F along M, an object of $D^b(\mathbf{k}_{T_MX})$.

Let F be an object of $D^b(\mathbf{k}_X)$; we denote by $SS(F)$ its microsupport, a closed $\mathbb{R}^+$-conic involutive subset of T^*X. For $p \in T^*X$, $D^b(\mathbf{k}_X; p)$ denotes the localization of $D^b(\mathbf{k}_X)$ by the full triangulated subcategory consisting of objects F such that $p \notin SS(F)$.

Let X be a finite-dimensional complex manifold. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions, by $\mathcal{D}_X$ the sheaf of linear holomorphic differential operators of finite order and by $\mathcal{D}_X(\cdot)$ the filtration by the order. Given a coherent $\mathcal{D}_X$-module $\mathcal{M}$, we denote by $\mathrm{Char}(\mathcal{M})$ its characteristic variety.

Let Y be a closed submanifold, let τ be the projection of T_YX on Y and let V_Y^* denote the V-filtration on $\mathcal{D}_X$ with respect to Y. Let $\mathcal{D}_{[T_YX]}$ denote the sheaf of differential operators on T_YX with polynomial coefficients with respect to the fibers of τ. Let θ denote the Euler operator on T_YX. Recall that $\mathcal{M}$ is regular along Y if for any local section u of $\mathcal{M}$ there exists a non trivial polynomial b of degree m such that

$$b(\theta)u \in (V_Y^1(\mathcal{D}_X) \cap \mathcal{D}_X(m))u.$$

Following Kashiwara in [8], given an appropriate good V_Y^*-filtration on $\mathcal{M}$, the specialized of $\mathcal{M}$ along Y, $\nu_Y(\mathcal{M})$, is the coherent $\mathcal{D}_{T_YX}$-module generated by the associated graded module. When Y is a hypersurface, one defines a coherent $\mathcal{D}_Y$-module, the nearby-cycles module $\psi_Y(\mathcal{M})$, as the degree zero homogeneous term of that graded module.

§3. Review on normal cones in cotangent bundles

For the reader's convenience we shall recall here some operations on conic subsets in cotangent bundles defined on [13].

Let X be a real manifold, (x) a system of local coordinates on X and denote by $(x;\xi)$ the associated coordinates on T^*X. Given two conic subsets A and B of T^*X, one defines the sum

$$A+B=\{(x;\xi)\in T^*X; \xi=\xi_1+\xi_2, \text{ for some } (x;\xi_1)\in A \text{ and } (x;\xi_2)\in B\}.$$

When A and B are closed, $A\widehat{+}B$ is the closed conic set containing $A+B$, described as follows: $(x_0;\xi_0)$ belongs to $A\widehat{+}B$ if and only if there exists sequences $\{(x_n;\xi_n)\}_n$ in A and $\{(y_n;\eta_n)\}_n$ in B such that:

$$\begin{cases} x_n, y_n \underset{n}{\longrightarrow} x_0, \\ \xi_n+\eta_n \underset{n}{\longrightarrow} \xi_0, \\ |x_n-y_n||\xi_n| \underset{n}{\longrightarrow} 0. \end{cases}$$

Let M be a submanifold of X. Let (x',x'') be a system of local coordinates on X such that $M=\{(x',x''); x'=0\}$ and let $(x',x'';\xi',\xi'')$ denote the associated coordinates on T^*X. Given a subset Λ of T^*X we describe the normal cone to Λ along T^*_MX, $C_{T^*_MX}(\Lambda)$, as follows: $(x'_0,x''_0;\xi'_0,\xi''_0)\in C_{T^*_MX}(\Lambda)$ if and only if there exist sequences of real positive numbers $\{c_n\}_n$ and $\{(x'_n,x''_n;\xi'_n,\xi''_n)\}_n$ in Λ such that:

$$\begin{cases} (x'_n,x''_n;\xi'_n,\xi''_n) \underset{n}{\longrightarrow} (0,x''_0;\xi'_0,0), \\ c_n(x'_n;\xi''_n) \underset{n}{\longrightarrow} (x'_0;\xi''_0). \end{cases}$$

Thanks to the Hamiltonian isomorphism, one gets an embedding of T^*M into $T_{T^*_MX}T^*X$ and, for a conic subset Λ of T^*X, the set $T^*M\cap C_{T^*_MX}(\Lambda)$ is described as follows: $(x'_0,x''_0;\xi'_0,\xi''_0)\in T^*M\cap C_{T^*_MX}(\Lambda)$ if and only if there exists a sequence $\{(x'_n,x''_n;\xi'_n,\xi''_n)\}_n$ in Λ such that:

$$\begin{cases} (x''_n;\xi''_n) \underset{n}{\longrightarrow} (x''_0;\xi''_0), \\ |x'_n| \underset{n}{\longrightarrow} 0 \\ |x'_n||\xi'_n| \underset{n}{\longrightarrow} 0. \end{cases}$$

Let $f : Y \to X$ be a morphism of manifolds. The notion of $f^{\#}$, a correspondence introduced in [13] associating conic subsets of T^*Y to conic subsets of T^*X, is rather complicated and we refer the reader to [13] for the details. We just recall the following results:

PROPOSITION 3.1. (cf. Proposition 6.2.4 of [13]) *Let Λ be a conic subset of T^*X.*

(i) *Assume that $f : M \to X$ is a closed embedding. Then,*

$$f^{\#}(\Lambda) = T^*M \cap C_{T^*_M X}(\Lambda).$$

(ii) *Let (x) (resp. (y)) be a system of local coordinates on X (resp. Y) and let $(x;\xi)$ (resp. $(y;\eta)$) be the associated coordinates on T^*X (resp. T^*Y). Then*

$(y_0;\eta_0) \in f^{\#}(\Lambda)$ *if and only if there exist a sequence $\{(x_n;\xi_n)\}_n$ in Λ and a sequence $\{y_n\}_n$ in Y such that*

$$y_n \underset{n}{\to} y_0, x_n \underset{n}{\to} f(y_0), ({}^t f'(y_n) \cdot \xi_n) \underset{n}{\to} \eta_0, |x_n - f(y_n)||\xi_n| \underset{n}{\to} 0.$$

We shall also need the following description of $j^{\#}$ when j is an embedding:

LEMMA 3.2. *Let M be a closed submanifold of X and let j denote the embedding of M in X. Let Λ be a closed conic subset of T^*X. Then:*

$$j_d j_\pi^{-1}(\Lambda \widehat{+} T^*_M X) = T^*M \cap C_{T^*_M X}(\Lambda),$$

where we identify $T_{T^*_M X} T^*X$ and $T^*T_M X$ by the Hamiltonian isomorphism.

Proof. It is enough to prove that

$$j_d j_\pi^{-1}(\Lambda \widehat{+} T^*_M X) = j^{\#}(\Lambda).$$

Let $p \in j_d j_\pi^{-1}(\Lambda \widehat{+} T^*_M X)$ and let (x', x'') be a system of local coordinates on X in a neighborhood of p such that $M = \{(x); x' = 0\}$. Let $(x;\xi)$ denote the associated coordinates on T^*X. Suppose $p = (x''_0;\xi''_0)$.

Then there exists ξ'_0 such that $(0, x''_0;\xi'_0,\xi''_0) \in \Lambda \widehat{+} T^*_M X$. By definition of $\widehat{+}$, there exist sequences $\{(x'_n, x''_n;\xi'_n,\xi''_n)\}_n$ in Λ and $\{(0, y''_n;\eta'_n, 0)\}_n$ in

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$T^*_M X$ such that

$$\begin{cases} (x'_n, x''_n), (0, y''_n) \underset{n}{\longrightarrow} (0, x''_0), \\ \xi''_n \underset{n}{\longrightarrow} \xi''_0, \\ \xi'_n + \eta'_n \underset{n}{\longrightarrow} \xi'_0, \\ |(x'_n, x''_n) - (0, y''_n)||(\xi'_n, \xi''_n)| \underset{n}{\longrightarrow} 0. \end{cases}$$

Hence

$$\begin{cases} x''_n \underset{n}{\longrightarrow} x''_0, \\ x'_n \underset{n}{\longrightarrow} 0, \\ \xi''_n \underset{n}{\longrightarrow} \xi''_0, \\ |x'_n||\xi'_n| \underset{n}{\longrightarrow} 0 \end{cases}$$

and $(x''_0; \xi''_0) \in j^{\#}(\Lambda)$.

Conversely, let $p \in j^{\#}(\Lambda)$, $p = (x''_0; \xi''_0)$. Then there exists a sequence $\{(x'_n, x''_n; \xi'_n, \xi''_n)\}_n$ in Λ such that

$$\begin{cases} (x''_n; \xi''_n) \underset{n}{\longrightarrow} (x''_0; \xi''_0), \\ x'_n \underset{n}{\longrightarrow} 0, \\ |x'_n||\xi'_n| \underset{n}{\longrightarrow} 0. \end{cases}$$

The sequences $\{(x'_n, x''_n; \xi'_n, \xi''_n)\}_n$ in Λ and $\{(0, x''_n; -\xi'_n, 0)\}$ in $T^*_M X$ satisfy the necessary conditions so that $(0, x''_0; 0, \xi''_0) \in \Lambda \widehat{+} T^*_M X$, hence $(x''_0; \xi''_0) \in j_d j_\pi^{-1}(\Lambda \widehat{+} T^*_M X)$. ∎

Lemma 3.3. *Let Λ be a closed conic subset of T^*X and M a closed submanifold of X. One has:*

$$(\Lambda \widehat{+} T^*_M X) \widehat{+} T^*_M X = \Lambda \widehat{+} T^*_M X.$$

Proof. Let (x', x'') be a system of local coordinates on X such that $M = \{(x', x''); x' = 0\}$ and let $(x', x''; \xi', \xi'')$ be the associated coordinates on T^*X.

Let $(x_0;\xi_0) \in (\Lambda\widehat{+}T^*_M X)\widehat{+}T^*_M X$, then there exist sequences $\{(x_n;\xi_n)\}_n$ and $\{(y_n;\eta_n)\}_n$ in $\Lambda\widehat{+}T^*_M X$ and $T^*_M X$, respectively, such that

$$\begin{cases} x_n, y_n \underset{n}{\to} x_0, \\ \xi_n + \eta_n \underset{n}{\to} \xi_0, \\ |x_n - y_n||\xi_n| \underset{n}{\to} 0. \end{cases}$$

For each $n \in \mathbb{N}$, since $(x_n;\xi_n) \in \Lambda\widehat{+}T^*_M X$, there exist sequences $\{(x^n_m;\xi^n_m)\}_m$ in Λ and $\{(y^n_m;\eta^n_m)\}_m$ in $T^*_M X$ such that

$$\begin{cases} x^n_m, y^n_m \underset{m}{\longrightarrow} x_n, \\ \xi^n_m + \eta^n_m \underset{m}{\longrightarrow} \xi_n, \\ |x^n_m - y^n_m||\xi^n_m| \underset{m}{\longrightarrow} 0. \end{cases}$$

Hence we can find subsequences $\{(x_k;\xi_k)\}_k$ and $\{(y_k;\eta_k))\}_k$ of $\{(x^n_m;\xi^n_m)\}_{n,m}$ and $\{(y^n_m;\eta^n_m+\eta_n)\}_{m,n}$, respectively, such that

$$\begin{cases} x_k, y_k \underset{k}{\to} x_0, \\ \xi_k + \eta_k \underset{k}{\to} \xi_0, \\ |x_k - y_k||\xi_k| \underset{k}{\to} 0, \end{cases}$$

which gives $(x_0;\xi_0) \in \Lambda\widehat{+}T^*_M X$.

Conversely, since $\pi(\Lambda\widehat{+}T^*_M X) \subset M$, we get:

$$\Lambda\widehat{+}T^*_M X \subset (\Lambda\widehat{+}T^*_M X) + T^*_M X \subset (\Lambda\widehat{+}T^*_M X)\widehat{+}T^*_M X.$$

$\square$

Let us now assume that X is an open subset of $\mathbb{R}^n$ with the coordinates $(x) = (x_1, ..., x_n)$ and that M is the submanifold $\{(x', x'') \in X; (x') = (x_1, ..., x_d) = 0\}$. Let $\delta > 0$ and let γ be the closed convex proper cone given by:

$$\gamma = \{(x', x''); x_n \leq -\frac{1}{\delta}|(x', x_{d+1}, ..., x_{n-1})|\}.$$

Hence

$$\gamma^{\circ a} = \{(\xi', \xi''); \xi_n \geq \delta|(\xi', \xi_{d+1}, ..., \xi_{n-1})|\}.$$

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Moreover $(x+\gamma)\cap M = x+(\gamma\cap M)$, for each $x\in M$. Let $\mathbb{R}^+$ denote the set of real positive numbers and let us introduce the following notation: for any $\lambda\in\mathbb{R}^+$

$$\gamma_\lambda = \{(x',x')\in X; (\lambda^{-1}x', x'')\in\gamma\}$$
$$V_\lambda = \{(x',x''); (\lambda^{-1}x',x'')\in V\}.$$

Remark that if $\lambda<1$, $\mathrm{Int}(\gamma_\lambda^{\circ a})\supset\gamma^{\circ a}$.

LEMMA 3.4. *Let Λ be a conic closed subset of T^*X.*

Let $x''\in M\cap\pi(\Lambda)$ and assume that there is a compact neighborhood V of x'' such that

$$(V\times\gamma^{\circ a})\cap(\Lambda\widehat{+}T^*_MX)\subset T^*_XX.$$

Then, there exists a real positive number C such that for any λ and ϵ satisfying $0<\lambda,\epsilon<C$,

$$(V_{\lambda\epsilon}\times\gamma_\lambda^{\circ a})\cap\Lambda\subset T^*_XX.$$

Proof. We shall argue by contradiction. Therefore, we can find sequences $(\lambda_l)_{l\in\mathbb{N}}, (\epsilon_l)_{l\in\mathbb{N}}$ of positive numbers converging to 0, $(x'_l,x''_l;\xi'_l,\xi''_l)_{l\in\mathbb{N}}$ in Λ, $(\xi'_l,\xi''_l)\neq(0,0)$, such that $|x'_l|\le C'\epsilon_l\lambda_l$, for some positive constant C' only depending on V, and $(0,x''_l;\lambda_l\xi'_l,\xi''_l)\in V\times\gamma^{\circ a}$.

Since the n-component of $(\xi_l)_{l\in\mathbb{N}}$ is positive, after dividing (ξ'_l,ξ''_l) by $\xi_{l,n}$, we may assume that $\xi_{l,n}=1$ and that $(\lambda_l\xi'_l,\xi''_l)$ is a bounded sequence. Since $|\xi'_l||x'_l|\le C'\epsilon_l\lambda_l|\xi'_l|$ we get that $(|\xi'_l||x'_l|)_l$ converges to 0. Moreover, since x''_l is bounded, we may assume that x''_l converges to some $\check{x}''\in V\cap M$ and that $(\lambda_l\xi'_l,\xi''_l)$ converges to some $(\xi'_0,\xi''_0)\in\gamma^{\circ a}$, with $\xi_{0n}=1$. Considering the sequences $(x'_l,x''_l;\xi'_l,\xi''_l)_{l\in\mathbb{N}}\in\Lambda$ and $(0,x''_l;-\xi'_l+\lambda_l\xi'_l,0)_{l\in\mathbb{N}}\in T^*_MX$ we get that $(0,\check{x}'';\xi'_0,\xi''_0)\in(V\times\gamma^{\circ a})\cap(\Lambda\widehat{+}T^*_MX)$, which entails $\xi_{0n}=0$, a contradiction. ∎

Let Ω be a subset of X. We shall now recall the notion of conormal cone to Ω, $N^*(\Omega)$. It is the subset of T^*X defined as follows:

Given $x\in X$, we denote by $N_x(\Omega)$ the subset of T_xX consisting of vectors $v\neq 0$ such that, in a local chart in a neighborhood of x, there exist an open cone γ containing v and a neighborhood U of x such that

$$U\cap((\Omega\cap U)+\gamma)\subset\Omega.$$

Note that, in particular, $N_x(\Omega)=T_xX$ if and only if $x\notin\overline{\Omega}$ or $x\in\Omega$. We denote by $N(\Omega)$ the open convex cone of TX:

$$N(\Omega)=\bigcup_{x\in X}N_x(\Omega),$$

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and call it the strict normal cone to Ω.

Finally $N^*(\Omega)$, the conormal cone to Ω, is given by

$$N^*(\Omega) = \bigcup_{x \in X} (N_x^*(\Omega)),$$

where, for each $x \in \Omega$, $N_x^*(\Omega) = (N_x(\Omega))^\circ$.

§4. Review on the truncated microsupport

We shall now recall equivalent definitions of the truncated microsupport, following [11].

Given $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n)^*$ and $\varepsilon \in \mathbb{R}$ we set:

$$H_\varepsilon(x_0, \xi_0) = \{x \in \mathbb{R}^n; \langle x - x_0, \xi_0 \rangle > -\varepsilon\},$$

and if there is no risk of confusion we will write H_ε instead of $H_\varepsilon(x_0, \xi_0)$.

PROPOSITION 4.1. *Let X be a real analytic manifold and let $p \in T^*X$. Let $F \in D^b(\mathbf{k}_X)$, $k \in \mathbb{Z}$ and $\alpha \in \mathbb{N} \cup \{\infty, \omega\}$. Then the following conditions are equivalent:*

(i)$_k$ *There exist $F' \in D^{>k}(\mathbf{k}_X)$ and an isomorphism $F \simeq F'$ in $D^b(\mathbf{k}_X; p)$.*

(ii)$_k$ *There exist $F' \in D^{>k}(\mathbf{k}_X)$ and a morphism $F' \to F$ in $D^b(\mathbf{k}_X)$ which is an isomorphism in $D^b(\mathbf{k}_X; p)$.*

(iii)$_{k,\alpha}$ *There exists an open conic neighborhood U of p such that for any $x \in \pi(U)$ and any $\mathbb{R}$-valued C^α-function φ defined on a neighborhood of x such that $\varphi(x) = 0$, $d\varphi(x) \in U$, one has*

$$H^j_{\{\varphi \geq 0\}}(F)_x = 0, \textit{ for any } j \leq k.$$

When X is an open subset of $\mathbb{R}^n$ and $p = (x_0; \xi_0)$, the above conditions are also equivalent to:

(iv)$_k$ *There exist a proper closed convex cone $\gamma \subset \mathbb{R}^n$, $\varepsilon > 0$ and an open neighborhood W of x_0 with $\xi_0 \in \mathrm{Int}(\gamma^\circ)$ such that $(W + \gamma^a) \cap \overline{H_\varepsilon} \subset X$ and*

$$H^j(X; \mathbf{k}_{(x+\gamma^a) \cap H_\varepsilon} \otimes F) = 0, \textit{ for any } j \leq k \textit{ and } x \in W.$$

Remark 4.2. Note that when X is an open subset of $\mathbb{R}^n$ and $p = (x_0; \xi_0)$, the equivalent conditions of Proposition 4.1 are also equivalent to:

There exists some $F' \in D^b(\mathbf{k}_X)$ isomorphic to F in a neighborhood of x_0 and a closed proper convex cone γ in E, with $0 \in \gamma$ and $\xi_0 \in \mathrm{Int}\gamma^{\circ a}$, such that $R\phi_{\gamma *}(F') \in D^{>k}(\mathbf{k}_{X_\gamma})$.

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Definition 4.3. Let $F \in D^b(\mathbf{k}_X)$. We define the closed conic subset $SS_k(F)$ of T^*X by: $p \notin SS_k(F)$ if and only if F satisfies the equivalent conditions in the preceding Proposition.

We shall need the following properties of the truncated microsupport also proved in [11]:

(i) Given a distinguished triangle $F' \to F \to F'' \xrightarrow{+1}$, one has

$$SS_k(F) \subset SS_k(F') \cup SS_k(F''), \tag{1}$$

$$(SS_k(F')\backslash SS_{k-1}(F'')) \cup (SS_k(F'')\backslash SS_{k+1}(F')) \subset SS_k(F). \tag{2}$$

(ii) For any $F \in D^b(\mathbf{k}_X)$, one has

$$SS_k(F) \cap T^*_X X = \pi(SS_k(F)) = \mathrm{supp}(\tau^{\leq k}(F)). \tag{3}$$

Proposition 4.4. *Let X and Y be two manifolds. Then for $F \in D^b(\mathbf{k}_X)$, $G \in D^b(\mathbf{k}_Y)$ and $k \in \mathbb{Z}$, one has:*

$$SS_k(F \boxtimes G) = \bigcup_{i+j=k} SS_i(F) \times SS_j(G).$$

Proposition 4.5. *Let Y and X be two manifolds, let $f\colon Y \to X$ be a morphism and let $G \in D^b(\mathbf{k}_Y)$ such that f is proper on the support of G. Then for any $k \in \mathbb{Z}$,*

$$SS_k(Rf_*(G)) \subset f_\pi f_d^{-1}(SS_k(G)). \tag{4}$$

The equality holds in the case f is a closed embedding.

Proposition 4.6. *Let Y and X be two manifolds and let $f\colon Y \to X$ be a smooth morphism. Let $F \in D^b(\mathbf{k}_X)$. Then*

$$SS_k(f^{-1}F) = f_d f_\pi^{-1}(SS_k(F)). \tag{5}$$

To end this section, we shall prove the following characterizations of the truncated microsupport not included in [11], which will be useful in the sequel.

LEMMA 4.7. *Let E be a real finite-dimensional vector space, X an open subset of E and let $F \in D^b(\mathbf{k}_X)$. Let U be an open subset of X and γ be a closed convex proper cone in E with $0 \in \gamma$. Assume that*

$$SS_k(F) \cap (U \times \mathrm{Int}(\gamma^{\circ a})) = \emptyset.$$

Then, given $(x_0, \xi_0) \in U \times Int(\gamma^{\circ a})$, $\varepsilon > 0$ and an open subset $W \subset X$ such that $(W + \gamma) \cap H_\varepsilon \subset\subset U$, one has:

(6) $$H^j(X; \mathbf{k}_{(x+\gamma)\cap H_\varepsilon} \otimes F) = 0, \text{ for any } x \in W + \gamma \text{ and } j \leq k.$$

Proof. We may assume that X is an open subset of $\mathbb{R}^n$.

Let $(x_0; \xi_0) \in U \times \mathrm{Int}(\gamma^{\circ a})$, $\varepsilon > 0$ and $W \subset X$ be an open subset such that $(W + \gamma) \cap H_\varepsilon \subset\subset U$. Let us prove (6).

By the microlocal cut-off lemma (Proposition 5.2.3 of [13]), we have a distinguished triangle

$$\phi_\gamma^{-1} R\phi_{\gamma *} F \to F \to G \xrightarrow{+1},$$

with $SS(G) \cap (X \times \mathrm{Int}(\gamma^{\circ a})) = \emptyset$. Therefore, setting $F' = \phi_\gamma^{-1} R\phi_{\gamma *} F$, one has

$$H^j(X; \mathbf{k}_{(x+\gamma)\cap H_\varepsilon} \otimes F) \simeq H^j(X; \mathbf{k}_{(x+\gamma)\cap H_\varepsilon} \otimes F'),$$

for any $x \in W + \gamma$ and $j \in \mathbb{Z}$, and $SS_k(F') \cap (U \times \mathrm{Int}(\gamma^{\circ a})) = \emptyset$. Hence we may replace F by F' to prove condition (6).

Arguing by induction on k, we may assume that (6) holds for $k - 1$ and hence $F \in D^{\geq k}(\mathbf{k}_X)$. Hence, given $x \in W + \gamma$,

$$H^k(X; \mathbf{k}_{(x+\gamma)\cap H_\varepsilon} \otimes F) \simeq \Gamma(X; \mathbf{k}_{(x+\gamma)\cap H_\varepsilon} \otimes H^k(F)).$$

Given $s \in \Gamma(X; \mathbf{k}_{(x+\gamma)\cap H_\varepsilon} \otimes H^k(F))$ we can extend s to a section

$$\widetilde{s} \in \Gamma(\Omega; \mathbf{k}_{H_\varepsilon} \otimes H^k(F)) \subset \Gamma(\Omega; H^k(F)),$$

where Ω is a γ-open neighborhood of $x + \gamma$ such that $\Omega \cap H_\varepsilon \subset\subset U$.

Set $S = \mathrm{supp}(\widetilde{s}) \subset \Omega \cap H_\varepsilon$. Since $H^k_{\{\varphi \geq 0\}}(F) \simeq \Gamma_{\{\varphi \geq 0\}}(H^k(F))$, for any real analytic function φ defined on $\mathbb{R}^n$, we get $S = \emptyset$ from the following Lemma, and hence $H^k(X; \mathbf{k}_{(x+\gamma)\cap H_\varepsilon} \otimes F) = 0$. ∎

LEMMA 4.8. ([11]) *Let γ be a proper closed convex cone in $\mathbb{R}^n$. Let Ω be a γ-open subset of $\mathbb{R}^n$ and let S be a closed subset of Ω such that $S \subset\subset \mathbb{R}^n$. Assume the following condition: for any $x \in \mathbb{R}^n$ and any real analytic function φ defined on $\mathbb{R}^n$, the three conditions $S \cap \{x; \varphi(x) < 0\} = \emptyset$, $\varphi(x) = 0$ and $d\varphi(x) \in \mathrm{Int}(\gamma^{\circ a})$ imply $x \notin S$. Then S is an empty set.*

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Corollary 4.9. *Let E be a real finite dimensional vector space, X an open subset of E and let $F \in D^b(\mathbf{k}_X)$. Let U be an open subset of X and γ a closed convex proper cone in E with $0 \in \gamma$. Assume that*

$$SS_k(F) \cap (U \times Int(\gamma^{\circ a})) = \emptyset.$$

Then, for each $x_0 \in U$ there exists an open neighborhood V of x_0 in U such that

$$R\phi_{\gamma *}(R\Gamma_{\Omega_1 \backslash \Omega_0}(F)) \in D^{>k}(\mathbf{k}_{X_\gamma}),$$

for every γ-open subsets Ω_1 and Ω_0 with $\Omega_0 \subset \Omega_1$ and $\Omega_1 \backslash \Omega_0 \subset\subset V$.

Proof. We may assume $X = \mathbb{R}^n$. Let $(x_0; \xi_0) \in (U \times \mathrm{Int}(\gamma^{\circ a}))$. We may find $\varepsilon > 0$ and a γ-open neighborhood Ω of x_0 such that $\Omega \cap H_\varepsilon \subset\subset U$. Then, by Lemma 4.7, one has:

$$H^j(X; \mathbf{k}_{(x+\gamma)\cap H_\varepsilon} \otimes F) = 0, \text{ for all } x \in \Omega \text{ and } j \leq k.$$

It follows that $(\phi_\gamma^{-1} R\phi_{\gamma *} F_{H_\varepsilon})_\Omega \in D^{>k}(\mathbf{k}_X)$.

Set $V = \Omega \cap H_\varepsilon$ and let $\Omega_0 \subset \Omega_1$ be two γ-open sets such that $\Omega_1 \backslash \Omega_0 \subset\subset V$.

One has:

$$\begin{aligned}
&R\phi_{\gamma *}(R\Gamma_{\Omega_1 \backslash \Omega_0}(F)) \simeq R\phi_{\gamma *}(R\Gamma_{\Omega_1 \backslash \Omega_0}(F_{H_\varepsilon})) \simeq R\Gamma_{\Omega_{1\gamma} \backslash \Omega_{0\gamma}} R\phi_{\gamma *}(F_{H_\varepsilon}) \simeq \\
&\simeq R\Gamma_{\Omega_{1\gamma} \backslash \Omega_{0\gamma}} R\phi_{\gamma *} \phi_\gamma^{-1} R\phi_{\gamma *}(F_{H_\varepsilon}) \simeq R\phi_{\gamma *} R\Gamma_{\Omega_1 \backslash \Omega_0}(\phi_\gamma^{-1} R\phi_{\gamma *}(F_{H_\varepsilon})) \simeq \\
&\simeq R\phi_{\gamma *} R\Gamma_{\Omega_1 \backslash \Omega_0}((\phi_\gamma^{-1} R\phi_{\gamma *}(F_{H_\varepsilon}))_\Omega) \in D^{>k}(\mathbf{k}_{X_\gamma}).
\end{aligned}$$

□

§5. Complements on functorial properties of the truncated microsupport

In order to prove the main results we need further functorial properties of the truncated microsupport similar to those of the microsupport itself but requiring adapted proofs.

Lemma 5.1. *Let X be a finite dimensional real vector space, γ a closed convex proper cone of X with $0 \in \gamma$, and Ω a γ^a-open subset of X such that, for any compact K of X, $\Omega \cap (K + \gamma)$ is relatively compact. Let $F \in D^b(\mathbf{k}_X)$ and assume $R\phi_{\gamma_*} F \in D^{>k}(\mathbf{k}_{X_\gamma})$. Then we have*

$$R\phi_{\gamma_*} F_\Omega \in D^{>k}(\mathbf{k}_{X_\gamma}).$$

Proof. The proof is contained in the proof of Lemma 5.4.3 (i) of [13]. □

PROPOSITION 5.2. *Let X be a manifold, $F \in D^b(\mathbf{k}_X)$ and Ω be an open subset of X.*

(i) *Assume $SS_k(F) \cap N^*(\Omega)^a \subset T^*_X X$. Then*

$$SS_k(R\Gamma_\Omega(F)) \subset N^*(\Omega) + SS_k(F).$$

(ii) *Assume $SS_k(F) \cap N^*(\Omega) \subset T^*_X X$. Then*

$$SS_k(F_\Omega) \subset N^*(\Omega)^a + SS_k(F).$$

Proof. The proof is an adaptation of the proof of Proposition 5.4.8 (i) and (ii) of [13], using Corollary 4.9, Remark 4.2 and Lemma 5.1 instead of Propositions 5.2.1, 5.1.1 and Lemma 5.4.3 of [13], respectively. □

PROPOSITION 5.3. *Let Ω be an open subset of X and let j be the embedding $\Omega \hookrightarrow X$. Let $F \in D^b(\mathbf{k}_\Omega)$. Then:*

(i) $SS_k(Rj_*F) \subset SS_k(F) \widehat{+} N^*(\Omega)$.

(ii) $SS_k(Rj_!F) \subset SS_k(F) \widehat{+} N^*(\Omega)^a$.

Proof. The proof is the stepwise adaptation of the proof of Proposition 6.3.1 of [13], using Propositions 5.2, 4.1 and Corollary 4.9 instead of Propositions 5.4.8, 5.1.1 and 5.2.1 of [13], respectively. □

§6. Proofs of the main results

6.1. Proofs of Theorems 1.1, 1.2 and Corollaries

Proof of Theorem 1.2. Let us first consider the case of the embedding of a closed submanifold of X:

PROPOSITION 6.1. *Let M be a closed submanifold of X and $F \in D^b(\mathbf{k}_X)$. Then*

$$SS_k(F|_M) \subset j_d j_\pi^{-1}(SS_k(F) \widehat{+} T^*_M X),$$

where j is the embedding of M in X.

Proof. Let d denote the codimension of M. Let $(x_1, ..., x_n)$ be a system of local coordinates on X such that $M = \{(x_1, ..., x_n); x_1 = ... = x_d = 0\}$ and let $(x;\xi)$ denote the associated coordinates on T^*X. Set $x' = (x_1, ..., x_d)$, $x'' = (x_{d+1}, ..., x_n)$.

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Let $(x_0'';\xi_0'') \in T^*M$ such that $(x_0'';\xi_0'') \notin j_d j_\pi^{-1}(SS_k(F)\widehat{+}T_M^*X)$. We shall prove that $(x_0'';\xi_0'') \notin SS_k(F|_M)$.

By the assumption, $(0,x_0'';\xi',\xi_0'') \notin SS_k(F)\widehat{+}T_M^*X$ for any $\xi' \in \mathbb{R}^d$. In particular, $(0,x_0'';0,\xi_0'') \notin SS_k(F)\widehat{+}T_M^*X$. We may assume that $(0,x_0'') \in \pi(SS_k(F)) \cap M$ and by (3), that $\xi_0'' \neq 0$.

Setting $x_0 = (0,x_0'')$, $\xi_0 = (0,\xi_0'')$ and $p = (x_0;\xi_0)$, there exists a closed convex proper cone γ such that $\mathrm{Int}(\gamma) \neq \emptyset$ and

$$\begin{cases} \xi_0 \in \mathrm{Int}(\gamma^{\circ a}), \\ (\{x_0\} \times \gamma^{\circ a}) \cap (SS_k(F)\widehat{+}T_M^*X) \subset T_X^*X. \end{cases} \tag{7}$$

Therefore we may find a neighborhood V of x_0 such that

$$(V \times \gamma^{\circ a}) \cap (SS_k(F)\widehat{+}T_M^*X) \subset T_X^*X. \tag{8}$$

In particular, $(\{x_0\} \times \gamma^{\circ a}) \cap SS_k(F) \subset \{(x_0;0)\}$. Therefore,

$$(\{x_0\} \times \mathrm{Int}(\gamma^{\circ a})) \cap SS_k(F) = \emptyset,$$

and we may choose V such that

$$(V \times \mathrm{Int}(\gamma^{\circ a})) \cap SS_k(F) = \emptyset. \tag{9}$$

and

$$(V \times \gamma^{\circ a}) \cap T_M^*X \subset T_X^*X. \tag{10}$$

After changing the local coordinates on X if necessary, we may also assume:

$$\begin{cases} \xi_0 = (0,...,0,1), \\ \gamma^{\circ a} = \{(\xi',\xi'');\xi_n \geq \delta|(\xi',\xi_{d+1},...,\xi_{n-1})|\}, \end{cases}$$

for some $\delta > 0$. Hence,

$$\gamma = \{(x',x'');x_n \leq -\frac{1}{\delta}|(x',x_{d+1},...,x_{n-1})|\},$$

and, for any $x \in M$, $(x+\gamma)\cap M = x+(\gamma\cap M)$. For $\varepsilon > 0$ let us denote by H_ε the open half-space $H_\varepsilon = \{x \in X; \langle x - x_0, \xi_0\rangle > -\varepsilon\}$. Let us choose $\varepsilon > 0$ and an open neighborhood $W \subset V$ of x_0 such that $(W+\gamma)\cap H_\varepsilon \subset\subset V$. Set $\gamma' = \gamma \cap M$, $V' = V \cap M$, $W' = W \cap M$ and $H_\varepsilon' = \{x'' \in M; \langle x'' - x_0'', \xi_0''\rangle >$

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$-\varepsilon\}$. Since γ' is a closed convex proper cone in M such that $\xi'_0 \in \mathrm{Int}(\gamma'^{\circ a})$ and W' is an open neighborhood of x'_0 in M, by Proposition 4.1 its enough to prove that there exists $\varepsilon' > 0$ such that $(W' + \gamma') \cap \overline{H'_{\varepsilon'}} \subset M$ and $H^j(M; \mathbf{k}_{(x+\gamma')\cap H'_{\varepsilon'}} \otimes F|_M) = 0$, for all $j \leq k$ and $x \in W'$.

This will be a consequence of Lemma 3.4 with $\Lambda = SS_k(F)$. We shall use the notation $\gamma_\lambda, V_\lambda$ introduced in Section 3. Let C be given by Lemma 3.4 and let us choose sequences $(\lambda_l)_{l\in\mathbb{N}}$, $(\epsilon_l)_{l\in\mathbb{N}}$ of real positive numbers, satisfying $0 < \epsilon_l, \lambda_l < C$, such that $(\lambda_l)_{l\in\mathbb{N}}$ converges to 0 and $(\epsilon_l)_{l\in\mathbb{N}}$ converges to C. Replacing the sequences by convenient ones we may assume from the beginning that $C < 1$.

Remark that $\gamma^{\circ a}_{\lambda_l} \supset \gamma^{\circ a}$ and that

$$V_{\lambda_l\epsilon_l} \cap M = V \cap M = V',$$

$$W_{\lambda_l\epsilon_l} \cap M = W \cap M = W',$$

$$W_{\lambda_l\epsilon_l} + \gamma_{\lambda_l\epsilon_l} \cap H_\varepsilon \subset V_{\lambda_l\epsilon_l},$$

$$(W_{\lambda_l\epsilon_l} + \gamma_{\lambda_l}) \cap H_{\epsilon_l\varepsilon} \subset V_{\lambda_l\epsilon_l}.$$

Let $x'' \in M \cap W$ be given, choose a sequence x''_l in W converging to x'' such that $x'' \in \mathrm{Int}(x''_l + \gamma')$ and note $H' = H \cap M$. Then, for any $j \geq k$, we have

$$H^j(M; \mathbf{k}_{(x''+\gamma')\cap H'_{C\varepsilon}} \otimes F|_M) \simeq \varinjlim_{l} H^j(X; \mathbf{k}_{(x''_l+\gamma_{\lambda_l})\cap H_{\epsilon_l\varepsilon}} \otimes F) = 0,$$

thanks to Lemma 4.7. Hence $(x''_0; \xi''_0) \notin SS_k(F|_M)$. $\square$

End of the proof of Theorem 1.2. Let us decompose f by the graph map

$$Y \underset{g}{\rightarrow} Y \times X \underset{h}{\rightarrow} X,\ f = h \circ g$$

where $g(y) = (y, f(y))$ and h is the second projection on $Y \times X$.

Identifying Y with the graph of f, we may assume that Y is a closed subvariety of $Y \times X$, and we get by Proposition 6.1 and Proposition 4.6,

$$SS_k(f^{-1}F) = SS_k(g^{-1}(h^{-1}F)) \subset$$

$$\subset g_d g_\pi^{-1}(h_d h_\pi^{-1}(SS_k(F)) \widehat{+} T^*_Y(Y \times X)).$$

We shall prove that

$$g_d g_\pi^{-1}(h_d h_\pi^{-1}(SS_k(F)) \widehat{+} T^*_Y(Y \times X)) = f^{\#}(SS_k(F)).$$

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Let (y) be a system of local coordinates on Y, (x) a system of local coordinates on X and let $(y;\xi)$, $(x;\eta)$ be the associated coordinates on T^*Y and T^*X, respectively.

Let $(y_0;\xi_0) \in g_d g_\pi^{-1}(h_d h_\pi^{-1}(SS_k(F))\widehat{+}T^*_Y(Y\times X))$, then there exists ξ,η such that $(y_0, f(y_0);\xi,\eta) \in h_d h_\pi^{-1}(SS_k(F))\widehat{+}T^*_Y(Y\times X)$ and $\xi_0 = \xi + {}^t f'(y_0)\cdot \eta$. Hence we may find sequences $\{(y_n, x_n;\xi_n,\eta_n)\}_n$ in $h_d h_\pi^{-1}(SS_k(F))$ and $\{(y'_n, f(y'_n);\xi'_n,\eta'_n)\}_n$ in $T^*_Y(Y\times X)$ such that

$$\begin{cases} (y_n, x_n), (y'_n, f(y'_n)) \underset{n}{\to} (y_0, f(y_0)), \\ (\xi_n,\eta_n) + (\xi'_n,\eta'_n) \underset{n}{\to} (\xi,\eta), \\ |(y_n,x_n) - (y'_n, f(y'_n))||(\xi_n,\eta_n)| \underset{n}{\to} 0. \end{cases}$$

One has $(x_n;\eta_n) \in SS_k(F)$, $\xi_n = 0$ and $\xi'_n + {}^t f'(y'_n)\cdot\eta'_n = 0$, for all $n\in\mathbb{N}$, and since ${}^t f'(y'_n)\cdot(\eta_n+\eta'_n) \underset{n}{\to} {}^t f'(y_0)\cdot\eta = \xi_0 - \xi$, it follows that ${}^t f'(y'_n)\cdot\eta_n \underset{n}{\to} \xi_0$.

Therefore we have sequences $\{(x_n;\eta_n)\}_n \in SS_k(F)$ and $\{y'_n\}_n$ in Y such that

$$\begin{cases} y_n \underset{n}{\to} y_0, x_n \underset{n}{\to} f(y_0), \\ {}^t f'(y'_n)\cdot\eta_n \underset{n}{\to} \xi_0 \\ |x_n - f(y'_n)||\eta_n| \underset{n}{\to} 0. \end{cases}$$

This gives $(y_0;\xi_0) \in f^{\#}(SS_k(F))$ and also the converse thanks to Proposition 3.1. ∎

Corollary 6.2. *Let M be a closed submanifold of X and $F \in D^b(\mathbf{k}_X)$. Then*

$$SS_k(F_M) \subset SS_k(F)\widehat{+}T^*_M X.$$

Proof. Let $j : M \hookrightarrow X$ denote the embedding of M on X. Then $F_M \simeq j_*(F|_M)$ and by Proposition 4.5 and Proposition 6.1

$$SS_k(F_M) = j_\pi j_d^{-1}(SS_k(F|_M)) \subset$$

$$\subset j_\pi j_d^{-1} j_d j_\pi^{-1}(SS_k(F)\widehat{+}T^*_M X) \subset SS_k(F)\widehat{+}T^*_M X.$$

∎

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PROPOSITION 6.3. *Let M be a closed submanifold of X, $U = X \setminus M$, j the embedding $U \hookrightarrow X$, ι the embedding of M in X and let $F \in D^b(\mathbf{k}_U)$. Then:*
(i) $SS_k(Rj_*F) \cap \pi^{-1}(M) \subset SS_k(F)\widehat{+}T^*_MX$,
(ii) $SS_k(Rj_!F) \cap \pi^{-1}(M) \subset SS_k(F)\widehat{+}T^*_MX$,
(iii) $SS_k((Rj_*F)|_M) \subset \iota_d\iota_\pi^{-1}(SS_k(F)\widehat{+}T^*_MX)$.

Proof. The proof of the two first conditions is analogous to the proof of the two first conditions of Proposition 6.3.2 of [13], replacing Proposition 5.4.4 and Theorem 6.3.1 of [13], by Propositions 4.5 and 5.3, respectively.

Let us now prove the third inequality. By Proposition 6.1 and (i),

$$\begin{aligned} SS_k((Rj_*F)|_M) &\subset \iota_d\iota_\pi^{-1}(SS_k(Rj_*F)\widehat{+}T^*_MX) \subset \\ &\subset \iota_d\iota_\pi^{-1}((SS_k(F)\widehat{+}T^*_MX)\widehat{+}T^*_MX). \end{aligned}$$

By Lemma 3.3

$$(SS_k(F)\widehat{+}T^*_MX)\widehat{+}T^*_MX = SS_k(F)\widehat{+}T^*_MX.$$

Hence

$$SS_k((Rj_*F)|_M) \subset \iota_d\iota_\pi^{-1}(SS_k(F)\widehat{+}T^*_MX).$$

∎

Note that, with Lemma 3.2 and Proposition 6.3 in hand, we obtain the analogue of Proposition 6.3.2 of [13].

COROLLARY 6.4. *Let M be a closed submanifold of X and $F \in D^b(\mathbf{k}_X)$. Then*

$$SS_k(R\Gamma_M(F)) \subset SS_k(F)\widehat{+}T^*_MX.$$

Proof. This is a consequence of Proposition 6.3, together with the distinguished triangle

$$R\Gamma_M(F) \to F \to R\Gamma_{X\setminus M}(F) \xrightarrow{+1} .$$

∎

COROLLARY 6.5. *Let M be a closed submanifold of X and $F \in D^b(\mathbf{k}_X)$. Assume that*

$$SS_k(F) \cap T^*_MX \subset T^*_XX.$$

Then

$$SS_k(F_M \otimes \omega_{M|X}) = SS_k(R\Gamma_M(F)).$$

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Proof. Let $\dot{\pi}$ be the restriction of π to the cotangent bundle deprived of the zero section. We have a distinguished triangle

$$F_M \otimes \omega_{M|X} \to R\Gamma_M(F) \to R\dot{\pi}(\mu_M(F)|_{\dot{T}^*_M X}) \underset{+1}{\to} .$$

Since by Theorem 5.1 of [11] and the assumption we have

$$\mathrm{supp}(\tau^{\leq k}(\mu_M(F))) \subset T^*_X X,$$

we get that $\mu_M(F)|_{\dot{T}^*_M X} \in D^{>k}(\mathbf{k}_{\dot{T}^*_M X})$. Hence the third term of the distinguished triangle above is an object of $D^{>k}(\mathbf{k}_X)$, which entails that

$$SS_k(R\dot{\pi}(\mu_M(F)|_{\dot{T}^*_M X})) = \emptyset.$$

∎

Corollary 6.6. *Let M be a closed submanifold of X and $F \in D^b(\mathbf{k}_X)$. Let j denote the embedding of M in X. Then*

$$SS_k(j^! F) \subset j_d j_\pi^{-1}(SS_k(F) \widehat{+} T^*_M X).$$

Proof. This is a consequence of Proposition 6.1 and Corollary 6.4 together with Lemma 3.3. ∎

Proof of Theorem 1.1. The proof is the adaptation step by step of the proof of Theorem 6.4.1 of [13], applying Proposition 4.6, and Proposition 6.3 instead of Proposition 5.4.5 and Proposition 6.3.2, respectively of [13]. ∎

Let now Y be a complex closed smooth hypersurface of X defined as the zero locus of a holomorphic function f. Let ψ_Y denote the functor of nearby cycles associated to Y. Then Y may be regarded as a submanifold of $T_Y X$ by a canonical section s such that $\psi_Y(F) \simeq s^{-1}\nu_Y(F)$. Once more we identify $T_{T^*_Y X} T^*X$, $T^*(T^*_Y X)$ and $T^*(T_Y X)$ (cf. Proposition 5.5.1 of [13]).

Recall that, in a system of linear coordinates $x = (x_1, ..., x_n)$ on X such that Y is defined by $x_1 = 0$, $s\colon Y \to T_Y X$ is the section $s(x_2, ..., x_n) = (x_2, ..., x_n; 1)$. With the local coordinates described above, and A being a conic closed subset of $T^*(T^*_Y X)$, we have:

$$s_d s_\pi^{-1}(A) = \{(x_2, ..., x_n; \xi_2, ..., \xi_n); \exists \xi_1, (x_2, ..., x_n, 1; \xi_2, ..., \xi_n, \xi_1) \in A\}.$$

Corollary 1.4 is an immediate consequence of Proposition 6.1.

The following estimate for the tensor product can be seen as a generalization of Proposition 6.1:

PROPOSITION 6.7. *Let F and G belong to $D^b(\mathbf{k}_X)$. Then:*

$$SS_k(F \otimes^{\mathbb{L}} G) \subset \bigcup_{i+j=k} (SS_i(F) \widehat{+} SS_j(G)).$$

Proof. Let $\delta_X : X \to X \times X$ be the diagonal embedding.

Since $F \otimes^{\mathbb{L}} G \simeq \delta_X^{-1}(F \boxtimes^{\mathbb{L}} G)$, the result follows from Proposition 6.1 and Proposition 4.4. ∎

6.2. Application to $\mathcal{D}$-modules

Let X be a complex finite dimensional manifold. One of the important problems in the theory of $\mathcal{D}$-modules is the relation between the characteristic variety of a system $\mathcal{M}$ and that of its induced system $\mathcal{M}_Y$ along a closed submanifold Y, which was completely solved in the non characteristic case by M. Kashiwara as well as in a more general situation treated in [15], which includes the case where $\mathcal{M}$ is regular along Y in the sense of [10]. Similarly, in the case of a smooth complex hypersurface, it is interesting to relate $\mathrm{Char}(\mathcal{M})$ and $\mathrm{Char}(\psi_Y(\mathcal{M}))$, where ψ_Y denotes the functor of nearby cycles.

Let d be the codimension of Y, denote by j the embedding $Y \to X$ and by π' the projection $T^*Y \to Y$. Given an homogeneous involutive subvariety V of T^*X of codimension $\geq d$, we shall say that Y is orthogonal to V if there exists a smooth involutive submanifold V^* containing V such that Y and V^* are orthogonal. More precisely, there exist a set $\{f_1, ..., f_d\}$ of homogeneous functions of degree zero vanishing on $\pi^{-1}(Y)$, such that the differential df_i are linearly independent on $\pi^{-1}(Y)$, and a set $\{g_1, ..., g_p\}, p \geq d$, of homogeneous functions of degree one linearly independent on V^* such that the matrix of the Poisson brackets $[\{f_i, g_j\}]|_{V^*}$ has everywhere rank d.

As before, F will denote the complex $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Let $SS(F) = \bigcup_\alpha V_\alpha$ be the decomposition of $SS(F)$ in its irreducible involutive components in a neighborhood of $p \in T^*X$. Let us denote by Y_α the variety $\pi(V_\alpha)$.

Recall that in Theorem 6.7 of [11] it is proved that, for any k, $SS_k(F) = (\bigcup_{codim Y_\alpha < k} V_\alpha) \cup (\bigcup_{codim Y_\alpha = k} T^*_{Y_\alpha} X)$.

Proof of Theorem 1.4. The first assertion is an immediate consequence of Theorem 1.1 and the second follows from the regularity of $\mathcal{M}$. ∎

Proof of Corollary 1.5. It is a consequence of Corollary 1.3 and the regularity of $\mathcal{M}$. ∎

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Proof of Theorem 1.6. Since $\mathcal{M}$ is regular along Y, one has the isomorphism

$$R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y,$$

and the first part is an immediate consequence of Proposition 6.1. Let us now prove the second assertion. It will be a consequence of the Lemma below:

LEMMA 6.8. *Assume that the homogeneous involutive variety V is irreducible and that Y is orthogonal to V.*

Then :

(i) $V' = j_d(j_\pi^{-1}(V))$ *is an irreducible homogeneous involutive subvariety of T^*Y.*

(ii) *The codimension of $\pi'(V')$ is equal to the codimension of $\pi(V)$.*

(iii) *When V is the characteristic variety of a coherent $\mathcal{D}_X$-module, the orthogonality of Y implies that Y is non characteristic for $\mathcal{M}$.*

Proof. Let V^* be a smooth involutive manifold containing V such that Y is orthogonal to V^*. Since the assertions can be checked locally, by a standard reasoning we may consider a system $(x;\xi)$ of local symplectic coordinates on T^*X in a neighborhood of $p \in V \cap \pi^{-1}(Y) = j_\pi^{-1}(V)$, such that Y is the submanifold $\{(x) = (x_1, ..., x_n); x_1 = ... = x_d = 0\}$ and V^* is defined in T^*X by the equations $\xi_1 = ... = \xi_d = g_{d+1}(x''; \xi'') = ... = g_p(x''; \xi'') = 0$, where we set $(x') = (x_1, ..., x_d)$ (resp. $(\xi') = (\xi_1, ..., \xi_d)$), $(x'') = (x_{d+1}, ..., x_n)$ (resp. $(\xi'') = (\xi_{d+1}, ..., \xi_n)$). Therefore, the irreducible ideal of definition $I(V)$ is generated by a set of functions

$$\{\xi_1, ..., \xi_d, g_{d+1}(x''; \xi''), ..., g_p(x''; \xi''), h_{p+1}(x''; \xi''), ..., h_{p+l}(x''; \xi'')\},$$

for some $l \geq 0$. Hence $I(V')$ is generated in $\mathcal{O}_{T^*Y}$ by the set of functions

$$\{g_{d+1}(x''; \xi''), ..., g_p(x''; \xi''), h_{p+1}(x''; \xi''), ..., h_{p+l}(x''; \xi'')\},$$

which entails (i), (ii) and (iii). $\square$

Since Y is non characteristic, we have

$$SS(F|_Y) = \mathrm{Char}(\mathcal{M}_Y) = j_d j_\pi^{-1}(SS(F)).$$

On the other side, since $SS_k(F) \cap T_Y^*X \subset T_X^*X$, we get from the first assertion that $SS_k(F|_Y) \subset j_d j_\pi^{-1}(SS_k(F))$, for any k. Moreover, setting

$V'_\alpha = j_d j_\pi^{-1}(V_\alpha)$, by the preceding Lemma, for any α such that $codim Y_\alpha \leq k$, V'_α is an irreducible component of $SS(F|_Y)$. Therefore by Theorem 6.7 of [11], for any $i \leq k$,

$$SS_i(F|_Y) \supset j_d j_\pi^{-1}(SS_i(F)).$$

□

EXAMPLE 6.9. Let $X = \mathbb{C}^n$, with $n \geq 3$, endowed with the coordinates $(x_1, ..., x_n)$. Let Y be the hypersurface $\{x_n = 0\}$ and $\Omega = \{x \in X; \mathrm{Re}(x_1 - x_{n-1}) < 0\}$. Let $\mathcal{J}$ be a coherent left ideal of $\mathcal{D}_X$ and set $\mathcal{M} = \mathcal{D}_X/\mathcal{J}$. Assume that there exist in $\mathcal{J}$ an operator P in the Weierstrass form with respect to the derivation D_{x_n} and an operator Q such that the principal symbol of Q, $\sigma(Q)$, is of the form

$$\sigma(Q) = x_1 q(x_1, ..., x_{n-1}; \xi_1, ..., \xi_{n-1}),$$

and q does not vanish on $T^*_{\delta\Omega}X$. Then,$T^*_{\delta\Omega}X \cap SS_1(\mathcal{M}) \subset \{0\}$ and, setting $\Omega' = \Omega \cap Y$, Ω' has smooth boundary $\delta\Omega'$. By Theorem 1.3, $T^*_{\delta\Omega'}Y \cap SS_1(\mathcal{M}_Y) \subset \{0\}$. Therefore

$$\mathcal{H}\mathrm{om}_{\mathcal{D}_Y}(\mathcal{M}_Y, H^1_{\{\mathrm{Re}(x_1 - x_{n-1}) \geq 0\}}(\mathcal{O}_Y))|_{\delta\Omega'} = 0.$$

Proof of Corollary 1.5. As proved in [8], we have the isomorphism

$$R\mathcal{H}om_{\mathcal{D}_Y}(\psi_Y(\mathcal{M}), \mathcal{O}_Y) \simeq \psi_Y(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)).$$

It is then enough to use Proposition 6.1. □

Let M be a real analytic manifold of dimension n, X a complex analytic manifold complexifying M and $\mathcal{M}$ a coherent $\mathcal{D}_X$-module.

Let $\mathcal{A}_M$ denote the sheaf of real analytic functions on M. Remark that $\mathcal{A}_M = \mathcal{O}_X|_M$. Let $\mathcal{B}_M$ denote the sheaf of Sato's hyperfunctions on M. Recall that

$$\mathcal{B}_M \simeq R\Gamma_M(\mathcal{O}_X) \otimes or_{M/X}[n].$$

Proof of Proposition 1.7. One has

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M) \simeq F|_M.$$

Therefore, by Proposition 6.1

$$SS_k(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)) \subset j_d j_\pi^{-1}(SS_k(F) \widehat{+} T^*_M X)$$

□

Let us remark that a variant of the preceding result was obtained in [19] for $k = 1$ using directly the properties of holomorphic functions.

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Proof of Corollary 1.8. The first part is an immediate consequence of Corollary 6.5. The second follows from Proposition 1.7 and Theorem 6.7 of [11]. ∎

Example 6.10. Let $M = \mathbb{R}^n$, with $n \geq 2$, endowed with the coordinates $x = (x_1, ..., x_n)$. Let $\Omega = \{x \in M; \phi(x) < 0\}$ for some real C^1-function. Let $X = \mathbb{C}^n$ and $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module defined by $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$ where P is a differential operator. Assume that the principal symbol $\sigma(P)$ is of the form

$$\sigma(P) = a(x)q(x;\xi),$$

where $a(x)$ is a holomorphic function and q does not vanish on $T^*_M X$, more precisely, q is the principal symbol of an elliptic operator. Recall that $SS_1(F) \subset \overline{\{(x;\xi); a(x) = 0, \xi \in \mathbb{C}da(x)\}} \cup q^{-1}(0)$. This entails that $T^*_M X \cap SS_1(F) \subset T^*_X X$ and hence

$$SS_1(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)) =$$

$$= SS_1(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)) \subset j_d j_\pi^{-1}(SS_1(F)).$$

Assume that $d\phi(x)$ is not in $\overline{\mathbb{C}da(x)}$ for any $x \in \delta\Omega \cap a^{-1}(0)$. Then $T^*_{\delta\Omega} M \cap SS_1(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)) \subset T^*_M M$. In other words

$$\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, H^1_{\{\phi(x)\geq 0\}}(\mathcal{A}_M))|_{\delta\Omega}$$

$$= \mathcal{E}\mathrm{xt}^1_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\{\phi(x)\geq 0\}}(\mathcal{B}_M))|_{\delta\Omega} = 0.$$

Remark 6.11. In general we do not have an interesting estimate for $SS_k(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M))$. Let j denote the inclusion $M \hookrightarrow X$. Then $\mathcal{B}_M \simeq j^! \mathcal{O}_X \otimes or_{M/X}[n]$ and $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \simeq j^!(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X))[n]$. By Corollary 6.6, one gets

$$SS_k(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)) = SS_{k+n}(j^! R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \subset$$

$$\subset SS_{k+n}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \widehat{+} T^*_M X.$$

By Theorem 6.7 of [11],

$$SS_{k+n}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) = SS(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) = \mathrm{Char}(\mathcal{M}).$$

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Hence we get

$$SS_k(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M)) \subset \mathrm{Char}(\mathcal{M})\widehat{+}T^*_M X \text{ for any k} \geq 0,$$

in other words, if M is hyperbolic for $\mathcal{M}$ then

$$SS_k(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M)|_M) \subset T^*_M M.$$

But this is well known and is an example that the notion of truncated microsupport does not work well under Fourier Transform.

Let $D^b_{\mathbb{C}-c}(\mathbf{k}_X)$ denote the full subcategory of $D^b(\mathbf{k}_X)$ consisting of objects with $\mathbb{C}$-constructible cohomology, that is, the objects $F \in D^b(\mathbf{k}_X)$ for which there exists a complex analytic stratification $X = \bigcup X_\alpha$ such that the sheaf $H^j(F)|_{X_\alpha}$ is locally constant of finite rank, for every $j \in \mathbb{Z}$ and α.

A perverse sheaf is an object F of $D^b_{\mathbb{C}-c}(\mathbf{k}_X)$ satisfying the following two conditions:

(a) for any complex submanifold Y of X of codimension d, $H^j_Y(F)|_Y$ is zero for $j < d$;

(b) for any $j \in \mathbb{Z}$, $H^j(F)$ is supported by a complex analytic subset of codimension $\geq j$.

P. Schapira proved in [18] that, when F is a perverse object of $D^b_{\mathbb{C}-c}(\mathbf{k}_X)$,

$$H^j(R\Gamma_S(F))_x = 0, \text{ for } j \geq 2n,$$

for any closed subanalytic subset S of X and any $x \in X$ being non isolated in S.

PROPOSITION 6.12. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module Then*

$$SS_{n-1}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M)) = SS(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M)).$$

Proof. Let φ be a real analytic function defined on X and $x_0 \in X$ such that $\varphi(x_0) = 0$. Then the set $\{x \in X; \varphi(x) \geq 0\}$ is a closed subanalytic subset of X. Assume that $\mathcal{M}$ is holonomic. By the Riemann-Hilbert correspondence ([7]), F is perverse.

Hence,

$$H^j(R\Gamma_{\{\varphi\geq 0\}}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M))_{x_0} \simeq H^{j+n}(R\Gamma_{\{\varphi\geq 0\}\cap M}(F))_{x_0} = 0,$$

for every $j \geq n$. By Proposition 4.1,

$$SS_{n-1}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M)) = SS(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M)),$$

under the assumption that $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module.

To treat the general case, we argue as in the proof of Theorem 2 of [18]. Let us denote by * the functor $\mathcal{N} \mapsto \mathcal{N}^* = \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{N}, \mathcal{D}_X)$. Recall that Kashiwara proved in [6] that if $\mathcal{M}$ is coherent, then $\mathcal{M}^*$ is holonomic, $\mathcal{M}^{***} \simeq \mathcal{M}^*$ and $\mathcal{M}^{**}$ is a submodule of $\mathcal{M}$. Defining the coherent $\mathcal{D}_X$-module $\mathcal{L}$ by the exact sequence:

$$0 \to \mathcal{M}^{**} \to \mathcal{M} \to \mathcal{L} \to 0,$$

one gets $\mathcal{L}^* = 0$ and so $\mathcal{L}$ locally admits a projective resolution of length $n-1$. Therefore,

$$H^j(R\Gamma_{\{\varphi \geq 0\}} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M))_{x_0} \simeq H^j(R\Gamma_{\{\varphi \geq 0\}} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^{**}, \mathcal{B}_M))_{x_0} = 0,$$

for $j \geq n$.

This proves

$$SS_{n-1}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)) = SS(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)),$$

for every coherent $\mathcal{D}_X$-module $\mathcal{M}$. ∎

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