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ONE SIDED SF-RINGS WITH CERTAIN CHAIN CONDITIONS

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ABSTRACT. We prove that with some weak chain conditions, left SF-rings are semi-simple Artinian or regular. We also prove that MERT left SF-rings are really regular.

A Ring R is called a *left* (*right*) SF-*ring* if all simple left (right) R-modules are flat. This paper investigates left SF-rings with certain chain conditions. It shows that with some weak chain conditions, left SF-rings are semisimple Artinian rings or regular rings. The rest of this paper settles some open questions. Yue Chi Ming asked if MERT right SF-rings are regular. J. Zhang and X. Du [12] answered it recently in the positive. The next question is if MELT right SF-rings are regular. J. Zhang and X. Du [12] assert that this is still open. Some recent papers show that these rings are regular if some weak conditions are added (see [9] and [12]). Here we point out that these conditions are unnecessary because MELT right SF-rings are really regular. Finally, we give an example of a left hereditary non-semisimple ring which contains an injective maximal left ideal. This settles a question proposed by Yue Chi Ming [9].

All rings throughout this paper are associative and have identities. A ring R satisfies PDCC$^{\perp}$ (the descending chain condition on the principal right annihilators) if there does not exist a properly descending infinite chain: $r(x_1) > r(x_2) > \cdots > r(x_n) > \cdots$, for any sequence $\{x_n\}_1^\infty \subset R$. Similarly we may define PACC$^{\perp}$, $^{\perp}$PDCC and $^{\perp}$PACC. A ring R satisfies left PACC (the ascending chain condition on the principal left ideals) if there does not exist a properly ascending infinite chain: $Rx_1 < Rx_2 < \cdots < Rx_n < \cdots$, for any sequence $\{x_n\}_1^\infty \subset R$. Similarly we may define right PACC and left (right) PDCC. Clearly the rings satisfying left (right) PDCC are just right (left) perfect rings. When R_R is p-injective (*i.e.* any R-homomorphism from a principal right ideal of R to R_R can be extended to an R-homomorphism from R_R to R_R). It is easy to show that R satisfies PACC$^{\perp}$ (resp. PDCC$^{\perp}$) if and only if R satisfies left PDCC (resp. right PACC). Therefore, speaking roughly, we say that PACC$^{\perp}$ is the dual of left PDCC *etc.* A ring R is called *left* (*right*) *quasi-duo* if all maximal left (right) ideals of R are two-sided. R is called an MELT (resp. MERT) ring if all essential maximal left (resp. right) ideals are two-sided. R is called (Von Neumann) *regular* if for every $x \in R$, there exists a $y \in R$, such that $x = xyx$. $J(R)$, $Z(R_R)$ and $\mathrm{Soc}(R_R)$ denote, respectively, the Jacobson radical, the right singular ideal and the right socle of R. For any subset X of R, we define $r(X) = \{r \in R \mid Xr = 0\}$.

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1. Left SF-rings with certain chain conditions.

LEMMA 1.1. *Let R be a ring and I is a left ideal of R; then the following are equivalent:*

(1) ${}_R(R/I)$ *is flat.*

(2) For every $x \in I$, $x \in xI$.

PROOF. See [1, 19.18]. ∎

REMARK. This lemma implies that all quotient rings of left SF-rings are left SF-rings.

LEMMA 1.2. *Let R be a left* SF*-ring; then*

(1) For every $x \in R$, $Rr(x) + Rx = R$.

(2) $Z(R_R) \subseteq J(R)$.

PROOF. (1) If $Rr(x)+Rx \neq R$, then there must exist a maximal left ideal M of R such that $Rr(x) + Rx \leq M < {}_RR$. This would yield $1 \in M$ from Lemma 1.1.

(2) For every $x \in Z(R_R)$, $r(1-x) = 0$. Thus from (1) we have $R(1-x) = R$, which implies that $Z(R_R)$ is a left quasi-regular ideal of R. Therefore $Z(R_R) \subseteq J(R)$. ∎

THEOREM 1.3. *For a left* SF*-ring R, the following are equivalent:*

(1) R is semisimple Artinian.

(2) R is left or right Noetherian.

(3) $R/J(R)$ is semisimple Artinian.

(4) R is semiprimitive and R_R has finite rank.

(5) R satisfies ${}^{\perp}$PACC.

(6) R satisfies PDCC$^{\perp}$.

(7) R satisfies left PACC.

PROOF. (2) ⇒ (3). When R is left Noetherian, from the well-known fact that finitely related flat modules are projective, we see that all simple left R-modules are projective. Therefore R must be semisimple Artinian. Now assume that R is right Noetherian; then the semiprime ring $R/J(R)$ is also a left SF and right Noetherian ring. Let Q denote the semisimple Artinian quotient ring. Take a $b^{-1} \in Q$ where $b \in R/J(R)$; then b is regular, so from Lemma 1.2(1) we see $b^{-1} \in R/J(R)$. Therefore $R/J(R)$ coincides with its semisimple Artinian quotient ring.

(3) ⇒ (1). If $R/J(R)$ is semisimple Artinian, then ${}_R\big(R/J(R)\big)$ is also semisimple. This implies that ${}_R\big(R/J(R)\big)$ is flat. Take an $x \in J(R)$; from Lemma 1.1 there is a $y \in J(R)$ such that $x = xy$, and so $x(1-y) = 0$. This implies $x = 0$ because $1-y$ is invertible. Therefore, $J(R) = 0$.

(4) ⇒ (1). In this case $Z(R_R) = 0$ and so the maximal right quotient ring Q of R exists and has also a finite rank. Therefore Q must be semisimple Artinian because it is regular. This implies that R has ACC$^{\perp}$ and so R is a semiprime Goldie ring. Thus by the same argument as (2) implies (1) we see that R must be semisimple Artinian.

(5) $\Rightarrow$ (1). Let M be a maximal left ideal of R. From Lemma 1.1 we have $x \in xM$ for every $x \in M$. Now take an $e \in M$ such that $l(1-e)$ is maximal among all $l(1-x)$ where $x \in M$.

CLAIM. *$l(1-e) = M$, i.e., $M = Re$, and $e^2 = e$.*

If there is a $y \in M$ such that $y(1-e) \neq 0$: Noting $y(1-e) \in M$, there exists an $e' \in M$, such that $y(1-e) = y(1-e)e'$, so $y = y(e+e'-ee')$. Denote $f = e+e'-ee' \in M$. Since $(1-f) = (1-e)(1-e')$, $y(1-f) = 0$ and $y(1-e) \neq 0$, we get $l(1-e) \subset l(1-f)$, a contradiction. Therefore, the claim is true.

Now from the above claim, ${}_R(R/M)$ is projective for every maximal left ideal M of R, so R must be semisimple Artinian.

NOTE. This proof is essentially the same as [8, p. 237].

(6) $\Rightarrow$ (1). Take a maximal left ideal M of R.

CLAIM 1. *For every $x \in M$, there exists an idempotent $e_x \in M$, such that $x = xe_x$.*

From Lemma 1.1 we have a sequence $\{x_n\}_1^\infty \subset M$ such that

$$x = xx_1,\ x_1 = x_1x_2,\ \ldots,\ x_k = x_kx_{k+1},\ \ldots.$$

This yields

$$r(x) \geq r(x_1) \geq r(x_2) \geq \cdots \geq r(x_k) \geq \cdots.$$

Therefore, there exists a positive integer n, such that $r(x_n) = r(x_{n+1})$. Denote $e_x = x_{n+1}$, then e_x is an idempotent in M and

$$x = xx_1 = xx_2x_3 = \cdots = xx_1\cdots x_n = xx_1\cdots x_ne_x = xe_x.$$

This completes the proof of Claim 1.

Now take an $e \in M$ such that $r(e)$ is minimal among all $r(x)$ where $x \in M$. From Claim 1 we may assume, without loss of generality, that e is an idempotent.

CLAIM 2. *$M = Re$.*

If $M \neq Re$, then there exists an $f \in M$, such that $f(1-e) \neq 0$. Again from Claim 1 we may assume, without loss of generality, that f is an idempotent and $r(f)$ is minimal like $r(e)$. Noting $f(1-e) \in M$, there is an $e' \in M$ such that $f(1-e) = f(1-e)e'$, i.e. $f = f(e+e'-ee')$. Since $e+e'-ee' \in M$, from the above assumption we have $r(f) = r(e+e'-ee')$. But clearly $r(e+e'-ee') \subseteq r(e)$, so we get $r(f) \subseteq r(e)$ and $r(f) = r(e)$ which implies $R(f) = Re$, a contradiction. Therefore, $M = Re$.

From these two claims, we see that R must be semisimple Artinian.

(7) $\Rightarrow$ (1). Again, we show that every maximal left ideal of R is generated by an idempotent. Let M be a maximal left ideal and $x \in M$. From Lemma 1.1 there is a sequence $\{x_n\}_1^\infty \subset M$ such that

$$x = xx_1,\ x_1 = x_1x_2,\ \ldots,\ x_k = x_kx_{k+1},\ \ldots.$$

Since the chain

$$Rx \subseteq Rx_1 \subseteq Rx_2 \subseteq \cdots \subseteq Rx_k \subseteq \cdots$$

stops for some n, we have $x_{n+1} = yx_n$ for some $y \in R$. Thus $x_n = x_n y x_n$ and it is easy to verify that $x = xe_x$, where $e_x = yx_n \in M$ and e_x is an idempotent. This shows that Claim 1 is also true in this case. Now take an $e \in M$ such that Re is maximal among all Rx where $x \in M$. From the above discussion, we can choose e to be an idempotent. If there is an $f \in M$ such that $f \neq fe$, then from the above discussion we may assume, without loss of generality, that f is an idempotent and Rf is maximal like Re. Thus by exactly dualizing the proof of Claim 2 we have $Re = Rf$, a contradiction. This shows $Re = M$. Therefore R is semisimple Artinian. This completes the proof of the theorem. ■

Goursand and Valette [7] show that for a regular ring R, all primitive factor rings of R are Artinian if and only if all homogeneous semisimple right R-modules are injective. We generalize this to the left SF-rings.

PROPOSITION 1.4. *For a left* SF*-ring R, the following are equivalent:*

(1) All right primitive factor rings of R are Artinian.

(2) All homogeneous semisimple right R-modules are injective.

If either of these two conditions holds, then R is a regular and V-ring.

PROOF. (1) $\Rightarrow$ (2). Take a maximal right ideal M of R, and let $P = r\big((R/M)_R\big)$. From the given condition and that R/P is also a left SF-ring, R/P must be semisimple Artinian by Theorem 1.3. This implies that ${}_R(R/P)$ is also semisimple, so ${}_R(R/P)$ is flat and so $(R/M)_R$ is injective because $(R/M)_{R/P}$ is (see [5, 6.17]). Thus R is a right V-ring, and so from a result of G. Baccella [2] R is regular. Therefore (2) is true from [5, 6.18].

(2) $\Rightarrow$ (1). Take a right primitive ideal P of R.

CLAIM. *$(R/P)_{R/P}$ has finite rank.*

Assume that there exists a sequence $\{x_n\}_1^\infty \subset R/P$ such that each $x_n \neq 0$ in R/P and $\oplus_{n=1}^\infty x_n R \subset R/P$. Take a faithful simple right R/P-module A; then for every x_n there exists an $a_n \in A$, such that $a_n x_n \neq 0$. Now let $B = \oplus_{n=1}^\infty A_n$ where each $A_n = A$; then B_R is injective. The rest of the proof is the same as [5, 6.18] which yields a contradiction. Therefore R/P must have a finite rank.

From Theorem 1.3 we see that R/P must be (semisimple) Artinian. ■

2. **Answering some questions.**

LEMMA 2.1. *If R is a left* SF*-ring and* $\big(J(R)\big)^2 = 0$, *then* $J(R) = Z(R_R)$.

PROOF. If there exists an x in $J(R)$ which is not in $Z(R_R)$, then there is a $0 \neq y \in R$ such that

$$r(x) \cap yR = 0.$$

Since $\big(J(R)\big)^2 = 0$, y is not contained in $J(R)$. Therefore there is a maximal left ideal M of R such that y is not contained in M. But $xy \in M$ implies there is an m such that

$xy = xym$ so that $xy(1 - m) = 0$. This means $y(1 - m) = 0$ and so $y = ym \in M$, a contradiction.

Therefore, together with Lemma 1.2, $J(R) = Z(R_R)$. ■

With this lemma, we answer the following open question (see [9, Proposition 2] and [12, Theorem 4 and 5]):

PROPOSITION 2.2. *An* MERT *left* SF-*ring is regular.*

PROOF. Let R be such a ring; then $R/\operatorname{Soc}(R_R)$ is a right quasi-duo and left SF-ring. Therefore $R/\operatorname{Soc}(R_R)$ is a strongly regular ring from [11, 4.10]. Thus $J(R) \subseteq \operatorname{Soc}(R_R)$, which implies $\big(J(R)\big)^2 = 0$.

Assume $Z(R_R) \neq 0$. Take a nonzero $x \in Z(R_R)$ and a maximal right ideal M of R which contains $r(x)$. Since $x \in r(x)$ and M is also a maximal left ideal of R, there is an $m \in M$ such that $x = xm$ so that $x(1 - m) = 0$ and $1 - m \in r(x) \subseteq M$ so $1 \in M$, which is impossible. So $Z(R_R) = 0$, and from Lemma 2.1 $J(R) = 0$. Therefore, from the well-known fact, which says that $\operatorname{Soc}(R_R)$ is a regular ideal of R for a semiprime ring R, R must be regular. ■

PROPOSITION 2.3. *If R is a left* SF-*ring and $R/\operatorname{Soc}(R_R)$ satisfies one of seven conditions listed in Theorem 1.3, then R is regular.*

PROOF. Let $S = \operatorname{Soc}(R_R)$. Then R/S is semisimple Artinian which implies ${}_R(R/S)$ is flat and $\big(J(R)\big)^2 = 0$.

Assume that there is an $x \in Z(R_R)$ which is not zero. Since $(R/S)_R$ is Artinian, $S \leq_e R_R$, and so there is an $r \in R$ such that $0 \neq xr \in S$. Thus from Lemma 1.1 there is an $s \in S$ such that $xr = (xr)s = x(rs) = 0$, because $rs \in S$. This contradiction shows that $Z(R_R) = 0$. Therefore R must be semiprime and so R must be regular. ■

A ring R is called a *right* SI-*ring* if all singular right R-modules are injective.

COROLLARY 2.4. *A left* SF *right* SI-*ring R must be regular.*

PROOF. $R/\operatorname{Soc}(R_R)$ is right Noetherian from [6,3.6]. ■

EXAMPLE 2.5. Let R be upper triangular matrix ring over a division ring D. R is Artinian and hereditary (see [4, 4.8]). Denote $e = e_{2,2}, f = e_{1,1}$. Now we can easily verify that both e and f are primitive idempotents and

$$\operatorname{Soc}({}_R Re) \cong {}_R(Rf/Jf), \quad \operatorname{Soc}(f R_R) \cong (eR/eJ)_R.$$

Therefore Re is an injective left ideal of R from Fuller-theorem [3, 31.3]. Clearly Re is also a maximal left ideal of R. This answers a question of Yue Chi Ming [9].

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