ON THE IRREDUCIBLE LATTICES OF ORDERS

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1. Introduction. We shall use the following notation:

R = Dedekind domain;

K = quotient field of R;

R_p = ring of p-adic integers in K, p being a prime ideal in R;

A = finite-dimensional separable K-algebra;

G = R-order in A (for the definition cf. (**3**)).

All modules that occur are assumed to be finitely generated unitary left modules, unless otherwise specified. By a *G-lattice* we mean a G-module which is torsion-free as R-module. A G-lattice is called *irreducible* if it does not contain a proper G-submodule of smaller R-rank. If p is a prime ideal in R we shall write $G_p = R_p \otimes_R G$; $M_p = R_p \otimes_R M$ for a G-lattice M, and $KM = K \otimes_R M$. Two G-lattices M and N are said to lie in the same *genus* (notation $M \vee N$) if $M_p \cong N_p$ for every prime ideal p in R.

For any A-module L, let $S(L)$ be the collection of G-lattices M, for which $KM \cong L$. Suppose that $S(L)$ splits into $r_g(L)$ genera, and into $r_i(L)$ classes under G-isomorphism. Maranda (**6**) has shown: If L is an absolutely irreducible A-module, then

$$r_i(L) = h \cdot r_g(L), \tag{1}$$

where h is the class number of K. Moreover, he listed all G-lattices which are in the same genus as $M \in S(L)$.

Our aim in this paper is to extend the results of Maranda (**6**). We shall describe (for a certain type of R-orders) all irreducible G-lattices in terms of irreducible lattices over maximal orders containing G. In § 2 we show that for considerations of irreducible G-lattices it suffices to look at orders in simple separable algebras. In § 3 we show that the irreducible G-lattices are also lattices over maximal orders in A, if for all irreducible G-lattices, $\mathrm{End}_G(M)$ is the same maximal order. In § 4 we apply the results of § 3 to extend Maranda's results; if L is an absolutely irreducible G-lattice, then we describe $S(L)$ explicitly. However, the applications are not restricted to absolutely irreducible A-modules.

Convention. Homomorphisms will be written opposite to the scalars.

2. Reduction to orders in simple algebras. If H is any R-order in A containing G, and if M is an H-lattice, we write M_H and M_G to indicate whether M should be considered as an H-lattice or as a G-lattice, respectively.

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PROPOSITION 1. *If M and N are H-lattices, then*

$$\mathrm{Hom}_H(M_H, N_H) = \mathrm{Hom}_G(M_G, N_G).$$

Proof. We have the inclusion

$$\mathrm{Hom}_H(M_H, N_H) \subset \mathrm{Hom}_G(M_G, N_G).$$

To show the reverse inclusion, we pick $0 \neq r \in R$ such that $rH \subset G$. For $f \in \mathrm{Hom}_G(M_G, N_G)$ we have:

$$r((xm)f) = (rxm)f = rx(mf), \qquad x \in H, m \in M.$$

Since N is R-torsion-free, $f \in \mathrm{Hom}_H(M_H, N_H)$.

For the remainder of this section we shall denote by $\mathrm{Irr}(G)$ the set of isomorphism classes of irreducible G-lattices.

PROPOSITION 2. *We have an injection*

$$F\colon \mathrm{Irr}(H) \to \mathrm{Irr}(G), \qquad F\colon (M_H) \to (M_G),$$

where (M) denotes the isomorphism class of M.

Proof. This map is well-defined, and $(M_G) \in \mathrm{Irr}(G)$ if $(M_H) \in \mathrm{Irr}(H)$, since M is an irreducible G-lattice if and only if KM is an irreducible A-module. Using Proposition 1, we conclude that F is injective.

LEMMA 3. *Let e_i, $i = 1, \ldots, n$, be the set of mutually orthogonal central primitive idempotents in A. Then*

$$H = \sum_{i=1}^{n} \oplus Ge_i$$

is an R-order in A containing G, and $F\colon \mathrm{Irr}(H) \to \mathrm{Irr}(G)$ is a bijection.

Proof. The e_i are integral over R, and $\sum_{i=1}^n e_i = 1$; therefore H is an R-order in A containing G. Because of Proposition 2, it only remains to show that F is surjective. Let M be an irreducible G-lattice such that KM corresponds to e_k. Then

$$e_i m' = \delta_{ik} m' \quad \text{for every } m' \in KM,$$

δ_{ik} is the Kronecker symbol. Since $1 \otimes_R M$ is canonically isomorphic to M, we may assume that $M \subset KM$, so that

$$e_i m = \delta_{ik} m \quad \text{for ever} \quad m \in M,$$

i.e., M is an H-lattice, and F is surjective.

Remark 4. By means of Lemma 3, one knows all irreducible G-lattices once the irreducible H-lattices are known, where

$$H = \sum_{i=1}^{n} \oplus Ge_i.$$

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However,

$$\operatorname{Irr}(H) = \bigcup_{i=1}^{n} \operatorname{Irr}(Ge_i)$$

is the disjoint union of a finite number of sets. Therefore we may restrict our attention to orders in simple algebras.

Example 5. Let $\mathfrak{G}$ be a finite abelian group of order g, and suppose that K splits $\mathfrak{G}$. If $X \cong \mathfrak{G}$ is the character group of $\mathfrak{G}$, then

$\operatorname{Irr}(R\mathfrak{G}) = \{(I_k e_\chi)\colon \chi \in X,\ I_k$ are representatives of the different ideal classes in R, and e_χ is the primitive idempotent to $\chi\}$.

Proof.

$$e_\chi = \frac{1}{g} \sum_{\mathfrak{g} \in \mathfrak{G}} \chi(\mathfrak{g}^{-1})\mathfrak{g}, \qquad \chi \in X.$$

We use the bijection in Lemma 3:

$$\operatorname{Irr}(H) \to \operatorname{Irr}(R\mathfrak{G}),$$

where $H = \sum_{\chi \in X} \oplus R\mathfrak{G}e_\chi$. However, $R\mathfrak{G}e_\chi = Re_\chi$ is the maximal R-order in Ke_χ. Thus

$$\operatorname{Irr}(Re_\chi) = \{(I_k e_\chi),\ k = 1, \ldots \text{ (class number of } R)\},$$

and by Remark 4 we conclude that

$$\operatorname{Irr}(R\mathfrak{G}) = \{(I_k e_\chi)\colon \chi \in X,\ k = 1, \ldots \text{ (class number of } R)\}.$$

3. Irreducible lattices of orders in simple algebras. Let G be an R-order in the simple separable finite-dimensional K-algebra $A = (D)_n$, D a skew-field of finite dimension over K. We put $C = G \cap D$, viewing D as embedded in A. Then C is an R-order in D. Let

$\{B_j\}$ $(j \in J)$ = different maximal R-orders in A containing G,
M_j = a fixed irreducible B_j-lattice, for every $j \in J$.

Then

$C_j = \operatorname{End}_{B_j}(M_j)$ is a maximal R-order in D;
$\{I_k\}$, $k \in J(C_j)$ = representatives of the different classes of left C_j-ideals in D.

With this notation we can write down a full set of non-isomorphic irreducible B_j-lattices for every $j \in J$:

$$(2) \qquad \operatorname{Irr}(B_j) = \{(M_j \otimes_{C_j} I_k)\colon k \in J(C_j)\};$$

cf. (**1**; **8**).

THEOREM 6. *Let* $\operatorname{Irr}(G)$ *denote the set of isomorphism classes of irreducible* G*-lattices. Then*

(i) $\operatorname{card}(\operatorname{Irr}(G)) \geqq \sum_{j \in J} \operatorname{card}(J(C_j))$;

(ii) *We have equality in* (i) *if* $C = \operatorname{End}_G(M)$ *for every irreducible* G*-lattice* M;

(iii) *In the latter case, we can give all irreducible* G*-lattices explicitly: Let* $\{I_k\}$, $k \in J(C)$, *be representatives of the different classes of left* C*-ideals in* D; *then*

$$\operatorname{Irr}(G) = \{(M_j \otimes_C I_k): j \in J, k \in J(C)\}.$$

Moreover, in this case we have:

$$\operatorname{card}(\operatorname{Irr}(G)) = (\operatorname{card}(J))(\operatorname{card}(J(C)));$$

(iv) *If we have equality in* (i), *then there are* $\operatorname{card}(J)$ *genera of irreducible* G*-lattices, and in each genus there are* $\operatorname{card}(J(C))$ *different isomorphism classes of irreducible* G*-lattices. Moreover,*

$$\{M \otimes_C I_k: k \in J(C)\}$$

are the non-isomorphic irreducible G*-lattices which lie in the same genus as the irreducible* G*-lattice* M, *and representatives of the different genera of irreducible* G*-lattices are the* G*-lattices*

$$\{M_j: j \in J\}.$$

The *proof of Theorem* 6 is done in several steps, as follows.

PROPOSITION 7. *Let* M *be an irreducible* B_j*-lattice,* N *an irreducible* B_k*-lattice,* $j, k \in J$, $j \neq k$, *then* M_G *and* N_G *are not isomorphic as* G*-lattices.*

Proof. Assume that $M_G \cong_G N_G$, and let $f: M_G \to N_G$ be a G-isomorphism Then we make M into a B_k-lattice, denoted by M_k, by defining

$$b_k m_k = (b_k(mf))f^{-1}, \qquad b_k \in B_k, m_k \in M_k, m_k = m.$$

It is easily checked that the action of B_j on M and the action of B_k on M_k coincide on $B_j \cap B_k \supset G$. From (**1**, Theorem 3.9) it follows that

$$C_j = \operatorname{End}_{B_j}(M), \qquad B_j = \operatorname{End}_{C_j}(M),$$
$$C_k = \operatorname{End}_{B_k}(M_k), \qquad B_k = \operatorname{End}_{C_k}(M_k).$$

Now we apply Proposition 1 and conclude that

$$C_j = \operatorname{End}_{B_j}(M) = \operatorname{End}_G(M) = \operatorname{End}_{B_k}(M_k) = C_k;$$

thus $B_j = B_k$, and we have deduced a contradiction.

Proof of Theorem 6(i). Because of (2) and Proposition 7, the G-lattices

$$\{M_j \otimes_{C_j} I_k, k \in J(C_j), j \in J\}$$

are non-isomorphic irreducible G-lattices, whence the inequality (i) in Theorem 6 follows.

Proof of Theorem 6(ii). If $C = \mathrm{End}_G(M)$ for every irreducible G-lattice M, then we have equality in Theorem 6(i). The hypothesis implies that C is maximal: Let M be an irreducible B_j-lattice for some $j \in J$; then $\mathrm{End}_{B_j}(M) = \mathrm{End}_G(M) = C$ is a maximal R-order in D. To prove Theorem 6(ii) we have to show that every irreducible G-lattice is a B_j-lattice for some maximal order B_j, $j \in J$. Let M be an irreducible G-lattice. Then M is a right C-lattice, since $C = \mathrm{End}_G(M)$, and $B = \mathrm{End}_C(M)$ is a maximal R-order in

$$K \otimes_R \mathrm{End}_C(M) = \mathrm{End}_D(KM) = A;$$

cf. (**1**, Theorem 3.9). Since M was a G-lattice to start with, $G \subset B = \mathrm{End}_C(M)$, and M is a B-lattice in the usual fashion.

Proof of Theorem 6(iii). If Theorem 6(ii) holds, then $C_j = C$ for every $j \in J$ ($C_j = \mathrm{End}_{B_j}(M_j)$, cf. the beginning of § 3), and a full set of non-isomorphic irreducible G-lattices is given by

$$\{M_j \otimes_C I_k: j \in J, k \in J(C)\}.$$

Proof of Theorem 6(iv). We shall prove the following lemma, which is of interest in itself.

LEMMA 8. *Let M be an irreducible G-lattice such that M is also a B_j-lattice for some $j \in J$; let $C_j = \mathrm{End}_{B_j}(M)$. Then*

$$\{M \otimes_{C_j} I_k: k \in J(C_j)\}$$

are all the non-isomorphic G-lattices in the same genus as M.

For the notation, compare the beginning of § 3.

Proof. Since C_j is a maximal R-order in D, all the G-lattices $M \otimes_{C_j} I_k$ are non-isomorphic, and they lie in the same genus as M. Now let N be a G-lattice in the same genus as M_G. Then N_p is a $(B_j)_p$-lattice for every prime ideal p in R. However, this can only be if N is a B_j-lattice itself. Therefore, $N \cong M \otimes_{C_j} I_k$ for some $k \in J(C_j)$.

COROLLARY 9. *If M and N are irreducible G-lattices such that M is a B_j-lattice for some $j \in J$ and N is a B_k-lattice for some $k \in J$, then M_G is in the same genus as N_G if and only if $B_j = B_k$.*

COROLLARY 10. *If L is an irreducible A-module, then*

$$r_g(L) \geqq \mathrm{card}(J).$$

For the definition of $r_g(L)$, compare § 1.

The proof of Theorem 6(iv) follows now easily if one observes that we have equality in Theorem 6(i), i.e. every irreducible G-lattice is isomorphic to some B_j-lattice.

This completes the proof of Theorem 6.

4. Applications of Theorem 6 to some special orders. Let A be a separable finite-dimensional K-algebra.

LEMMA 11. *If R is a Dedekind domain such that the class number of R is finite and such that $(R:p)$ is finite for every prime ideal p in R, then there are only finitely many different maximal R-orders in A containing a fixed R-order G in A.*

Proof. There is only a finite number of non-isomorphic irreducible A-modules, say $L_1, \ldots, L_t$. Under the hypotheses on R, the Jordan-Zassenhaus theorem is valid (cf. **10**), i.e. for the R-order G, $S(L_i)$ (cf. § 1) contains only a finite number of non-isomorphic irreducible G-lattices. Now the result follows from Proposition 7 if one observes that every maximal R-order in A decomposes into a direct sum of maximal orders in the simple components of A. The main applications of Theorem 6 can be gained by using the following result.

LEMMA 12. *Let G be an R-order in the simple separable K-algebra $A = (K')_n$, K' an extension field of finite dimension over K. If $G \cap K' = C$ is the maximal R-order in K', then every irreducible G-lattice is an irreducible lattice for some maximal R-order in A containing G, i.e. Theorem* 6(iii), (iv) *can be applied.*

Proof. It only remains to show that $\mathrm{End}_G(M) = C$ for every irreducible G-lattice M; then the lemma follows from Theorem 6(ii). Since C is the only maximal R-order in D, $\mathrm{End}_G(M) \subset C$ for every irreducible G-lattice M. But since C is commutative and is contained in the centre of G, $\mathrm{End}_G(M) = C$.

For the remainder of the paper we adopt the following notation:
A is a separable finite-dimensional K-algebra;
L = irreducible A-module;
$D_L = \mathrm{End}_A(L)$;
e_L = central primitive idempotent corresponding to L;
$Ae_L = \mathrm{End}_D(L)$ = simple component of A corresponding to L.
For an R-order G in A we let:
$C_L = Ge_L \cap D_L$;
$B_j^L, j \in J_L$ = different maximal R-orders in Ae_L containing Ge_L;
M_j^L = irreducible B_j-lattice, $j \in J_L$;
I_k^L, $k \in J(C_L)$ = representatives of the classes of left C_L-ideals in D;
$S(L) = \{M: M = G\text{-lattice}, KM \cong L\}$.

THEOREM 13. *If D_L is commutative and if C_L is the maximal R-order in D, then*
(i) *all irreducible non-isomorphic G-lattices in $S(L)$ are given by*

$$\{M_j^L \otimes_{C_L} I_k^L, j \in J_L, k \in J(C_L)\},$$

(ii) $S(L)$ *splits into* $\mathrm{card}(J_L)$ *genera:*

$$\{M_j^L \otimes_{C_L} I_k, k \in J(C_L)\}, \qquad j \in J_L,$$

(iii) $r_i(L) = (\mathrm{card}(J(C)))r_g(L)$, $r_g(L) = \mathrm{card}(J_L)$,
(this is an extension of Maranda's results (**6**)).

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Remark 14. In the special case where L is an absolutely irreducible A-module, we obtain the well-known formula (1).

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