The Equation of Telegraphy.

By Miss M. C. Gray.

(*Read and Received 2nd November 1923.*)

§ 1. *Introduction.*

The equation of the propagation of electric signals along cables, generally known as *the equation of telegraphy*, may be written

$$\frac{\partial^2 V}{\partial t^2} + 2\gamma \frac{\partial V}{\partial t} = a^2 \frac{\partial^2 V}{\partial z^2} . \qquad \text{.....................} \quad (1)$$

Particular solutions of this equation, adapted to various purposes have been found by Heaviside,* Poincaré,† A. G. Webster,‡ T. W. Chaundy,§ and others. The object of the present paper is to unify the theory of the equation by exhibiting the relations which these solutions bear to each other, and by obtaining them as particular cases of a general solution. The derivation of new particular solutions by the solution of integral equations is also discussed.

§ 2. *The Relation of the Riemann-Green Solution to the General Solution.*

To reduce the equation (1) to the normal form of hyperbolic partial differential equations put

$$V = e^{-\gamma t} u$$

and we find

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial z^2} = \gamma^2 u .$$

---

* Electrical Papers, Vol. I., pp. 53 *et seq.* "On the Extra Current."

† *Comptes Rendus*, 117 (1893), pp. 1027 *et seq.* "On the Propagation of Electricity."

‡ "Electricity and Magnetism," pp. 540 *et seq.*

§ *Proc. London Math. Soc.* (2) XXI. (1922), pp. 214–234.

If we introduce new variables

$$\left.\begin{array}{l} x = z - at \\ y = z + at \end{array}\right\}$$

it becomes

$$\frac{\partial^2 u}{\partial x \, \partial y} + \frac{\gamma^2}{4a^2} u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2a)$$

or if we take

$$at = y$$
$$z = x$$
$$\frac{\gamma}{a} = k$$

it becomes

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + k^2 u = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2b)$$

Either $(2a)$ or $(2b)$ may be regarded as a normal form.

To find the general solution of the equation we start from the well-known general solution of Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

viz. $$V = \int_0^{2\pi} f(z + ix \cos u + iy \sin u, \, u) \, du,$$

of which a particular form is

$$V = \int_0^{2\pi} e^{k(z + ix \cos u + iy \sin u)} \phi(u) \, du \, . \quad \ldots\ldots\ldots\ldots \quad (4)$$

In (3) substitute $iy = y'$, and the equation becomes

$$\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y'^2} + \frac{\partial^2 V}{\partial z^2} = 0 \, .$$

To make this coincide with the telegraphy equation $(2b)$ we must have

$$\frac{\partial^2 V}{\partial z^2} = k^2 z \, .$$

Hence we now substitute $V = e^{kz} f(x, \, y')$ and the equation for f is

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y'^2} + k^2 f = 0 \, .$$

https://doi.org/10.1017/S0013091500036099 Published online by Cambridge University Press

Also from equation (4) we must have

$$e^{kz} f = \int_0^{2\pi} e^{k(z+ix\cos u + y'\sin u)} \phi(u)\, du$$

$$i.e. \qquad f = \int_0^{2\pi} e^{ikx\cos u + k\,y\sin u} \phi(u)\, du \quad \ldots\ldots\ldots\ldots (5b)$$

which *we take as our general solution.*

For the equation (2*a*) the corresponding general solution will be

$$f = \int_0^{2\pi} e^{\frac{ikx}{2}(\cos\theta + i\sin\theta) + \frac{iky}{2}(\cos\theta - i\sin\theta)} \phi(\theta)\, d\theta \ldots\ldots \quad (5a)$$

when k again equals $\frac{\gamma}{a}$.

We now introduce the Riemann-Green function for the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + k^2 u = 0.$$

This is defined* as a function $G(x, y; \xi, \eta)$ having the two important properties:—

1. It satisfies the partial Differential Equation.
2. It has the value unity along the two characteristics through the arbitrary point (ξ, η).

To specify the solution of the equation we must know the values of u and one of its derivates (and therefore really both derivates) along a curve AB. Let A and B be the points in which the characteristics through (ξ, η) meet this curve. Then, if G is the Riemann-Green function for the equation, the general solution may be written

$$u(\xi, \eta) = \tfrac{1}{2}u_A + \tfrac{1}{2}u_B + \tfrac{1}{2}\int_A^B \left[\left(G\frac{\partial u}{\partial x} - u\frac{\partial G}{\partial x}\right)dy + \left(G\frac{\partial u}{\partial y} - u\frac{\partial G}{\partial y}\right)dx\right].$$

The Riemann-Green function corresponding to our general solution (5 *b*) is

$$G(x, y; \xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} e^{ik(x-\xi)\cos\theta + k(y-\eta)\sin\theta}\, d\theta \ldots\ldots\ldots (6c)$$

---

* *Cf.* Riemann, *Ges. Werke* (1876), pp. 158 *et seq.*;
or Darboux, *Théorie Génerale des Surfaces*, t. II., pp. 75 *et seq.*

since this satisfies the equation, and has the value unity along the characteristics

$$\left.\begin{array}{l} x+y=\xi+\eta \\ x-y=\xi-\eta \end{array}\right\}$$

for $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{m(\cos\theta \pm i\sin\theta)}\, d\theta$ has the value unity whatever m may be.

The Riemann-Green function for the equation (2b) is usually written

$$G(x, y\,;\ \xi, \eta)=J_0\,\{k\sqrt{(x-\xi)^2-(y-\eta)^2}\}\ldots\ldots\ldots\ldots(6b)$$

while that for equation (2a) is

$$G(x, y\,;\ \xi, \eta)=J_0\,\{k\sqrt{(x-\xi)(y-\eta)}\}\ \ldots\ldots\ldots\ldots\ldots(6a)$$

where J_0 is the Bessel function of order zero.

The equivalence of (6b) and (6c) can be shown immediately by substituting in (6c) the values

$$x-\xi=\rho\cos\phi$$

$$y-\eta=i\rho\sin\phi,$$

$$\text{so}\quad G(x, y\,;\ \xi, \eta)=\frac{1}{2\pi}\int_0^{2\pi} e^{ik\rho\cos(\theta-\phi)}\, d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi} e^{ik\rho\cos u}\ du.$$

Also $e^{ik\rho\cos u}=J_0(k\rho)+2i\cos u\,J_1(k\rho)+2i^2\cos 2u\,J_2(k\rho)+\ldots$

$$\therefore\quad G(x, y\,;\ \xi, \eta)=\frac{1}{2\pi}\int_0^{2\pi} J_0(k\rho)\,du$$

$$=J_0(k\rho)$$

$$=J_0\,\{k\sqrt{(x-\xi)^2-(y-\eta)^2}\}.$$

Now the general solution corresponding to the Riemann-Green function (6c) is

$$u(\xi, \eta)=\tfrac{1}{2}u_A+\tfrac{1}{2}u_B+\tfrac{1}{2}\int_A^B G\left(\frac{\partial u}{\partial x}\,dy+\frac{\partial u}{\partial y}\,dx\right)-u\left(\frac{\partial G}{\partial x}dy+\frac{\partial G}{\partial y}\,dx\right)$$

https://doi.org/10.1017/S0013091500036099 Published online by Cambridge University Press

$$= \tfrac{1}{2}u_A + \tfrac{1}{2}u_B + \frac{1}{4\pi}\int_A^B \left[\int_0^{2\pi} e^{ik(x-\xi)\cos\theta + k(y-\eta)\sin\theta}\, d\theta \right] \left\{\frac{\partial u}{\partial x}dy + \frac{\partial u}{\partial y}dx\right\}$$

$$- \frac{1}{4\pi}\int_A^B \left[u \int_0^{2\pi} e^{ik(x-\xi)\cos\theta + k(y-\eta)\sin\theta} d\theta\right] \{ik\cos\theta\, dy + k\sin\theta\, dx\}.$$

Now suppose that along AB we have the values

$$x = f_1(t),\ y = f_2(t),\ u = f_3(t),$$

$$\frac{\partial u}{\partial x} = \psi_1(t), \frac{\partial u}{\partial y} = \psi_2(t);$$

then $u(\xi, \eta) = \tfrac{1}{2}u_A + \tfrac{1}{2}u_B\ \frac{1}{4\pi}\int_A^B \int_0^{2\pi} e^{-ik\xi\cos\theta - k\eta\sin\theta}$

$$[e^{ikf_1(t)\cos\theta + kf_2(t)\sin\theta} F(\theta, t)]\, d\theta\, dt$$

where $F(\theta, t) = \psi_1(t) f_2'(t) + \psi_2(t) f_1'(t)$

$$- (ik\cos\theta f_2'(t) + k\sin\theta f_1'(t)) f_3(t).$$

Now, change the order of integration; this gives

$$u(\xi, \eta) = \tfrac{1}{2}u_A + \tfrac{1}{2}u_B + \frac{1}{4\pi}\int_0^{2\pi} e^{-ik\xi\cos\theta - k\eta\sin\theta} \left[\int_A^B e^{ikf_1(t)\cos\theta + kf_2(t)\sin\theta} F(\theta, t)\, dt\right] d\theta$$

$$= \tfrac{1}{2}u_A + \tfrac{1}{2}u_B + \frac{1}{4\pi}\int_0^{2\pi} e^{-ik\xi\cos\theta - k\eta\sin\theta}\, \psi(\theta)\, d\theta.$$

Since the original equation involves only terms of even order the general solution may be written

$$u = \int_0^{2\pi} e^{\pm(ikx\cos\theta + ky\sin\theta)} f(\theta)\, d\theta;$$

hence the Riemann-Green method of solution leads to a solution similar in form to the general solution; and as the values of z and its first derivates may be chosen arbitrarily along the curve $A\,B$, the function $\psi(\theta)$ may be regarded as arbitrary. Thus *the Riemann-Green solution is exhibited as an equivalent form of the general solution.*

§ 3. *Heaviside's Solution.*

Heaviside considers the equation

$$\frac{\partial^2 v}{\partial x^2} = ck\frac{\partial v}{\partial t} + sc\frac{\partial^2 v}{\partial t^2}$$

in which we must take $\frac{k}{s} = 2\gamma$

$$\frac{1}{cs} = a^2$$

to reduce it to the form

$$\frac{\partial^2 v}{\partial t^2} + 2\gamma\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial x^2}.$$

His general solution, obtained by assuming that we have initially a periodic variation $V\begin{matrix}\sin\\ \cos\end{matrix}\frac{n\pi x}{l}$ is

$$v = V\begin{matrix}\sin\\ \cos\end{matrix}\frac{n\pi x}{l}\, e^{-t/2\alpha}$$
$$\left\{A\, e^{\frac{t}{2\alpha}\sqrt{1-4n^2\pi^2\alpha/\beta}} + (1-A)\, e^{-t/2\alpha\sqrt{1-4n^2\pi^2\alpha/\beta}}\right\}$$

where $\alpha = \frac{s}{k} = \frac{1}{2\gamma}$ and $\beta = ckl^2 = \frac{2\gamma l^2}{a^2}$.

If we now substitute $at = y$, and $\cos\theta = \frac{na\pi}{l\gamma}$, v reduces to the form

$$v = V e^{\frac{-\gamma}{a}y}$$
$$\left\{A\, e^{i\frac{\gamma}{a}x\cos\theta + \frac{\gamma}{a}y\sin\theta} + (1-A)\, e^{i\frac{\gamma}{a}x\cos\theta - \frac{\gamma}{a}y\sin\theta}\right\},$$

and our general solution is

$$v = e^{\frac{-\gamma}{a}y}\int_0^{2\pi} e^{i\frac{\gamma}{a}x\cos\theta + \frac{\gamma}{a}y\sin\theta}\,\phi(\theta)\,d\theta.$$

Thus *the general solution is given by summing an infinite number of these particular solutions, each multiplied by the corresponding value of the arbitrary function* $\phi(\theta)$.

https://doi.org/10.1017/S0013091500036099 Published online by Cambridge University Press

§ 4. *Poincaré's Solution.*

There is also a solution due to Poincaré who considers the equation in the form

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} + z = 0,$$

so that in our general solution we must write $k = 1$.

The solution given by Poincaré is

$$z = \int_{-\infty}^{+\infty} e^{iqx} \left[\theta \cos y \sqrt{q^2 - 1} + \theta_1 \frac{\sin y \sqrt{q^2 - 1}}{\sqrt{q^2 - 1}} \right] dq \quad \ldots\ldots (7)$$

where for the time variable $y = 0$ we have given initial values of z and $\frac{\partial z}{\partial y}$:—

$$z \text{ reduces to } f(x) = \int_{-\infty}^{+\infty} \theta(q)\, e^{iqx}\, dq$$

$$\text{and } \frac{\partial z}{\partial y} \text{ reduces to } f_1(x) = \int_{-\infty}^{+\infty} \theta_1(q)\, e^{iqx}\, dq\,.$$

Consider first the second term of the solution (7). Since it has singularities at the points ± 1, we must integrate along a contour of the form $ABCDE$. To evaluate this, suppose $x + y > 0$, and integrate round the closed contour $ABCDEMA$ (Fig. I), where

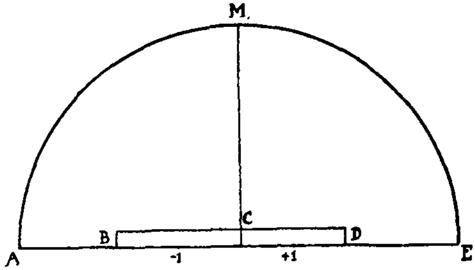

Fig. I.

EMA is a circle of very large radius and centre the origin. The integral round the contour is zero, and also the integral round the semicircle tends to zero as the radius tends to infinity.

$\therefore$ Integral along $ABCDE = 0$.

https://doi.org/10.1017/S0013091500036099 Published online by Cambridge University Press

Now for $x+y<0$, we retain the path $ABCDE$ and adjoin to it the semicircle $EM'A$ (Fig. II). The integral round the semicircle

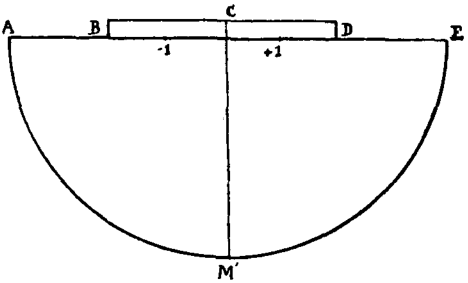

FIG. II.

will again tend to zero, though the integral round the whole contour is not now zero, but equals that round the contour $OABCDEO$. (Fig. III).

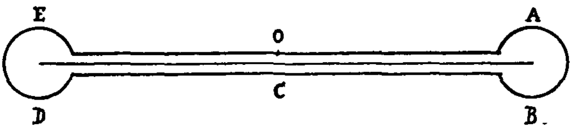

FIG. III

Hence, if we substitute $q=\cos\phi$, ϕ must vary from 0 to 2π, and the integral becomes

$$I_2=\int_{-\infty}^{+\infty}\frac{\theta_1}{2i}\left\{e^{iy\sqrt{q^2-1}}-e^{-iy\sqrt{q^2-1}}\right\}\frac{e^{iqx}\,dq}{\sqrt{q^2-1}}$$

$$=\tfrac{1}{2}\int_0^{2\pi}\theta_1\left\{e^{ix\cos\phi+y\sin\phi}-e^{ix\cos\phi-y\sin\phi}\right\}d\phi$$

and *each of these terms is a particular case of the general solution.*

If we now consider the first term of the solution (7), we see that it may be derived from the second term by interchanging θ and θ_1, and differentiating with respect to y. Hence to evaluate I_1 we perform the same operations on the above value of I_2 which gives

$$I_1=\int_0^{2\pi}\frac{\theta}{2}\left\{e^{ix\cos\phi+y\sin\phi}+e^{ix\cos\phi-y\sin\phi}\right\}\sin\phi\,d\phi,$$

https://doi.org/10.1017/S0013091500036099 Published online by Cambridge University Press

and again each of these terms is a particular case of the general solution.

Thus *Poincaré's solution is exhibited as the sum of four terms, each of which is a particular form of the solution* (5*b*).

§ 5. *Webster's Solution.*

We pass on now to the more complicated solution given by Webster. He starts from the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + b^2 u$$

with the initial conditions $u = F(x)$ and $\frac{\partial u}{\partial t} = g(x)$ but works out the solution for the more general equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + b^2 u,$$

where we may put at any time u independent of y. We now make u satisfy the auxiliary equation $a^2 \frac{\partial^2 u}{\partial z^2} = b^2 u$ so that the equation becomes

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

and the initial conditions are transformed into

$$u = \Phi(x, y, z), \quad \frac{\partial u}{\partial t} = \phi(x, y, z).$$

Then Poisson's formula * gives as a solution of this equation

$$u(x, y, z) = \frac{1}{4\pi} \frac{\partial}{\partial t} \iint t \Phi(x + at \cos \alpha, \ y + at \cos \beta, \ z + at \cos \gamma) \, d\omega$$

$$+ \frac{1}{4\pi} \iint t \phi(x + at \cos \alpha, \ y + at \cos \beta, \ z + at \cos \gamma) \, d\omega.$$

* *Nouveaux Mémoires de l'Acad. des Sciences*, t. III.

For the telegraphic equation Webster, integrating over a hemisphere for constant z, and then making y vanish, reduces this finally to the form

$$u = \tfrac{1}{2}\{F(x+at)+F(x-at)\} + \frac{1}{2a}\int_{-at}^{+at} F(x+\lambda)\frac{\partial}{\partial t} J_0\left(ib\sqrt{t^2-\frac{\lambda}{a^2}}\right)d\lambda$$
$$+\frac{1}{2a}\int_{-at}^{+at} g(x+\lambda) J_0\left(ib\sqrt{t^2-\frac{\lambda^2}{a^2}}\right)d\lambda.$$

To connect this with our standard solution we consider the Riemann-Green function

$$G(x, y;\ \xi, \eta) = J_0\left\{\frac{b}{a}\sqrt{(x-\xi)^2-(y-\eta)^2}\right\}$$

which gives the general solution

$$u = \tfrac{1}{2}u_A + \tfrac{1}{2}u_B + \tfrac{1}{2}\int_A^B J_0\left\{\frac{b}{a}\sqrt{(x-\xi)^2-(y-\eta)^2}\right\}\left\{\frac{\partial u}{\partial x}dy + \frac{\partial u}{\partial y}dx\right\}$$
$$-\tfrac{1}{2}\int_A^B u\left\{\frac{\partial}{\partial x}dy + \frac{\partial}{\partial y}dx\right\} J_0\left\{\frac{b}{a}\sqrt{(x-\xi)^2-(y-\eta)^2}\right\}.$$

In this we may substitute, following Webster,

$$x-\xi = at\cos\theta,\quad y-\eta = at\sin\theta.$$

$$\therefore\quad u = \tfrac{1}{2}u_A + \tfrac{1}{2}u_B - \tfrac{1}{2}\int_0^\pi J_0\{bt\sqrt{\cos 2\theta}\}\,.\,2\sin\theta\cos\theta\frac{\partial u}{\partial\theta}d\theta$$
$$+\tfrac{1}{2}\int_0^\pi u\,.\,2\sin\theta\cos\theta\frac{\partial}{\partial\theta}J_0\{bt\sqrt{\cos 2\theta}\}\,d\theta.$$

Now change the variable from θ to α, given by $\cos 2\theta = -\sin^2\alpha$

$$\therefore\quad u = \tfrac{1}{2}\{F(x+at)+F(x-at)\}$$
$$-\tfrac{1}{2}\int_0^\pi J_0(ibt\sin\alpha)\sqrt{1-\sin^4\alpha}\frac{du}{d\alpha}d\alpha$$
$$+\tfrac{1}{2}\int_0^\pi u\sqrt{1-\sin^4\alpha}\frac{d}{d\alpha}J_0(ibt\sin\alpha)\,d\alpha.$$

https://doi.org/10.1017/S0013091500036099 Published online by Cambridge University Press

Finally substitute $at \cos \alpha = \lambda$; this gives

$$u = \tfrac{1}{2}\{F(x+at) + F(x-at)\}$$

$$+ \tfrac{1}{2}\int_{-at}^{+at} J_0\left(ib\sqrt{t^2 - \frac{\lambda^2}{a^2}}\right)\frac{\lambda^2}{a^2t^2}\left(2 - \frac{\lambda^2}{a^2t^2}\right)\frac{du}{d\lambda}\, d\lambda$$

$$- \tfrac{1}{2}\int_{-at}^{+at} u\, \frac{\lambda^2}{a^2t^2}\left(2 - \frac{\lambda^2}{a^2t^2}\right)\frac{d}{d\lambda} J_0(ib\sqrt{t^2 - \lambda^2/a^2})\, d\lambda.$$

And $\dfrac{dJ_0}{d\lambda} = -\dfrac{\lambda}{a^2t}\dfrac{dJ_0}{dt}$.

$$\therefore \quad u = \tfrac{1}{2}\{F(x+at) + F(x-at)\}$$

$$+ \tfrac{1}{2}\int_{-at}^{+at} J_0(ib\sqrt{t^2 - \lambda^2/a^2})\frac{\lambda^2}{a^2t^2}\left(2 - \frac{\lambda^2}{a^2t^2}\right)\frac{\partial u}{\partial\lambda}\, d\lambda$$

$$+ \tfrac{1}{2}\int_{-at}^{+at} u\, \frac{\lambda^3}{a^4t^3}\left(2 - \frac{\lambda^2}{a^2t^2}\right)\frac{\partial J_0(ib\sqrt{t^2 - \lambda^2/a^2})}{\partial t}\, d\lambda.$$

This solution is evidently identical with Webster's solution, since u and $\dfrac{\partial u}{\partial \lambda}$ may be chosen arbitrarily along the curve AB, to which a definite form has been assigned. Thus ***Webster's solution is really a particular case of the Riemann-Green solution***, which we have already shown to be equivalent to the general solution.

§ 6. *T. W. Chaundy's Solution.*

Chaundy obtains a solution of the telegraphy equation in an entirely different manner, but his actual result is very similar to Webster's, so need only be briefly considered. He starts from the equation

$$\frac{\partial^2 z}{\partial x \partial y} + z = 0$$

and obtains the solution in series form reducing it finally to

$$z = k \,.\, J_0(xy) + \int_0^x \phi(u)\, J_0(2\sqrt{(x-u)\,y})\, du$$

$$+ \int_0^y \psi(u)\, J_0(2\sqrt{x(y-u)})\, du$$

https://doi.org/10.1017/S0013091500036099 Published online by Cambridge University Press

But the Riemann-Green function corresponding to this equation is

$$G(x, y; \xi, \eta) = J_0\{2\sqrt{(x-\xi)(y-\eta)}\}$$

which leads to the general solution

$$z = \tfrac{1}{2}z_A + \tfrac{1}{2}z_B + \tfrac{1}{2}\int_A^B J_0\{2\sqrt{(x-\xi)(y-\eta)}\}\left\{\frac{\partial z}{\partial x}\,dy + \frac{\partial z}{\partial y}\,dx\right\} + \tfrac{1}{2}\int_A^B z\left(\frac{\partial J_0}{\partial x}\,dy + \frac{\partial J_0}{\partial y}\,dx\right)$$

and we see at once that the term

$$\tfrac{1}{2}\int_A^B J_0\{2\sqrt{(x-\xi)(y-\eta)}\}\left\{\frac{\partial z}{\partial x}\,dy + \frac{\partial z}{\partial y}\,dx\right\}$$

can be reduced to either of the last two terms in Chaundy's solution, if we take (ξ, η) to be the point $(u, 0)$ or $(0, u)$. The term $J_0(xy)$ is simply the Riemann-Green function corresponding to the origin. Hence Chaundy's solution is the sum of three terms each of which is a particular form of the Riemann-Green solution. *This solution is, therefore, also a particular form of the general solution.*

Chaundy refers also to another solution

$$u = \int_\gamma \phi(t)\, e^{ixt + iy/t}\, dt$$

where γ is a simple closed contour surrounding the origin; this is built up from the elementary solution

$$e^{ixt + iy/t}.$$

If we take γ to be a circle of unit radius with centre at the origin, we have evidently $t = e^{i\theta}$, and therefore

$$u = \int_0^{2\pi} \Phi(\theta)\, e^{ix(\cos\theta + i\sin\theta) + iy(\cos\theta - i\sin\theta)}\, d\theta,$$

which is exactly equivalent to the general solution (5*a*) since in Chaundy's equation $\dfrac{k^2}{4} = 1$. Hence this solution is rather more general than the one we have been considering, as the contour γ need not be circular. Practically, however, the two solutions may be regarded as identical.

§ 7. *Derivation of Particular Solutions from the General Solution.*

Since all these solutions can thus be shown to be particular cases of the general solution, it would seem natural to expect that there should be some method of evaluating the arbitrary function $f(\theta)$ when the initial values of z and its time derivate are assigned. By substituting this value of $f(\theta)$ in the general solution, we should obtain the so'ution corresponding to these given initial values.

Now the general solution of our equation is

$$z=\int_0^{2\pi} e^{ikx\cos\theta + ky\sin\theta} f(\theta)\, d\theta$$

where y is proportional to the time coordinate.

For $y=0$, therefore, we have

$$(z)_0=\int_0^{2\pi} e^{ikx\cos\theta} f(\theta)\, d\theta,$$

and $$\left(\frac{\partial z}{\partial y}\right)_0=\int_0^{2\pi} e^{ikx\cos\theta} k\sin\theta f(\theta)\, d\theta$$

if we may differentiate under the integral sign.

Then if the given initial values are

$$(z)_0=F(x)$$
$$\left(\frac{\partial z}{\partial y}\right)_0=F_1(x)$$

$f(\theta)$ will be found from the equations

$$\int_0^{2\pi} e^{ikx\cos\theta} f(\theta)\, d\theta=F(x).$$
$$\int_0^{2\pi} e^{ikx\cos\theta} k\sin\theta f(\theta)\, d\theta=F_1(x).$$

In the case of Poincaré's initial conditions the equations are immediately soluble, for they are

$$\int_0^{2\pi} e^{ix\cos\theta} f(\theta)\, d\theta=\int_{-\infty}^{+\infty} \theta(q)\, e^{iqx}\, dq \;\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$\int_0^{2\pi} e^{ix\cos\theta} \sin\theta f(\theta)\, d\theta=\int_{-\infty}^{+\infty} \theta_1(q)\, e^{iqx}\, dq \;\ldots\ldots\ldots(9)$$

In (8) write $q = \cos\phi$, and we find as before

$$\int_{-\infty}^{+\infty} \theta(q)\, e^{iqx}\, dq = -\int_0^{2\pi} \theta(\cos\phi) e^{ix\cos\phi} \sin\phi\, d\phi$$

$$\therefore \quad f(\phi) = -\sin\phi\, \theta(\cos\phi).$$

Similarly from (9) we have

$$\int_{-\infty}^{+\infty} \theta_1(q) e^{iqx}\, dq = -\int_0^{2\pi} \theta_1(\cos\phi)\, e^{ix\cos\phi} \sin\phi\, d\phi\,.$$

$\therefore$ We may also have

$$f(\phi) = -\theta_1(\cos\phi)\,.$$

This gives as our solution, with Poincaré's initial conditions

$$z = -\int_0^{2\pi} e^{ix\cos\phi + y\sin\phi}\, \theta(\cos\phi) \sin\phi\, d\phi$$

$$-\int_0^{2\pi} e^{ix\cos\phi + y\sin\phi}\, \theta_1(\cos\phi)\, d\phi.$$

$$= \int_{-\infty}^{+\infty} e^{ixu + iy\sqrt{u^2-1}}\, \theta(u)\, du + \int_{-\infty}^{+\infty} e^{ixu + iy\sqrt{u^2-1}}\, \theta_1(u) \frac{du}{\sqrt{u^2-1}}.$$

This solution is evidently equivalent to the form given by Poincaré.

In general the integral equation

$$\int_0^{2\pi} e^{ix\cos\theta} f(\theta)\, d\theta = F(x)$$

can be solved in the form of a series, provided that $F(x)$ satisfies the conditions necessary for its expansion in a series of Bessel functions.

Thus we have

$$F(x) = a_0 J_0(x) + \sum_{n=1}^{\infty} a_n J_n(x).$$

Also we have $e^{ix\cos\theta} = J_0(x) + \sum_{n=1}^{\infty} 2i^n J_n(x) \cos n\theta.$

We further assume that $f(\theta)$ can be expanded as a Fourier cosine series

$$f(\theta) = b_0 + \sum_{n=1}^{\infty} b_n \cos n\theta.$$

Then

$$\int_0^{2\pi} e^{ix\cos\theta} f(\theta)\,d\theta = \int_0^{2\pi} d\theta \left\{J_0(x) + \sum_{n=1}^{\infty} 2i^n \cos n\theta\, J_n(x)\right\} \left\{b_0 + \sum_{n=1}^{\infty} b_n \cos n\theta\right\}$$

$$= \int_0^{2\pi} d\theta \left\{b_0 J_0(x) + 2\sum_{n=1}^{\infty} i^n b_n \cos^2 n\theta\, J_n(x)\right\},$$

since the terms in $\cos m\theta \cos n\theta$ $(m \neq n)$ and those in $\cos m\theta$ vanish on integration.

$$\therefore \int_0^{2\pi} e^{ix\cos\theta} f(\theta)\,d\theta = \int_0^{2\pi} d\theta \left\{b_0 J_0(x) + \sum_{n=1}^{\infty} i^n b_n J_n(x)(1+\cos 2n\theta)\right\}$$

$$= 2\pi\, b_0 J_0(x) + 2\pi \sum_{n=1}^{\infty} i^n b_n J_n(x).$$

Thus we are led to the identity

$$2\pi\, b_0 J_0(x) + 2\pi \sum_{n=1}^{\infty} i^n b_n J_n(x) = a_0 J_0(x) + \sum_{n=1}^{\infty} a_n J_n(x);$$

and equating coefficients we have

$$b_0 = \frac{a_0}{2\pi}.$$

$$b_n = \frac{a_n}{2\pi i^n}.$$

$$\therefore f(\theta) = \frac{1}{2\pi}\left(a_0 + \Sigma \frac{a_n}{i^n} \cos n\theta\right).$$

Thus the equation is theoretically solved, and we can find the value of $f(\theta)$ for any given initial conditions.