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On Weakly Tight Families

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Abstract. Using ideas from Shelah's recent proof that a completely separable maximal almost disjoint family exists when $\mathfrak{c} < \aleph_\omega$, we construct a weakly tight family under the hypothesis $\mathfrak{s} \leq \mathfrak{b} < \aleph_\omega$. The case when $\mathfrak{s} < \mathfrak{b}$ is handled in ZFC and does not require $\mathfrak{b} < \aleph_\omega$, while an additional PCF type hypothesis, which holds when $\mathfrak{b} < \aleph_\omega$, is used to treat the case $\mathfrak{s} = \mathfrak{b}$. The notion of a weakly tight family is a natural weakening of the well-studied notion of a Cohen indestructible maximal almost disjoint family. It was introduced by Hrušák and García Ferreira [8], who applied it to the Katétov order on almost disjoint families.

1 Introduction

Recall that two infinite subsets a and b of ω are said to be *almost disjoint or a.d.* if $a \cap b$ is finite. We say that a family $\mathcal{A} \subset [\omega]^\omega$ is *almost disjoint or a.d.* if its elements are pairwise a.d. A *Maximal Almost Disjoint or MAD* family is an infinite a.d. family $\mathcal{A} \subset [\omega]^\omega$ such that $\forall b \in [\omega]^\omega\, \exists a \in \mathcal{A}\, [|a \cap b| = \omega]$.

MAD families have been intensively studied in set theory. They have several applications in set theory as well as general topology. For instance, the technique of almost disjoint coding has been used in forcing theory (see [10]) and MAD families are used in the construction of the Isbell–Mrówka space in topology (see [13] and [14]). Almost disjoint families also have applications to geometric Banach space theory, operator algebras, and group theory. See [16] for a general survey of some recent results and open problems regarding MAD families.

Particular attention has been focused on the existence and size of MAD families with strong combinatorial properties. These combinatorial properties typically require the family to be "maximal" with respect to some additional criteria. The most well known is that of a completely separable MAD family. Recall that a MAD family $\mathcal{A} \subset [\omega]^\omega$ is said to be *completely separable* if for any $b \in \mathcal{I}^+(\mathcal{A})$, there is $a \in \mathcal{A}$ with $a \subset b$. Here $\mathcal{I}(\mathcal{A})$ denotes the ideal on ω generated by $\mathcal{A}$, and for any ideal $\mathcal{I}$ on ω, $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$.

In most cases, it is unknown whether MAD families with these strong combinatorial properties can be constructed in ZFC. In fact, almost all known constructions of such families use an assumption of the form $\mathfrak{x} = \mathfrak{c}$, where $\mathfrak{x}$ is some appropriately chosen cardinal invariant. Here, the assumption $\mathfrak{p} = \mathfrak{c}$ serves as a limiting case, sufficing for virtually all known constructions of this sort. A small number of examples

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are known that do not use a hypothesis of the form $\mathfrak{x} = \mathfrak{c}$, among them the construction of completely separable families first considered in [5], and that of van Douwen families posed in [12].

The work of Balcar, Dočkálková, and Simon proved that a completely separable MAD family can be constructed from any of the assumptions $\mathfrak{b} = \mathfrak{d}$, $\mathfrak{s} = \omega_1$, or $\mathfrak{d} \leq \mathfrak{a}$. See [1], [2], and [20] for this work. Then Shelah [18] recently achieved a breakthrough by constructing such a family from $\mathfrak{c} < \aleph_\omega$.

Recall that an a.d. family of total functions $\mathcal{A} \subset \omega^\omega$ is said to be *van Douwen* if for each p, an infinite partial function from ω to ω, there is $f \in \mathcal{A}$ such that $|p \cap f| = \omega$. Raghavan [17] showed how to get such an object just in ZFC alone.

Another prominent example of a strong combinatorial property which has been considered for a.d. families is that of indestructibility.

Definition 1 Let $\mathbb{P}$ be a notion of forcing and let $\mathcal{A} \subset [\omega]^\omega$ be a MAD family. We will say that $\mathcal{A}$ *is* $\mathbb{P}$*-indestructible* if $\Vdash_{\mathbb{P}} \mathcal{A}$ is MAD.

There is no forcing notion $\mathbb{P}$ adding a new real for which a ZFC construction of a $\mathbb{P}$-indestructible MAD family is known. A Sacks indestructible MAD family is provably the weakest such object in the sense that if $\mathcal{A} \subset [\omega]^\omega$ is a MAD family that is $\mathbb{P}$-indestructible for some $\mathbb{P}$ which adds a new real, then $\mathcal{A}$ is also Sacks indestructible. It is not too hard to see that if $\mathfrak{a} < \mathfrak{c}$, then any MAD family of size $\mathfrak{a}$ is Sacks indestructible. However, the only known constructions of a Sacks indestructible MAD family of size $\mathfrak{c}$ use either $\mathfrak{b} = \mathfrak{c}$ or $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$. It remains an open problem whether such families (of any size) can be built in ZFC. Another basic example of a $\mathbb{P}$ that adds a real is Cohen forcing. Cohen indestructibility is closely related to another combinatorial property of MAD families first considered by Malykhin [11].

Definition 2 An a.d. family $\mathcal{A} \subset [\omega]^\omega$ is called $\aleph_0$*-MAD* or *tight* or *strongly MAD* if for every countable collection $\{b_n : n \in \omega\} \subset \mathcal{I}^+(\mathcal{A})$ there is $a \in \mathcal{A}$ such that $\forall n \in \omega[|b_n \cap a| = \omega]$.

It is not too difficult to see that there is a Cohen indestructible MAD family iff an $\aleph_0$-MAD family exists. The only known construction of an $\aleph_0$-MAD family (of size $\mathfrak{c}$) uses $\mathfrak{b} = \mathfrak{c}$, and it is a long-standing open problem whether their existence can be proved in ZFC. It is shown in [15] that the weak Freese–Nation property of $\mathcal{P}(\omega)$ (wFN($\mathcal{P}(\omega)$)), which is shown to hold in [6] in any model gotten by adding fewer than $\aleph_\omega$ Cohen reals to a ground model satisfying CH, implies that all $\aleph_0$-MAD families have size at most $\aleph_1$. $\aleph_0$-MAD families have been studied in [9] and [7]. Also, Brendle and Yatabe [4] have provided combinatorial characterizations of $\mathbb{P}$-indestructibility for many other standard posets $\mathbb{P}$.

Hrušák and García Ferreira [8] introduced the following natural weakening of an $\aleph_0$-MAD family.

Definition 3 An a.d. family $\mathcal{A} \subset [\omega]^\omega$ is called *weakly tight* if for every countable collection $\{b_n : n \in \omega\} \subset \mathcal{I}^+(\mathcal{A})$, there is $a \in \mathcal{A}$ such that $\exists^\infty n \in \omega[|b_n \cap a| = \omega]$.

They proved that such families are almost maximal in the Katétov order on a.d.

families. Given a.d. families $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ is *Katétov below* $\mathcal{B}$ and write $\mathcal{A} \leq_K \mathcal{B}$ if there is a function $f \in \omega^\omega$ such that $\forall a \in \mathcal{I}(\mathcal{A})[f^{-1}(a) \in \mathcal{I}(\mathcal{B})]$. They showed that if $\mathcal{A}$ is weakly tight, then for any other MAD family $\mathcal{B}$, if $\mathcal{A} \leq_K \mathcal{B}$, then there is a $c \in \mathcal{I}^+(\mathcal{A})$ such that $\mathcal{B} \leq_K \{a \cap c : a \in \mathcal{A}\}$. It is unknown whether it is consistent to have a MAD family that is Katétov maximal.

One more application of weakly tight families is stated here. Let $\mathrm{Sym}(\omega)$ be the symmetric group on ω and let $\mathrm{Sym}_{<\omega}(\omega)$ denote the subgroup of permutations with finite support. If there is a weakly tight family, then it is possible to build a subgroup $\mathrm{Sym}_{<\omega}(\omega) \leq G \leq \mathrm{Sym}(\omega)$ such that $G/\mathrm{Sym}_{<\omega}(\omega)$ is a maximal Abelian subgroup of $\mathrm{Sym}(\omega)/\mathrm{Sym}_{<\omega}(\omega)$ with the property that for any $\{g_n : n \in \omega\} \subset \mathrm{Sym}(\omega)$, if $\forall n \in \omega \exists^\infty g \in G\big[[g, g_n] \neq \text{ id mod } \mathrm{Sym}_{<\omega}(\omega)\big]$, then there is a $g \in G$ such that $\exists^\infty n \in \omega\big[[g, g_n] \neq \text{ id mod } \mathrm{Sym}_{<\omega}(\omega)\big]$.

Till now, it was only known how to get weakly tight families of size $\mathfrak{c}$ from $\mathfrak{b} = \mathfrak{c}$. It was also known how to construct ones of size possibly less than $\mathfrak{c}$ from either $\mathfrak{a} < \mathrm{cov}(\mathcal{M})$ or $\diamondsuit(\mathfrak{d})$. These methods fail to distinguish weakly tight families in any way from $\aleph_0$ MAD families.

In this paper, we prove that weakly tight families exist when $\mathfrak{s} < \mathfrak{b}$, and that they also exist when $\mathfrak{s} = \mathfrak{b}$ provided that a certain PCF type hypothesis holds. By a PCF type hypothesis, we mean a hypothesis about $\mathrm{cf}(\langle[\kappa]^\omega, \subset\rangle)$ for some cardinal κ. Such hypotheses typically hold below $\aleph_\omega$. Our construction is a modification of Shelah [18], which in turn is a modification of the classic constructions of Balcar, Dočkálková, and Simon. Shelah [18] shows that there is a completely separable MAD family in any of the following three situations:

Case 1: $\mathfrak{s} < \mathfrak{a}$;
Case 2: $\mathfrak{s} = \mathfrak{a}$ and a certain PCF type hypothesis holds;
Case 3: $\mathfrak{a} < \mathfrak{s}$ plus a stronger PCF type assumption.

Therefore, we prove the exact analogues of Shelah's cases 1 and 2 for weakly tight families, except that we compare $\mathfrak{s}$ to $\mathfrak{b}$ instead of $\mathfrak{a}$. However, we cannot prove the analogue of case 2, and we conjecture that it cannot be done (see Conjecture 23).

However, our approach is somewhat different from [18]. We first introduce a new cardinal invariant $\mathfrak{s}_{\omega,\omega}$, and prove outright in ZFC that a weakly tight family exists if $\mathfrak{s}_{\omega,\omega} \leq \mathfrak{b}$.

Definition 4 $\mathfrak{s}_{\omega,\omega}$ is the least κ such that there is a family $\{e_\alpha : \alpha < \kappa\} \subset [\omega]^\omega$ such that for any collection $\{b_n : n \in \omega\} \subset [\omega]^\omega$, there exists $\alpha < \kappa$ such that $\exists^\infty n \in \omega[|b_n \cap e^0_\alpha| = \omega]$ and $\exists^\infty n \in \omega[|b_n \cap e^1_\alpha| = \omega]$.

An advantage of our approach is that it shows that the PCF hypothesis can be eliminated from case 2 so long as one is willing to replace $\mathfrak{s}$ with $\mathfrak{s}_{\omega,\omega}$.[1] Indeed, our proof shows that this will also work for completely separable MAD families—*i.e.*, we can prove (in ZFC) that they exist under $\mathfrak{s}_{\omega,\omega} \leq \mathfrak{a}$. Also, it is easy to show, by the same argument as for $\mathfrak{s}$, that $\mathfrak{s}_{\omega,\omega} \leq \mathfrak{d}$. So, as a corollary, we get in ZFC that weakly

[1] In work with Mildenberger, we have recently shown that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. So the use of PCF hypotheses can be entirely eliminated from case 2, both in Shelah's construction and in ours. This work will appear in a future publication.

tight families exist under $\mathfrak{b} = \mathfrak{d}$. We don't know if $\mathfrak{b} = \mathfrak{d}$ yields the PCF assumption $P(\mathfrak{b})$ (see Definition 14) used in the proof of the $\mathfrak{s} = \mathfrak{b}$ case. In Section 3 we first show in ZFC that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$ when $\mathfrak{s} < \mathfrak{b}$, thus getting case 1 as a corollary. Then we show that if $\kappa = \mathfrak{s} = \mathfrak{b}$ and $P(\kappa)$ holds, then a weakly tight family can be constructed. Here $P(\kappa)$ is our PCF type hypothesis, and it appears to be slightly stronger than the one used by Shelah for his case 2. $P(\kappa)$ is always true for $\kappa < \aleph_\omega$, so we get weakly tight families when $\mathfrak{s} = \omega_1$.

It is also worth pointing out here that one cannot construct an $\aleph_0$-MAD family of size $\mathfrak{c}$ from $\mathfrak{s} \leq \mathfrak{b} < \aleph_\omega$, because in the Cohen model there is a weakly tight MAD family of size $\mathfrak{c}$, but no $\aleph_0$-MAD families of that size.

We now make some general remarks on the basic method. Suppose $\kappa = \mathfrak{s}$. First each node η of $2^{<\kappa}$ is labelled with a subset of ω, say e_η. Each member of the a.d. family under construction is "associated" with a node, and the idea is that whenever two sets are associated with incomparable nodes, they are automatically a.d. This is ensured by specifying at each node of $2^{<\kappa}$ a collection of subsets of ω that are "allowed" to be associated with that node. Then most of the argument goes into showing that at any stage $\alpha < \mathfrak{c}$ there is a perfect set of nodes with which a_α is allowed to be associated. Here a_α is the member of the a.d. family constructed at stage α. This means that a_α can be associated with a node that is incomparable with "most" (all but fewer than $\mathfrak{s}$) of the nodes with which some a_β has already been associated. So a_α will be automatically a.d. from most of the previous a_β.

For constructing a completely separable MAD family, we can simply require that a set a is allowed to be associated with a node η iff for each node $\tau \subsetneq \eta$, a is either almost included in e_τ or almost disjoint from e_τ, depending on which way η went at $\operatorname{dom}(\tau)$. However, this requirement is too strong for building a weakly tight family. Recall that a *partitioner* of an a.d. family $\mathcal{A}$ is a set $b \in \mathcal{I}^+(\mathcal{A})$ with the property that $\forall a \in \mathcal{A}[a \subset^* b \vee |a \cap b| < \omega]$. It is clear that any $\mathcal{A}$ that is subject to the above mentioned constraint will have an infinite pairwise disjoint family of partitioners. However, such an $\mathcal{A}$ must necessarily fail to be weakly tight. We deal with this using two innovations. First, each member of the a.d. family will be associated with a countable collection of nodes instead of one single node, and will be the union of a countable sequence of infinite subsets of ω. Second, each such countable sequence will be associated with its own node, and the collection $\mathcal{I}_\eta$ of countable sequences allowable at a node η will be defined so as to ensure almost disjointness (see Definition 6).

We believe that these adaptations we have introduced for building a weakly tight family will be of use in getting other kinds of MAD families with few partitioners (see Conjecture 24) by helping us to replace assumptions of the form $\mathfrak{x} = \mathfrak{c}$ with weaker hypotheses of the form $\mathfrak{x} \leq \mathfrak{y}$. Eventually they should either show us how to do a ZFC construction or tell us where to look for a consistency proof.

2 The Main Construction

In this section we give the PCF free construction of a weakly tight family.

Theorem 5 *If $\mathfrak{s}_{\omega,\omega} \leq \mathfrak{b}$, then there is a weakly tight family of size $\mathfrak{c}$. In particular, such* *families exist if* $\mathfrak{b} = \mathfrak{d}$ *or if* $\mathfrak{s}_{\omega,\omega} = \omega_1$.

The construction given here and the one in Section 3 are very similar; we could have presented a single, unified construction, and then derived the two results as corollaries. However, we have chosen to separate them because we feel that the construction presented in this section is the easiest one to follow, and a reader who has understood it should have no difficulty in assimilating the modifications made to it in Section 3. We first establish some notation.

For any $e \subset \omega$, we use e^0 to denote e and e^1 to denote $\omega \setminus e$. Next, we give the definition of $\mathcal{I}_\eta$, which should be thought of as the collection of sequences of sets that are allowable at η.

Definition 6 We say that a sequence $\vec{C} = \langle c_n : n \in \omega \rangle \subset [\omega]^\omega$ is a *p.w.d.* if for any $n \neq m$, $c_n \cap c_m = 0$. Define

$$\mathcal{C} = \{\vec{C} : \vec{C} \text{ is a p.w.d.}\}.$$

Let κ be an infinite cardinal, and let $\langle e_\alpha : \alpha < \kappa \rangle \subset [\omega]^\omega$. For an $\eta \in 2^{\leq \kappa}$, we define

$$\mathcal{I}_\eta(\langle e_\alpha : \alpha < \kappa \rangle) = \{\vec{C} \in \mathcal{C} : \forall \beta < \operatorname{dom}(\eta) \forall^\infty n \in \omega [\vec{C}(n) \subset e_\beta^{\eta(\beta)}]\}.$$

We will often omit the $\langle e_\alpha : \alpha < \kappa \rangle$ because it will be clear from the context.

Lemma 7 *Let* $\langle e_\alpha : \alpha < \kappa \rangle$ *witness* $\kappa = \mathfrak{s}_{\omega,\omega}$. *Let* $\mathcal{A} \subset [\omega]^\omega$ *be any a.d. family. Then for each* $b \in \mathcal{I}^+(\mathcal{A})$, *there is an* $\alpha < \kappa$ *such that* $b \cap e_\alpha^0 \in \mathcal{I}^+(\mathcal{A})$ *and* $b \cap e_\alpha^1 \in \mathcal{I}^+(\mathcal{A})$.

Proof There are two cases to consider. Suppose first that there are only finitely many $a \in \mathcal{A}$ with $|b \cap a| = \omega$. Then since $b \in \mathcal{I}^+(\mathcal{A})$, there is a $c \in [b]^\omega$ that is a.d. from every member of $\mathcal{A}$. Now choose $\alpha < \kappa$ such that $|c \cap e_\alpha^0| = |c \cap e_\alpha^1| = \omega$. It is clear that this α is as required.

Next, suppose that there is an infinite collection $\{a_n : n \in \omega\} \subset \mathcal{A}$ with $|b \cap a_n| = \omega$ for each $n \in \omega$. Put $c_n = b \cap a_n$ and choose $\alpha < \kappa$ such that $\exists^\infty n \in \omega[|c_n \cap e_\alpha^0| = \omega]$ and $\exists^\infty n \in \omega[|c_n \cap e_\alpha^1| = \omega]$. Now both $b \cap e_\alpha^0$ and $b \cap e_\alpha^1$ are in $\mathcal{I}^+(\mathcal{A})$ because they both have infinite intersection with infinitely many members of $\mathcal{A}$. ■

Fix a sequence $\langle e_\alpha : \alpha < \kappa \rangle$ witnessing $\kappa = \mathfrak{s}_{\omega,\omega}$. We will construct an increasing sequence of subtrees of $2^{<\kappa}$ by induction on $\mathfrak{c}$. The weakly tight family $\mathcal{A} \subset [\omega]^\omega$ will be constructed along with these subtrees. At a stage $\alpha < \mathfrak{c}$, we are given an increasing sequence $\langle \mathcal{T}_\beta : \beta < \alpha \rangle$ of subtrees of $2^{<\kappa}$, as well as an almost disjoint family $\{a_\beta : \beta < \alpha\}$. Thus $\mathcal{T}^\alpha = \bigcup_{\beta < \alpha} \mathcal{T}_\beta$ is a subtree of $2^{<\kappa}$. Now we ensure that for each $\beta < \alpha$, $a_\beta = \bigcup_{n \in \omega} d_n^\beta$, where $\vec{D}^\beta = \langle d_n^\beta : n \in \omega \rangle$ is a p.w.d. Moreover, to each a_β and each d_n^β, we associate nodes $\eta(a_\beta) \in \mathcal{T}_\beta$ and $\eta(d_n^\beta) \in \mathcal{T}_\beta$ in such a way that the following conditions are satisfied:

$$(\dagger_{a_\beta}) \qquad \vec{D}^\beta \in \mathcal{I}_{\eta(a_\beta)},$$

$$(\dagger_{d_n^\beta}) \qquad \forall \gamma < \operatorname{dom}\left(\eta(d_n^\beta)\right) [d_n^\beta \subset^* e_\gamma^{\eta(d_n^\beta)(\gamma)}].$$

It will also be important that $\eta(a_\beta) \neq \eta(a_\gamma)$ for all $\gamma < \beta < \alpha$, that $\eta(d^\beta_n) \neq \eta(d^\gamma_m)$ for all $\langle \beta, n\rangle \neq \langle \gamma, m\rangle$ where $\beta, \gamma < \alpha$ and $n, m \in \omega$, and also that $\eta(a_\beta) \neq \eta(d^\gamma_m)$ for all $\beta, \gamma < \alpha$ and $m \in \omega$. Moreoever, for each $\beta < \alpha$, $\mathcal{T}_\beta$ is simply equal to

$$\mathcal{T}^\beta \cup \{\sigma \in 2^{<\kappa} : \sigma \subset \eta(a_\beta)\} \cup \{\sigma \in 2^{<\kappa} : \exists n \in \omega[\sigma \subset \eta(d^\beta_n)]\},$$

whence

$$\mathcal{T}^\alpha = \left\{ \sigma \in 2^{<\kappa} : \exists \xi < \alpha \left[\sigma \subset \eta(a_\xi) \vee \exists n \in \omega[\sigma \subset \eta(d^\xi_n)] \right] \right\}.$$

Thus, T^α is always a union of fewer than $\mathfrak{c}$ chains.

The next lemma says that at each stage $\alpha < \mathfrak{c}$, it is not the case that $\{a_\beta : \beta < \alpha\}$ is already a MAD family "somewhere", *i.e.*, there is no positive set on which this family is already MAD. Having this be the case is, of course, essential if we are to meet all our $\mathfrak{c}$ many requirements. This lemma is already sufficient for constructing a completely separable MAD family from $\mathfrak{s}_{\omega,\omega} \leq \mathfrak{a}$. For a weakly tight family, we need an analogue of this for p.w.d.s (Lemma 10).

Lemma 8 *Let $b \in \mathcal{I}^+(\{a_\beta : \beta < \alpha\})$. Let $\mathcal{T}^\alpha \subset \mathcal{T}$ be a subtree of $2^{<\kappa}$ that is a union of fewer than $\mathfrak{c}$ chains. There is a $c \in [b]^\omega$ that is a.d. from a_β for every $\beta < \alpha$, and a $\tau \in (2^{<\kappa}) \setminus \mathcal{T}$ such that $\forall \delta < \operatorname{dom}(\tau)[c \subset^* e^{\tau(\delta)}_\delta]$.*

Proof Put $\mathcal{A}_\alpha = \{a_\beta : \beta < \alpha\}$. Build a perfect subtree $\mathcal{P} = \{\sigma_s : s \in 2^{<\omega}\}$ of $2^{<\kappa}$ as follows. To obtain σ_0 apply Lemma 7 to find the least $\gamma_0 < \kappa$ such that both $b \cap e^0_{\gamma_0}$ and $b \cap e^1_{\gamma_0}$ are in $\mathcal{I}^+(\mathcal{A}_\alpha)$. It follows that it is possible to define $\sigma_0 \colon \gamma_0 \to 2$ by $\sigma_0(\delta) = i$ iff $b \cap e^i_\delta \in \mathcal{I}^+(\mathcal{A}_\alpha)$ for each $\delta < \gamma_0$. Now suppose $\{\sigma_s : s \in 2^{\leq n}\} \subset 2^{<\kappa}$ and $\{\gamma_s : s \in 2^{\leq n}\} \subset \kappa$ have been constructed. For $s \in 2^{\leq n+1}$, define $e(s)$ as follows. Let $e(0)$ denote ω. Given $e(s)$ for $s \in 2^{\leq n}$, let $e(s^\frown\langle i\rangle) = e(s) \cap e^i_{\gamma_s}$. Note that $e(s^\frown\langle i\rangle) \subset e(s)$. Now assume that the following properties hold:

(1) $\forall s \in 2^{\leq n}[\operatorname{dom}(\sigma_s) = \gamma_s]$ and $\forall s \in 2^{<n}[\sigma_{s^\frown\langle i\rangle} \supset \sigma_s{}^\frown\langle i\rangle]$.
(2) $\forall s \in 2^{\leq n}[b \cap e(s^\frown\langle 0\rangle) \in \mathcal{I}^+(\mathcal{A}_\alpha)$ and $b \cap e(s^\frown\langle 1\rangle) \in \mathcal{I}^+(\mathcal{A}_\alpha)]$
(3) $\forall s \in 2^{\leq n}\, \forall \delta < \gamma_s\, [\sigma_s(\delta) = i$ iff $b \cap e(s) \cap e^i_\delta \in \mathcal{I}^+(\mathcal{A}_\alpha)]$.

Note that condition (3) entails that for each $s \in 2^{\leq n}$ and $\delta < \gamma_s$, $b \cap e(s) \cap e^{1-\sigma_s(\delta)}_\delta \notin \mathcal{I}^+(\mathcal{A}_\alpha)$. Now, given $s \in 2^{\leq n}$ and $i \in 2$, apply Lemma 7 to find the least $\gamma < \kappa$ such that both $b \cap e(s^\frown\langle i\rangle) \cap e^0_\gamma$ and $b \cap e(s^\frown\langle i\rangle) \cap e^1_\gamma$ are in $\mathcal{I}^+(\mathcal{A}_\alpha)$. Again, for each $\delta < \gamma$, there is a unique $j \in 2$ such that $b \cap e(s^\frown\langle i\rangle) \cap e^j_\delta \in \mathcal{I}^+(\mathcal{A}_\alpha)$. Moreover, by (3), for each $\delta < \gamma_s$, $b \cap e(s^\frown\langle i\rangle) \cap e^{1-\sigma_s(\delta)}_\delta \notin \mathcal{I}^+(\mathcal{A}_\alpha)$. Also, $b \cap e(s^\frown\langle i\rangle) \cap e^{1-i}_{\gamma_s} = 0$. Therefore, $\gamma > \gamma_s$. Thus if we define $\gamma_{s^\frown\langle i\rangle} = \gamma$ and $\sigma_{s^\frown\langle i\rangle} \colon \gamma_{s^\frown\langle i\rangle} \to 2$ by $\sigma_{s^\frown\langle i\rangle}(\delta) = j$ iff $b \cap e(s^\frown\langle i\rangle) \cap e^j_\delta \in \mathcal{I}^+(\mathcal{A}_\alpha)$ for each $\delta < \gamma_{s^\frown\langle i\rangle}$, then $\sigma_{s^\frown\langle i\rangle} \supset \sigma_s^\frown\langle i\rangle$ and conditions (1)–(3) hold.

Now, since $\mathcal{T}$ is a union of fewer than $\mathfrak{c}$ chains, there is $f \in 2^\omega$ such that $\tau = \bigcup_{n\in\omega} \sigma_{f\restriction n} \notin \mathcal{T}$. Notice that $b \cap e(f\restriction 0) \supset b \cap e(f\restriction 1) \supset \cdots$ is a decreasing sequence of sets in $\mathcal{I}^+(\mathcal{A}_\alpha)$. Therefore, we may choose $b_0 \in [b]^\omega$ such that $b_0 \in \mathcal{I}^+(\mathcal{A}_\alpha)$, and $b_0 \subset^* b \cap e(f\restriction n)$ for each $n \in \omega$. We claim that for all $\delta < \gamma = \sup\{\gamma_{f\restriction n} : n \in \omega\}$, $b_0 \cap e^{1-\tau(\delta)}_\delta \notin \mathcal{I}^+(\mathcal{A}_\alpha)$. Indeed, if $\delta < \gamma$, then $\delta < \gamma_{f\restriction n}$ for some $n \in \omega$, and so by (3), $b \cap e(f\restriction n) \cap e_\delta^{1-\tau(\delta)} \notin \mathcal{I}^+(\mathcal{A}_\alpha)$. And since $b_0 \subset^* b \cap e(f\restriction n)$, the claim follows. Therefore, for each $\delta < \gamma$, there is a finite set $\mathcal{F}_\delta \subset \mathcal{A}_\alpha$ such that $b_0 \cap e_\delta^{1-\tau(\delta)} \subset^* \bigcup \mathcal{F}_\delta$. Put $\mathcal{F} = \bigcup_{\delta<\gamma} \mathcal{F}_\delta$, and observe that $|\mathcal{F}| \le |\gamma|$. Observe also that since $\gamma_{f\restriction n} < \gamma_{f\restriction(n+1)}$, γ is a limit ordinal and that $\mathrm{cf}(\gamma) = \omega$. Next, put

$$\mathcal{G} = \left\{ a_\beta : [\beta < \alpha] \wedge \left[\eta(a_\beta) \subset \tau \vee \exists n \in \omega[\eta(d_n^\beta) \subset \tau] \right] \right\},$$

and note that $|\mathcal{G}| \le |\gamma|$ and $|\mathcal{F} \cup \mathcal{G}| \le |\gamma|$. Now, if there exists a set $c \in [b_0]^\omega$ which is a.d. from every $a \in \mathcal{F} \cup \mathcal{G}$, then for each $\delta < \gamma$, $c \cap e_\delta^{1-\tau(\delta)}$ is finite, and hence $c \subset^* e_\delta^{\tau(\delta)}$. We claim that such a c must be a.d. from every $a_\beta \in \mathcal{A}_\alpha$. Fix $a_\beta \in \mathcal{A}_\alpha$, and recall that $a_\beta = \bigcup_{n\in\omega} d_n^\beta$, where $\vec{D}^\beta = \langle d_n^\beta : n \in \omega \rangle$ is a p.w.d. Since $\tau \notin \mathcal{T}$, $\eta(a_\beta) \not\supset \tau$, and there is no $n \in \omega$ such that $\eta(d_n^\beta) \supset \tau$. If either $\eta(a_\beta) \subset \tau$, or there exists an $n \in \omega$ such that $\eta(d_n^\beta) \subset \tau$, then $a_\beta \in \mathcal{G}$, and $c \cap a_\beta$ is finite. So suppose that there is a $\delta < \min\left\{ \gamma, \mathrm{dom}\big(\eta(a_\beta)\big) \right\}$ such that $\tau(\delta) \ne \eta(a_\beta)(\delta)$, and also that for each $n \in \omega$, there is a $\delta_n < \min\left\{ \gamma, \mathrm{dom}\big(\eta(d_n^\beta)\big) \right\}$ such that $\tau(\delta_n) \ne \eta(d_n^\beta)(\delta_n)$. Since $\vec{D}^\beta \in \mathcal{I}_{\eta(a_\beta)}$ by $(\dagger_{a_\beta})$, there is a $k \in \omega$ so that $\forall n \ge k[d_n^\beta \subset e_\delta^{\eta(a_\beta)(\delta)}]$, and $c \subset^* e_\delta^{1-\eta(a_\beta)(\delta)}$. Therefore, $\bigcup_{n\ge k} d_n^\beta \subset e_\delta^{\eta(a_\beta)(\delta)}$, and so $c \cap (\bigcup_{n\ge k} d_n^\beta) \subset c \cap e_\delta^{\eta(a_\beta)(\delta)}$, which is finite. Thus,

$$c \cap a_\beta =^* c \cap \Big(\bigcup_{n<k} d_n^\beta \Big),$$

and so it suffices to show that $c \cap d_n^\beta$ is finite for each $n < k$. But for each such n, $c \subset^* e_{\delta_n}^{\tau(\delta_n)}$, while $d_n^\beta \subset^* e_{\delta_n}^{1-\tau(\delta_n)}$ because of $(\dagger_{d_n^\beta})$, giving us the desired conclusion.

We next argue that there must be a $c \in [b_0]^\omega$ which is a.d. from every $a \in \mathcal{F} \cup \mathcal{G}$. There are two cases to consider here. First, suppose that $\mathrm{cf}(\mathfrak{s}_{\omega,\omega}) \ne \omega$. In this case, $\gamma < \mathfrak{s}_{\omega,\omega} \le \mathfrak{b} \le \mathfrak{a}$, and so $|\mathcal{F} \cup \mathcal{G}| < \mathfrak{a}$. Since $b_0 \in \mathcal{I}^+(\mathcal{A}_\alpha)$, there is a c as required. Also, since $\mathrm{dom}(\tau) = \gamma$, we have that $\tau \in (2^{<\kappa}) \setminus \mathcal{T}$, which is as required.

Next, suppose that $\mathrm{cf}(\mathfrak{s}_{\omega,\omega}) = \omega$. Then γ could equal $\mathfrak{s}_{\omega,\omega}$ *a priori*. However, we claim that this cannot happen. To see this, note that since $\mathfrak{b}$ is regular, in this case, we have that $\mathfrak{s}_{\omega,\omega} < \mathfrak{b}$, and so $|\mathcal{F} \cup \mathcal{G}| \le \mathfrak{s}_{\omega,\omega} < \mathfrak{b} \le \mathfrak{a}$. So again, since $b_0 \in \mathcal{I}^+(\mathcal{A}_\alpha)$, there is $c \in [b_0]^\omega$ which is a.d. from every $a \in \mathcal{F} \cup \mathcal{G}$. Now we have argued above that for any such c, $\forall \delta < \gamma[c \subset^* e_\delta^{\tau(\delta)}]$. So if $\gamma = \mathfrak{s}_{\omega,\omega}$, then there would be no $\delta < \mathfrak{s}_{\omega,\omega}$ such that e_δ split c, contradicting the definition of $\mathfrak{s}_{\omega,\omega}$. Therefore, $\gamma < \mathfrak{s}_{\omega,\omega} = \kappa$, and again $\tau \in \big(2^{<\kappa}\big) \setminus \mathcal{T}$, as needed. ∎

Definition 9 We say that a p.w.d. $\vec{D}$ *refines* another such p.w.d. $\vec{C}$, and write $\vec{D} \prec \vec{C}$, if there is a sequence $\langle k_n : n \in \omega \rangle \subset \omega$ such that $\forall n \in \omega[k_{n+1} > k_n]$ and $\forall n \in \omega[\vec{D}(n) \subset \vec{C}(k_n)]$. Given $e \in [\omega]^\omega$ and $i \in \omega$, $e(i)$ denotes the i-th element of e. Given a p.w.d. $\vec{C}$, and $e \in [\omega]^\omega$, $\vec{C}\restriction e$ is the p.w.d. defined by $(\vec{C}\restriction e)(n) = \vec{C}\big(e(n)\big)$ for each $n \in \omega$. It is clear that $\prec$ is a transitive relation, and that $\forall \vec{C} \in \mathcal{C} \forall e \in [\omega]^\omega[\vec{C}\restriction e \prec \vec{C}]$.

The next lemma is the analogue of Lemma 8 for p.w.d.s. It is here that comparing $\mathfrak{s}$ to $\mathfrak{b}$ rather than to $\mathfrak{a}$ becomes important.

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Lemma 10 *Let $\mathcal{A}_\alpha = \{a_\beta : \beta < \alpha\}$. Suppose that $\vec{C}$ is a p.w.d. such that for each $n \in \omega$, $\vec{C}(n)$ is a.d. from every member of $\mathcal{A}_\alpha$. There is an $\eta \in 2^{<\kappa}$ and a $\vec{D} \in \mathfrak{I}_\eta$ such that*

(i) $\vec{D} \prec \vec{C}$,
(ii) $\exists^\infty n \in \omega[|\vec{D}(n) \cap e^0_{\mathrm{dom}(\eta)}| = \omega]$,
(iii) $\exists^\infty n \in \omega[|\vec{D}(n) \cap e^1_{\mathrm{dom}(\eta)}| = \omega]$.

Proof By definition of $\mathfrak{s}_{\omega,\omega}$, there is a $\gamma < \kappa$ such that $\exists^\infty n \in \omega[|\vec{C}(n) \cap e^0_\gamma| = \omega]$ and $\exists^\infty n \in \omega[|\vec{C}(n) \cap e^1_\gamma| = \omega]$. Choose the least such γ. So for each $\delta < \gamma$, there is a unique $j \in 2$ such that $\exists^\infty n \in \omega[|\vec{C}(n) \cap e^j_\delta| = \omega]$. Define $\eta\colon \gamma \to 2$ by $\eta(\delta) = j$ iff $\exists^\infty n \in \omega[|\vec{C}(n) \cap e^j_\delta| = \omega]$ for all $\delta < \gamma$. To get $\vec{D}$, note that for each $\delta < \gamma$, there is a $k_\delta \in \omega$ such that $\forall n \geq k_\delta[|\vec{C}(n) \cap e^{1-\eta(\delta)}_\delta| < \omega]$. So we can define a function $f_\delta\colon \omega \to \omega$ by $f_\delta(n) = \max\left(\vec{C}(n) \cap e^{1-\eta(\delta)}_\delta\right)$ for each $n \geq k_\delta$, and $f_\delta(n) = 0$, for each $n < k_\delta$. Since $\gamma < \mathfrak{s}_{\omega,\omega} \leq \mathfrak{b}$, find a function $f \in \omega^\omega$ such that for each $\delta < \gamma$, $\forall^\infty n \in \omega[f(n) > f_\delta(n)]$. Put $\vec{D}(n) = \vec{C}(n) \setminus f(n)$. It is clear that $\vec{D} \prec \vec{C}$. Also, since $\vec{D}(n) =^* \vec{C}(n)$, (2) and (3) are satisfied by the choice of γ. Finally to see that $\vec{D} \in \mathfrak{I}_\eta$, fix $\delta < \gamma$. There is an $m \in \omega$ such that $\forall n \geq m[f(n) > f_\delta(n)]$. Now suppose that $n \geq \max\{k_\delta, m\}$. Then if $l \in \vec{D}(n)$, then $l \in \vec{C}(n)$ and $l > \max\left(\vec{C}(n) \cap e^{1-\eta(\delta)}_\delta\right)$, whence $l \in e^{\eta(\delta)}_\delta$. Thus we have shown that $\forall^\infty n \in \omega[\vec{D}(n) \subset e^{\eta(\delta)}_\delta]$. ∎

The next lemma is easy, but plays a crucial role in the construction, and depends a lot on having the right definition of $\mathfrak{I}_\eta$. It is a sticking point in further applications of this technique that needs to be resolved each time by finding a definition of $\mathfrak{I}_\eta$ that is appropriate for the specific type of a.d. family being sought.

Lemma 11 *Suppose that $\langle \sigma_n : n \in \omega \rangle \subset 2^{<\kappa}$, $\langle \gamma_n : n \in \omega \rangle \subset \kappa$, and $\langle \vec{C}_n : n \in \omega \rangle \subset \mathfrak{C}$ are sequences such*

(i) $\forall n \in \omega[\mathrm{dom}(\sigma_n) = \gamma_n$ *and* $\gamma_{n+1} > \gamma_n$ *and* $\sigma_{n+1} \supset \sigma_n]$,
(ii) $\forall n \in \omega[\vec{C}_n \in \mathfrak{I}_{\sigma_n}$ *and* $\vec{C}_{n+1} \prec \vec{C}_n]$.

Then there is a p.w.d. $\vec{D} \in \mathfrak{I}_\sigma$, where $\sigma = \bigcup_{n\in\omega}\sigma_n$, with the property that $\forall n \in \omega \left[\left(\vec{D}\restriction[n,\omega)\right) \prec \vec{C}_n\right]$.

Proof Simply define a p.w.d. $\vec{D}$ by $\vec{D}(n) = \vec{C}_n(n)$. Note that $\vec{D}$ is indeed a p.w.d. because if $n < l$, then since $\vec{C}_l \prec \vec{C}_n$, $\vec{D}(l) = \vec{C}_l(l) \subset \vec{C}_n(k_l)$ for some $k_l \geq l > n$. Therefore, $\vec{C}_n(n) \cap \vec{C}_n(k_l) = 0$, and so $\vec{D}(n) \cap \vec{D}(l) = 0$. Put $\gamma = \sup\{\gamma_n : n \in \omega\}$, and note that $\gamma \leq \kappa$ is a limit ordinal with $\mathrm{cf}(\gamma) = \omega$. Now we claim that for each $\delta < \gamma$, $\forall^\infty n \in \omega[\vec{D}(n) \subset e^{\sigma(\delta)}_\delta]$. Indeed, given $\delta < \gamma$, fix $i \in \omega$ such that $\delta < \gamma_i$. Now there is an $m \in \omega$ such that $\forall n \geq m[\vec{C}_i(n) \subset e^{\sigma(\delta)}_\delta]$. Suppose $n \geq \max\{m, i\}$. Then since $\vec{C}_n \prec \vec{C}_i$, there is a $k_n \geq n$ such that $\vec{D}(n) = \vec{C}_n(n) \subset \vec{C}_i(k_n) \subset e^{\sigma(\delta)}_\delta$. It is also clear that $\vec{D}\restriction[n,\omega) \prec \vec{C}_n$ holds for each $n \in \omega$. ∎

Proof of Theorem 5 The argument will be similar in structure to the proof of Lemma 8. Suppose that at stage $\alpha < \mathfrak{c}$, we are given a collection $\{b_n : n \in \omega\} \subset [\omega]^\omega$ such that for each $n \in \omega$, $b_n \in \mathfrak{I}^+(\mathcal{A}_\alpha)$, where $\mathcal{A}_\alpha = \{a_\beta : \beta < \alpha\}$. We want to find an a_α that is a.d. from $\mathcal{A}_\alpha$ with the property that $|a_\alpha \cap b_n| = \omega$ for infinitely many $n \in \omega$. Moreover, we want to enlarge $\mathcal{T}^\alpha$ to a bigger subtree, $\mathcal{T}_\alpha$, of $2^{<\kappa}$, as well as find a p.w.d. $\vec{D}^\alpha$, and nodes $\eta(a_\alpha)$ and $\eta\big(\vec{D}^\alpha(n)\big)$ in $\mathcal{T}_\alpha$, in such a way that $a_\alpha = \bigcup_{n\in\omega} \vec{D}^\alpha(n)$, $(\dagger_{a_\alpha})$, and $(\dagger_{\vec{D}^\alpha(n)})$ hold.

First find $c_n \in [b_n]^\omega$ and nodes $\tau_n \in 2^{<\kappa}$ as follows. Given $\{c_i : i < n\}$ and $\{\tau_i : i < n\}$, apply Lemma 8 with b_n as b and $\mathcal{T}^\alpha \cup \{\tau_i \restriction \delta : i < n \wedge \delta \leq \operatorname{dom}(\tau_i)\}$ as $\mathcal{T}$ to find $c_n \in [b_n]^\omega$ that is a.d. from every $a \in \mathcal{A}_\alpha$ and a node $\tau_n \in (2^{<\kappa}) \setminus \mathcal{T}$ such that

$$(*_1) \qquad \forall \delta < \operatorname{dom}(\tau_n)[c_n \subset^* e_\delta^{\tau_n(\delta)}].$$

We may also assume, by shrinking them further if necessary, that $c_n \cap c_m = 0$ for all $n \neq m$. Now construct a perfect subtree $\mathcal{P} = \{\sigma_s : s \in 2^{<\omega}\}$ of $2^{<\kappa}$ together with a collection of ordinals $\{\gamma_s : s \in 2^{<\omega}\} \subset \kappa$, and a collection of p.w.d.s $\{\vec{C}_s : s \in 2^{<\omega}\}$ so that the following conditions are satisfied.

(1) $\forall s \in 2^{<\omega} \forall i \in 2[\operatorname{dom}(\sigma_s) = \gamma_s \wedge \sigma_{s^\frown\langle i\rangle} \supset \sigma_s{}^\frown\langle i\rangle]$.

(2) $\forall s \in 2^{<\omega}\big[\exists^\infty n \in \omega[|\vec{C}_s(n) \cap e^0_{\gamma_s}| = \omega] \wedge \exists^\infty n \in \omega[|\vec{C}_s(n) \cap e^1_{\gamma_s}| = \omega]\big]$.

(3) $\forall s \in 2^{<\omega} \forall i \in 2[\vec{C}_s \in \mathcal{I}_{\sigma_s} \wedge \vec{C}_{s^\frown\langle i\rangle} \prec \vec{C}_s]$.

To start with, define a p.w.d. $\vec{E}_0$ by $\vec{E}_0(n) = c_n$. Now suppose that $\vec{E}_s \prec \vec{E}_0$ is given for some $s \in 2^{<\omega}$. To obtain σ_s, apply Lemma 10 to $\vec{E}_s$ to find $\sigma_s \in 2^{<\kappa}$ and a $\vec{C}_s \prec \vec{E}_s$ such that $\vec{C}_s \in \mathcal{I}_{\sigma_s}$, and $\exists^\infty n \in \omega[|\vec{C}_s(n) \cap e^0_{\gamma_s}| = \omega]$ and $\exists^\infty n \in \omega[|\vec{C}_s(n) \cap e^1_{\gamma_s}| = \omega]$, where $\gamma_s = \operatorname{dom}(\sigma_s)$. Now, for each $i \in 2$, let $\langle n^i_j : j \in \omega\rangle$ enumerate $\{n \in \omega : |\vec{C}_s(n) \cap e^i_{\gamma_s}| = \omega\}$ in strictly increasing order and define a p.w.d. $\vec{E}_{s^\frown\langle i\rangle}$ by $\vec{E}_{s^\frown\langle i\rangle}(j) = \vec{C}_s(n^i_j) \cap e^i_{\gamma_s}$. It is clear that (2) is satisfied. (3) will be satisfied because $\vec{E}_{s^\frown\langle i\rangle} \prec \vec{C}_s$, and therefore, $\vec{C}_{s^\frown\langle i\rangle} \prec \vec{E}_{s^\frown\langle i\rangle} \prec \vec{C}_s$. To see that (1) holds, note that $\vec{E}_{s^\frown\langle i\rangle} \in \mathcal{I}_{((\sigma_s)^\frown\langle i\rangle)}$. Since $\gamma_{s^\frown\langle i\rangle} = \operatorname{dom}(\sigma_{s^\frown\langle i\rangle})$ is chosen in such a way that $\exists^\infty n \in \omega[|\vec{C}_{s^\frown\langle i\rangle}(n) \cap e^0_{\gamma_{(s^\frown\langle i\rangle)}}| = \omega]$ and $\exists^\infty n \in \omega[|\vec{C}_{s^\frown\langle i\rangle}(n) \cap e^1_{\gamma_{(s^\frown\langle i\rangle)}}| = \omega]$, it follows that $\gamma_{s^\frown\langle i\rangle} > \gamma_s$. Moreover, since $\vec{C}_{s^\frown\langle i\rangle} \in \mathcal{I}_{\sigma_{(s^\frown\langle i\rangle)}}$, if there is a $\delta \leq \gamma_s$ such that $(\sigma_s)^\frown\langle i\rangle(\delta) \neq \sigma_{s^\frown\langle i\rangle}(\delta)$, then there would be an $n \in \omega$ such that $\vec{C}_{s^\frown\langle i\rangle}(n) \subset e^0_\delta$ and $\vec{C}_{s^\frown\langle i\rangle}(n) \subset e^1_\delta$, which is impossible. Therefore, $\sigma_{s^\frown\langle i\rangle} \supset (\sigma_s)^\frown\langle i\rangle$, and so (1) is satisfied.

Now put $\mathcal{T} = \mathcal{T}^\alpha \cup \{\tau_n \restriction \delta : n < \omega \wedge \delta \leq \operatorname{dom}(\tau_n)\}$, and note that $\mathcal{T}$ is the union of fewer than $\mathfrak{c}$ chains. Therefore, there is an $f \in 2^\omega$ such that $\tau = \bigcup_{n\in\omega} \sigma_{f\restriction n} \notin \mathcal{T}$. By (1)–(3), we have that $\operatorname{dom}(\sigma_{f\restriction n}) = \gamma_{f\restriction n}$, that $\gamma_{f\restriction n+1} > \gamma_{f\restriction n}$, that $\sigma_{f\restriction n+1} \supset \sigma_{f\restriction n}$, that $\vec{C}_{f\restriction n} \in \mathcal{I}_{(\sigma_{f\restriction n})}$, and that $\vec{C}_{f\restriction n+1} \prec \vec{C}_{f\restriction n}$. So the hypotheses of Lemma 11 are satisfied and we can find a p.w.d. $\vec{E} \in \mathcal{I}_\tau$ with $\vec{E} \prec \vec{C}_0 \prec \vec{E}_0$. We set $\eta(a_\alpha) = \tau$. Notice that $\operatorname{dom}(\tau) = \gamma = \sup\{\gamma_{f\restriction n} : n \in \omega\}$. Clearly, $\gamma \leq \kappa$ is a limit ordinal, and $\operatorname{cf}(\gamma) = \omega$. To see that $\gamma \neq \kappa$, we argue as in Lemma 8. If $\gamma = \kappa$, then since $\vec{E} \in \mathcal{I}_\tau$, there is no $\delta < \kappa$ so that $\exists^\infty n \in \omega[|\vec{E}(n) \cap e^0_\delta| = \omega]$ and $\exists^\infty n \in \omega[|\vec{E}(n) \cap e^1_\delta| = \omega]$, contradicting the definition of $\mathfrak{s}_{\omega,\omega}$. Thus $\gamma < \kappa$, and so $\eta(a_\alpha) \in 2^{<\kappa}$, as needed.

Next, to define $\vec{D}^\alpha$, proceed as follows. Since $\vec{E} \prec \vec{E}_0$, $\vec{E}(n)$ is a.d. from $\mathcal{A}_\alpha$ for each $n \in \omega$. For each $\delta < \gamma$, if either there exists $\beta < \alpha$ such that $\eta(a_\beta) = \tau \restriction \delta$ or there exists a $\beta < \alpha$ and $m \in \omega$ with $\eta(d^\beta_m) = \tau \restriction \delta$, we define a function $f_\delta \in \omega^\omega$ as follows. Given $n \in \omega$, we set $f_\delta(n) = \max\big(\vec{E}(n) \cap a_\beta\big)$, where β, assuming it exists, is the unique $\beta < \alpha$ such that either $\eta(a_\beta) = \tau{\restriction}\delta$ or $\eta(d^\beta_m) = \tau{\restriction}\delta$ for some $m \in \omega$. Notice that since $\gamma < \mathfrak{s}_{\omega,\omega} \leq \mathfrak{b}$, we can find a function $f \in \omega^\omega$ such that

$$\forall \delta < \gamma \quad [\text{if } f_\delta \text{ is defined, then } f_\delta <^* f].$$

Now define $\vec{D}^\alpha$ by $\vec{D}^\alpha(n) = \vec{E}(n) \setminus f(n)$ for each $n \in \omega$. It is clear that $\vec{D}^\alpha \prec \vec{E}$, and therefore, $\vec{D}^\alpha \in \mathcal{I}_\tau$. So $(\dagger_{a_\alpha})$ is satisfied. Next, we put

$$(*_3) \qquad a_\alpha = \bigcup_{n\in\omega} \vec{D}^\alpha(n).$$

Suppose that the relation $\vec{D}^\alpha \prec \vec{E}_0$ is witnessed by the sequence $\langle k_n : n \in \omega\rangle$. Notice that for each $n \in \omega$, $\vec{D}^\alpha(n) \in [b_{k_n}]^\omega$, and hence that $|a_\alpha \cap b_{k_n}| = \omega$. Now, for each $n \in \omega$, set $\eta\big(\vec{D}^\alpha(n)\big) = \tau_{k_n}$. By $(*_1)$, we have that for each $n \in \omega$, $\forall \delta < \operatorname{dom}\big(\eta\big(\vec{D}^\alpha(n)\big)\big)\,[\vec{D}^\alpha(n) \subset^* e_\delta^{\eta(\vec{D}^\alpha(n))(\delta)}]$, hence $(\dagger_{\vec{D}^\alpha(n)})$ is satisfied. Note also, that for each $i \in \omega$, $\tau_i \notin \mathcal{T}^\alpha$, and therefore, for each $\beta < \alpha$ and $m \in \omega$, $\eta\big(\vec{D}^\alpha(n)\big) \neq \eta(a_\beta)$, and $\eta\big(\vec{D}^\alpha(n)\big) \neq \eta(d^\beta_m)$. Also, since $\tau_i \neq \tau_j$ whenever $i \neq j$, we have that $\eta\big(\vec{D}^\alpha(n)\big) \neq \eta\big(\vec{D}^\alpha(m)\big)$ whenever $n \neq m$. And similarly, since $\eta(a_\alpha)$ is not in $\mathcal{T}^\alpha \cup \{\tau_n{\restriction}\delta : n < \omega \wedge \delta \leq \operatorname{dom}(\tau_n)\}$, we have that $\eta(a_\alpha) \neq \eta\big(\vec{D}^\alpha(n)\big)$, for any $n \in \omega$, and also that for any $\beta < \alpha$ and $m \in \omega$, $\eta(a_\alpha) \neq \eta(a_\beta)$, and $\eta(a_\alpha) \neq \eta(d^\beta_m)$. Therefore, we may set

$$(*_4) \qquad \mathcal{T}_\alpha = \mathcal{T}^\alpha \cup \{\tau_{k_n}{\restriction}\delta : n < \omega \wedge \delta \leq \operatorname{dom}(\tau_{k_n})\} \cup \{\tau{\restriction}\delta : \delta \leq \operatorname{dom}(\tau)\}.$$

It only remains to be seen that $a_\alpha \cap a_\beta$ is finite for each $\beta < \alpha$. Fix $\beta < \alpha$. There are two cases to consider. Suppose first that either $\eta(a_\beta) \subsetneq \tau$ or that there is an $m \in \omega$ so that $\eta(d^\beta_m) \subsetneq \tau$. In this case, f_δ is defined as above, and $\exists k \in \omega \forall n \geq k[f(n) > f_\delta(n) = \max\big(\vec{E}(n) \cap a_\beta\big)]$. It follows that $a_\alpha \cap a_\beta \subset a_\beta \cap \big(\bigcup_{n<k} \vec{D}^\alpha(n)\big)$, which is finite.

Now suppose that for every $\delta < \gamma$, $\eta(a_\beta) \neq \tau{\restriction}\delta$, and also that for every $m \in \omega$ and every $\delta < \gamma$, $\eta(d^\beta_m) \neq \tau{\restriction}\delta$. Since $\tau \notin \mathcal{T}^\alpha$, it follows that $\tau \not\subset \eta(a_\beta)$, and also that for each $m \in \omega$, $\tau \not\subset \eta(d^\beta_m)$. Therefore, there is a $\delta < \min\big\{\gamma, \operatorname{dom}\big(\eta(a_\beta)\big)\big\}$ such that $\tau(\delta) \neq \eta(a_\beta)(\delta)$, as well as $\delta_m < \min\big\{\gamma, \operatorname{dom}\big(\eta(d^\beta_m)\big)\big\}$ such that $\tau(\delta_m) \neq \eta(d^\beta_m)(\delta_m)$, for each $m \in \omega$. Hence there are $k_\alpha \in \omega$ and $k_\beta \in \omega$ such that $\forall n \geq k_\beta[d^\beta_n \subset e_\delta^{1-\tau(\delta)}]$ and $\forall n \geq k_\alpha[\vec{D}^\alpha(n) \subset e_\delta^{\tau(\delta)}]$. Put $d = \bigcup_{n\geq k_\beta} d^\beta_n$. Notice that

$$a_\alpha \cap d = \Big(\bigcup_{n<k_\alpha} \big(\vec{D}^\alpha(n) \cap d\big)\Big) \cup \Big(\bigcup_{n\geq k_\alpha} \big(\vec{D}^\alpha(n) \cap d\big)\Big),$$

and this is finite because $\vec{D}^\alpha(n) \cap d = 0$ when $n \geq k_\alpha$, and $\vec{D}^\alpha(n) \cap d$ is finite for all $n \in \omega$ since $\vec{D}^\alpha(n)$ is a.d. from a_β. So it suffices to show that $a_\alpha \cap (\bigcup_{n<k_\beta} d^\beta_n)$ is finite, and for this it is enough to show that $a_\alpha \cap d^\beta_n$ is finite for every $n \in \omega$. To see this, fix $n \in \omega$. By assumption, there is a $k \in \omega$ such that $\forall m \geq k[\vec{D}^\alpha(m) \subset e_{\delta_n}^{\tau(\delta_n)}]$, while $d^\beta_n \subset^* e_{\delta_n}^{1-\tau(\delta_n)}$. It follows that $d^\beta_n \cap a_\alpha \subset \big(\bigcup_{m<k}\big(d^\beta_n \cap \vec{D}^\alpha(m)\big)\big) \cup (d^\beta_n \cap e_{\delta_n}^{\tau(\delta_n)})$, which is finite because $\vec{D}^\alpha(m)$ is a.d. from a_β, and hence from d^β_n. ■

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3 Using PCF-type Assumptions

In this section, we show that $\mathfrak{s}_{\omega,\omega}$ can be replaced in Theorem 5 by $\mathfrak{s}$ in the presence of a relatively weak PCF type hypothesis. This hypothesis is only needed when $\mathfrak{s} = \mathfrak{b}$; when $\mathfrak{s} < \mathfrak{b}$ we get a ZFC result. In fact, we are able to show that if $\mathfrak{s} < \mathfrak{b}$, then $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$, so Theorem 5 can be directly applied. This gives us an exact analogue of case 1 of Shelah's construction, where he gets a completely separable MAD family from $\mathfrak{s} < \mathfrak{a}$ without further hypotheses.

When $\mathfrak{s} = \mathfrak{b}$ we seem to need a slightly stronger hypothesis than the one used by Shelah. For his construction Shelah uses the following.

Definition 12 For a cardinal $\kappa > \omega$, $U(\kappa)$ is the following principle. There is a sequence $\langle u_\alpha : \omega \leq \alpha < \kappa \rangle$ such that

(i) $u_\alpha \subset \alpha$ and $|u_\alpha| = \omega$,
(ii) $\forall X \in [\kappa]^\kappa \exists \omega \leq \alpha < \kappa [|u_\alpha \cap X| = \omega]$.

It is easily seen that $U(\kappa)$ holds whenever $\kappa < \aleph_\omega$, and more generally whenever $\mathrm{cf}(\langle [\kappa]^\omega, \subset \rangle) = \kappa$. Shelah [18] (see Section 2) showed that if $\kappa = \mathfrak{s} = \mathfrak{a}$ and $U(\kappa)$ holds, then there is there is a completely separable MAD family. Our result will use the principle $P(\kappa)$ given below. But we first dispose of the easy case when $\mathfrak{s} < \mathfrak{b}$.

Theorem 13 *If $\mathfrak{s} < \mathfrak{b}$, then $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. So there is a weakly tight family of size $\mathfrak{c}$ under $\mathfrak{s} < \mathfrak{b}$.*

Proof Let $\langle e_\alpha : \alpha < \kappa \rangle$ witness that $\kappa = \mathfrak{s}$. Suppose $\{b_n : n \in \omega\} \subset [\omega]^\omega$ is a countable collection such that $\forall \alpha < \kappa \exists i \in 2 \forall^\infty n \in \omega [b_n \subset^* e^i_\alpha]$. By shrinking them if necessary we may assume that $b_n \cap b_m = 0$ whenever $n \neq m$. Now, for each $\alpha < \kappa$ define $f_\alpha \in \omega^\omega$ as follows. We know that there is a unique $i_\alpha \in 2$ such that there is a $k_\alpha \in \omega$ such that $\forall n \geq k_\alpha [|b_n \cap e^{i_\alpha}_\alpha| < \omega]$. We define $f_\alpha(n) = \max(b_n \cap e^{i_\alpha}_\alpha)$ if $n \geq k_\alpha$, and $f_\alpha(n) = 0$ if $n < k_\alpha$. As $\kappa < \mathfrak{b}$, there is a $f \in \omega^\omega$ with $f^* > f_\alpha$ for each $\alpha < \kappa$. Now, for each $n \in \omega$, choose $l_n \in b_n$ with $l_n \geq f(n)$. Since the b_n are pairwise disjoint, $c = \{l_n : n \in \omega\} \in [\omega]^\omega$. So by definition of $\mathfrak{s}$, there is $\alpha < \kappa$ such that $|c \cap e^0_\alpha| = |c \cap e^1_\alpha| = \omega$. In particular, $c \cap e^{i_\alpha}_\alpha$ is infinite. But we know that there is an $m_\alpha \in \omega$ such that $\forall n \geq m_\alpha [f_\alpha(n) < f(n)]$. So there exists $n \geq \max\{m_\alpha, k_\alpha\}$ with $l_n \in b_n \cap e^{i_\alpha}_\alpha$. But this is a contradiction because $l_n \leq f_\alpha(n) < f(n)$. ■

Definition 14 For a cardinal $\kappa > \omega$, $P(k)$ is the following principle. There is a sequence $\langle u_\alpha : \omega \leq \alpha < \kappa \rangle$ such that

(i) $u_\alpha \subset \alpha$ and $|u_\alpha| = \omega$,
(ii) $\forall \{X_n : n \in \omega\} \subset [\kappa]^\kappa \exists \omega \leq \alpha < \kappa \exists^\infty n \in \omega [u_\alpha \cap X_n \neq 0]$.

Again, it is easy to see that $P(\kappa)$ holds whenever $\mathrm{cf}(\langle [\kappa]^\omega, \subset \rangle) = \kappa$. Also, it is clear that $P(\kappa) \Rightarrow U(\kappa)$. We don't know whether these principles are different. We also do not know of a model where $\kappa = \mathfrak{s} = \mathfrak{b}$ and $P(\kappa)$ fails. Similarly, it is not known whether $U(\kappa)$ can fail when $\kappa = \mathfrak{s} = \mathfrak{a}$, which is the hypothesis relevant to case 2 of Shelah's construction.

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The argument for the next lemma is well known and fairly standard. It allows us to assume that the order type of each u_α is ω, and plays an important role in the construction below. We include a proof for the reader's convenience.

Lemma 15 *Suppose* $\mathfrak{b} \le \kappa$ *and* $P(\kappa)$ *holds. Then there is a family* $\langle u_\alpha : \omega \le \alpha < \kappa\rangle$ *as in Definition* 14 *with* $\operatorname{otp}(u_\alpha) = \omega$, *for each* $\omega \le \alpha < \kappa$.

Proof It is sufficient to show that for each set $y \subset \kappa$ with $|y| = \omega$ there is a family $\langle y_\gamma : \gamma < \kappa\rangle$ with

(a) $y_\gamma \subset y$ and $\operatorname{otp}(y_\gamma) = \omega$,
(b) $\forall x \in [y]^\omega \exists \gamma < \kappa[|x \cap y_\gamma| = \omega]$.

Clearly, we may assume that $\operatorname{otp}(y)$ is a limit ordinal. We will prove this claim by induction on $\operatorname{otp}(y)$. If $\operatorname{otp}(y) = \omega$, then there is nothing to do. For any $\xi < \operatorname{otp}(y)$, let $y(\xi)$ denote the ξ-th element of y. If $\operatorname{otp}(y) = \delta + \omega$ for some limit δ, then let $z = \{y(\xi) : \xi < \delta\}$ and let $\langle z_\gamma : \gamma < \kappa\rangle$ be a family satisfying (a) and (b) with respect to z. Now simply let $\langle y_\gamma : \gamma < \kappa\rangle$ be an enumeration of $\{y(\delta+n) : n < \omega\} \cup \{z_\gamma : \gamma < \kappa\}$. Suppose that $\operatorname{otp}(y)$ is a limit of limits. Let $\langle \delta_n : n \in \omega\rangle$ be an increasing sequence of limit ordinals converging to $\delta = \operatorname{otp}(y)$. Put $z_n = \{y(\xi) : \delta_{n-1} \le \xi < \delta_n\}$, where δ_{-1} is taken to be 0. Let $\langle z^n_\gamma : \gamma < \kappa\rangle$ be a family satisfying (a) and (b) with respect to z_n. Now let $\langle f_\alpha : \alpha < \mathfrak{b}\rangle$ be a family in ω^ω which is unbounded with respect to infinite partial functions from ω to ω, and let $\{\zeta^n_i : i \in \omega\}$ be an enumeration of z_n. For each $\alpha < \mathfrak{b}$, define a set ${y'}_\alpha = \{\zeta^n_i : i \le f_\alpha(n)\}$. Notice that $\operatorname{otp}({y'}_\alpha) = \omega$. Let $\langle y_\gamma : \gamma < \kappa\rangle$ enumerate $(\bigcup_{n\in\omega}\{z^n_\gamma : \gamma < \kappa\}) \cup \{{y'}_\alpha : \alpha < \mathfrak{b}\}$. We check that this family satisfies (b) with respect to y. Fix $x \in [y]^\omega$. If $x \cap z_n$ is infinite for some $n \in \omega$, then there is a $\gamma < \kappa$ so that $|x \cap z^n_\gamma| = \omega$. On the other hand, if $x \cap z_n$ is finite for each $n \in \omega$, then $\exists^\infty n \in \omega[x \cap z_n \neq 0]$. So we may pick a strictly increasing sequence $\langle k_n : n \in \omega\rangle \subset \omega$ and $\{i_{k_n} : n \in \omega\} \subset \omega$ such that $\zeta^{k_n}_{i_{k_n}} \in x$ for each $n \in \omega$. There is an $\alpha < \kappa$ so that $\exists^\infty n \in \omega[f_\alpha(k_n) \ge i_{k_n}]$. Now it is clear that $|x \cap {y'}_\alpha| = \omega$. ∎

Theorem 16 *Assume* $\kappa = \mathfrak{s} = \mathfrak{b}$ *and that* $P(\kappa)$ *holds. Then there is a weakly tight family of size* $\mathfrak{c}$. *In particular, such families exist if* $\mathfrak{s} \le \mathfrak{b} < \aleph_\omega$, *and in particular, when* $\mathfrak{s} = \omega_1$.

The proof of Theorem 16 is very similar to the proof of Theorem 5. The main difference will be that instead of using a sequence of sets $\langle e_\alpha : \alpha < \kappa\rangle$, we will construct a tree $\langle e_\eta : \eta \in 2^{<\kappa}\rangle$. So the pair of sets e^0, e^1 used at a node of the tree will now depend not just on the height of that node, but on all the pairs of sets that occur below that node. The idea is that along each cofinal branch ψ of the tree, each countable collection of κ-sized subsets ψ can be "captured" at some node η that lies on ψ using $P(\kappa)$. Then e_η is chosen in such a way that for any $\{b_n : n \in \omega\} \subset [\omega]^\omega$, if $\{X_n : n \in \omega\}$ is the countable collection of κ-sized subsets of ψ "captured" at η, where X_n is the set of nodes on ψ where b_n "hits the other side", then $\exists^\infty n \in \omega[|b_n \cap e_\eta^{1-\psi(\operatorname{dom}(\eta))}| = \omega]$. While the basic idea is the same as in cases 2 and 3 of Shelah's construction, there is one crucial difference here. An appropriate e_η is chosen in Shelah's construction using a $\mathfrak{b}$ family (quite similarly to what is done in Lemma 15), while we use an $\mathfrak{s}$ family for this. If we could replace the $\mathfrak{s}$ family in our construction by a $\mathfrak{b}$ family, then we would also be able to prove the analogue of

Shelah's case 3—*i.e.*, we would be able to get a weakly tight family from $\mathfrak{b} < \mathfrak{s} < \aleph_\omega$. But we suspect that there are fundamental reasons for not being able to do this (see Conjecture 23).

Proof of Theorem 16 First construct $\langle e_\eta : \eta \in 2^{<\kappa}\rangle \subset \mathcal{P}(\omega)$ as follows. Let $\kappa = \bigcup_{\alpha<\kappa} S_\alpha$ be a partition of κ so that $|S_\alpha| = \kappa$ and $S_\alpha \cap \alpha = 0$ for each $\alpha < \kappa$. Let $\langle u_\alpha : \omega \leq \alpha < \kappa\rangle$ witness that $P(\kappa)$ holds. By Lemma 15, we may assume that $\operatorname{otp}(u_\alpha) = \omega$. Now, for each $\alpha < \kappa$, let $\langle \tilde{e}_\gamma : \gamma \in S_\alpha\rangle$ witness that $\kappa = \mathfrak{s}$. We define e_η by induction on $\operatorname{dom}(\eta)$. Assume $\operatorname{dom}(\eta) = \gamma < \kappa$, and that for each $\beta < \gamma$, $e_{\eta\restriction\beta} \subset \omega$ has been defined. Suppose $\gamma \in S_\alpha$. If $\alpha < \omega$, then let $e_\eta = \tilde{e}_\gamma$. If $\alpha \geq \omega$, we proceed as follows. Since u_α has order type ω, enumerate it in strictly increasing order as $u_\alpha = \{\xi_i^\alpha : i < \omega\}$. Since $\gamma \geq \alpha > \xi_i^\alpha$, $e_{\eta\restriction\xi_i^\alpha}$ has already been defined. For each $i < \omega$, we put

$$c_i^\eta = e_{\eta\restriction\xi_i^\alpha}^{1-\eta(\xi_i^\alpha)} \cap \Big(\bigcap_{j<i} e_{\eta\restriction\xi_j^\alpha}^{\eta(\xi_j^\alpha)}\Big).$$

Notice that $c_i^\eta \cap c_j^\eta = 0$ for all $i \neq j$. We then define

$$e_\eta = \bigcup_{i\in\tilde{e}_\gamma} c_i^\eta.$$

This completes the definition of $\langle e_\eta : \eta \in 2^{<\kappa}\rangle$. The next lemma establishes the key property of this family, which will give the analogues of Lemmas 7, 8, and 10. ■

Lemma 17 *Let $\{b_n : n \in \omega\} \subset [\omega]^\omega$, and let $\psi \in 2^\kappa$. Then there is a $\gamma < \kappa$ such that $\exists^\infty n \in \omega[|b_n \cap e_{\psi\restriction\gamma}^{1-\psi(\gamma)}| = \omega]$.*

Proof Suppose not. Fix $\psi \in 2^\kappa$ such that for all $\gamma < \kappa$, $\forall^\infty n \in \omega[b_n \subset^* e_{\psi\restriction\gamma}^{\psi(\gamma)}]$. For each $n \in \omega$, define

$$X_n = \{\gamma < \kappa : |b_n \cap e_{\psi\restriction\gamma}^{1-\psi(\gamma)}| = \omega\}.$$

We claim that $|X_n| = \kappa$. Indeed, suppose, for a contradiction, that $|X_n| < \kappa$. Put $\mathcal{F} = \{e_{\psi\restriction\gamma} : \gamma \in X_n\}$. This is a family of subsets of ω of size less than $\kappa = \mathfrak{s}$. So we may find a $c \in [b_n]^\omega$ such that for each $\gamma \in X_n$, there is an $i \in 2$ so that $c \subset^* e_{\psi\restriction\gamma}^i$. However, $\langle \tilde{e}_\gamma : \gamma \in S_0\rangle$ enumerates a splitting family. So there is a $\gamma \in S_0$ satisfying $|c \cap e_{\psi\restriction\gamma}^0| = |c \cap e_{\psi\restriction\gamma}^1| = \omega$. In particular, $|b_n \cap e_{\psi\restriction\gamma}^{1-\psi(\gamma)}| = \omega$, and so $\gamma \in X_n$. But this is a contradiction because $c \subset^* e_{\psi\restriction\gamma}^i$.

Now choose $\omega \leq \alpha < \kappa$ such that $\exists^\infty n \in \omega[u_\alpha \cap X_n \neq 0]$. We choose two strictly increasing sequences $\langle k_m : m \in \omega\rangle \subset \omega$ and $\langle i_m : m \in \omega\rangle \subset \omega$ as follows. Let k_0 be the least $n \in \omega$ such that $u_\alpha \cap X_n \neq 0$, and let i_0 be the least $i \in \omega$ such that $\xi_i^\alpha \in X_{k_0}$. Suppose that k_m and i_m are given to us with $\xi_{i_m}^\alpha \in X_{k_m}$. Put

$$s = \{n \in \omega : \exists i \leq i_m |b_n \cap e_{\psi\restriction\xi_i^\alpha}^{1-\psi(\xi_i^\alpha)}| = \omega\}.$$

Since for each $i \leq i_m$, $\forall^\infty n \in \omega[b_n \subset^* e_{\psi\restriction\xi_i^\alpha}^{\psi(\xi_i^\alpha)}]$, s is a finite set. So we may choose $k_{m+1} \in \omega$ such that $u_\alpha \cap X_{k_{m+1}} \neq 0$ and such that $k_{m+1} > n$ for all $n \in s$. Observe that $k_m \in s$, and so $k_{m+1} > k_m$. Now i_{m+1} is defined to be the least $i \in \omega$ such that $\xi_i^\alpha \in X_{k_{m+1}}$. Since $k_{m+1} \notin s$, $i_{m+1} > i_m$. Notice that each i_m is defined so that $\xi_{i_m}^\alpha \in X_{k_m}$ and $\forall i < i_m[\xi_i^\alpha \notin X_{k_m}]$. It follows that for each $m \in \omega$

$$(*) \qquad \Big| b_{k_m} \cap e_{\psi\restriction\xi_{i_m}^\alpha}^{1-\psi(\xi_{i_m}^\alpha)} \cap \Big(\bigcap_{i<i_m} e_{\psi\restriction\xi_i^\alpha}^{\psi(\xi_i^\alpha)}\Big)\Big| = \omega.$$

Next, choose $\gamma \in S_\alpha$ such that $\exists^\infty m \in \omega[i_m \in \tilde{e}_\gamma^0]$ and $\exists^\infty m \in \omega[i_m \in \tilde{e}_\gamma^1]$. Note that $\gamma \geq \alpha$. Put $\eta = \psi\restriction\gamma$. It follows from $(*)$ that for each $m \in \omega$, $|b_{k_m} \cap c_{i_m}^\eta| = \omega$. Therefore, $\exists^\infty m \in \omega[|b_{k_m} \cap e_\eta^0| = \omega]$. On the other hand, since c_i^η and c_j^η are disjoint whenever $i \neq j$, we also get $\exists^\infty m \in \omega[|b_{k_m} \cap e_\eta^1| = \omega]$. But this contradicts our initial hypothesis about ψ, and we are done. ∎

Observe that Lemma 17 is not saying that $\langle e_{\psi\restriction\gamma} : \gamma < \kappa\rangle$ is an $\mathfrak{s}_{\omega,\omega}$ family for each $\psi \in 2^\kappa$. That would prove $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$, given $\kappa = \mathfrak{s} = \mathfrak{b}$ and $P(\kappa)$. For this, we would need $\gamma < \kappa$ so that $\exists^\infty n \in \omega[|b_n \cap e_{\psi\restriction\gamma}^{1-\psi(\gamma)}| = \omega]$ and $\exists^\infty n \in \omega[|b_n \cap e_{\psi\restriction\gamma}^{\psi(\gamma)}| = \omega]$, which is not proved. But Lemma 17 is still good enough for proving the following analogue of Lemma 7.

Lemma 18 *Let $\mathcal{A} \subset [\omega]^\omega$ be an a.d. family. Let $b \in \mathcal{I}^+(\mathcal{A})$, and let $\eta \in 2^{<\kappa}$. Assume that $\forall\beta < \operatorname{dom}(\eta)[b \cap e_{\eta\restriction\beta}^{1-\eta(\beta)} \notin \mathcal{I}^+(\mathcal{A})]$. Then there is a $\tau \in 2^{<\kappa}$ with $\tau \supset \eta$ such that*

(i) $\forall\beta < \operatorname{dom}(\tau)[b \cap e_{\tau\restriction\beta}^{1-\tau(\beta)} \notin \mathcal{I}^+(\mathcal{A})]$,

(ii) *$b \cap e_\tau^0 \in \mathcal{I}^+(\mathcal{A})$ and $b \cap e_\tau^1 \in \mathcal{I}^+(\mathcal{A})$.*

Proof Suppose not. In other words, assume that for any $\tau \in 2^{<\kappa}$, if $\tau \supset \eta$ and if $\forall\beta < \operatorname{dom}(\tau)[b \cap e_{\tau\restriction\beta}^{1-\tau(\beta)} \notin \mathcal{I}^+(\mathcal{A})]$, then there is an $i \in 2$ such that $b \cap e_\tau^i \notin \mathcal{I}^+(\mathcal{A})$. This allows us to build a $\psi \in 2^\kappa$ with $\eta \subset \psi$ and with the property that $\forall\beta < \kappa[b \cap e_{\psi\restriction\beta}^{1-\psi(\beta)} \notin \mathcal{I}^+(\mathcal{A})]$. Now there exists a collection $\{b_n : n \in \omega\} \subset [b]^\omega$ with the property that for any $c \in [\omega]^\omega$, if c has infinite intersection with infinitely many b_n, then $c \in \mathcal{I}^+(\mathcal{A})$. Applying Lemma 17 to ψ and $\{b_n : n \in \omega\}$, we get a $\gamma < \kappa$ such that $\exists^\infty n \in \omega[|b_n \cap e_{\psi\restriction\gamma}^{1-\psi(\gamma)}| = \omega]$. But since $b_n \subset b$, we have that $\exists^\infty n \in \omega[|b_n \cap b \cap e_{\psi\restriction\gamma}^{1-\psi(\gamma)}| = \omega]$. It follows that $b \cap e_{\psi\restriction\gamma}^{1-\psi(\gamma)} \in \mathcal{I}^+(\mathcal{A})$, contradicting the way we constructed ψ. ∎

The next definition specifies the analogue of $\mathcal{I}_\eta$ in the present context. It is simply the obvious modification of $\mathcal{I}_\eta$.

Definition 19 For any $\eta \in 2^{<\kappa}$, we define

$$J_\eta = \{\vec{C} \in \mathcal{C} : \forall\gamma < \operatorname{dom}(\eta)\forall^\infty n \in \omega[\vec{C}(n) \subset e_{\eta\restriction\gamma}^{\eta(\gamma)}]\}.$$

The next lemma proves the analogue of Lemma 10. That $\kappa = \mathfrak{b}$ is important here.

Lemma 20 *Let $\vec{C}$ be a p.w.d. and let $\eta \in 2^{<\kappa}$. Assume $\vec{C} \in J_\eta$. Then there exists $\tau \in 2^{<\kappa}$ with $\tau \supset \eta$ and $\vec{D} \prec \vec{C}$ such that*

(i) *$\vec{D} \in J_\tau$ and*

(ii) $\exists^\infty n \in \omega[|\vec{D}(n) \cap e^0_\tau| = \omega]$ *and* $\exists^\infty n \in \omega[|\vec{D}(n) \cap e^1_\tau| = \omega]$.

Proof Suppose not. In other words, for any $\tau \in 2^{<\kappa}$, *if* $\tau \supset \eta$, and *if* there exists a $\vec{D} \prec \vec{C}$ with $\vec{D} \in J_\tau$, *then* there is an $i \in 2$ such that $\forall^\infty n \in \omega[|\vec{D}(n) \cap e^i_\tau| < \omega]$. Now construct a $\psi \in 2^\kappa$ with the property that for each $\gamma < \kappa$,

$$(*_\gamma) \qquad \forall^\infty n \in \omega[|\vec{C}(n) \cap e^{1-\psi(\gamma)}_{\psi\restriction\gamma}| < \omega],$$

contradicting Lemma 17. To see that this can be done, put $\psi\restriction \mathrm{dom}(\eta) = \eta$, and suppose that for some $\mathrm{dom}(\eta) \le \gamma < \kappa$, $\psi\restriction\gamma$ has been defined so that $(*_\beta)$ holds for each $\beta < \gamma$. Since $\gamma < \kappa = \mathfrak{b}$, we can find $\vec{D} \prec \vec{C}$ with $\vec{D} \in J_{\psi\restriction\gamma}$ and with the property that $\forall n \in \omega[\vec{D}(n) =^* \vec{C}(n)]$. So by the hypothesis there is $i \in 2$ so that $\forall^\infty n \in \omega[|\vec{D}(n) \cap e^i_{\psi\restriction\gamma}| < \omega]$. But since $\vec{D}(n) =^* \vec{C}(n)$ for all $n \in \omega$, if we set $\psi(\gamma) = 1 - i$, then ψ will be as needed. ■

Proof of Theorem 16 (continued) Armed with Lemmas 18 and 20, proceed exactly as in Theorem 5. At a stage $\alpha < \mathfrak{c}$, $\mathcal{A}_\alpha = \langle a_\beta : \beta < \alpha\rangle$, $\langle \mathfrak{T}_\beta : \beta < \alpha\rangle$, $\mathfrak{T}^\alpha$, $\langle \vec{D}^\beta : \beta < \alpha\rangle$ are all exactly as before. Now the nodes $\eta(a_\beta)$ and $\eta\big(\vec{D}^\beta(n)\big)$ satisfy

$$(\dagger\dagger_{a_\beta}) \qquad \vec{D}^\beta \in J_{\eta(a_\beta)},$$

$$(\dagger\dagger_{\vec{D}^\beta(n)}) \qquad \forall\gamma < \mathrm{dom}\left(\eta\big(\vec{D}^\beta(n)\big)\right)\left[\vec{D}^\beta(n) \subset^* e^{\eta(\vec{D}^\beta(n))(\gamma)}_{\eta(\vec{D}^\beta(n))\restriction\gamma}\right].$$

Given any $b \in \mathfrak{I}^+(\mathcal{A}_\alpha)$, apply Lemma 18 to construct $\{\sigma_s : s \in 2^{<\omega}\} \subset 2^{<\kappa}$, $\{b_s : s \in 2^{<\omega}\} \subset \mathfrak{I}^+(\mathcal{A})$, and $\{\gamma_s : s \in 2^{<\omega}\} \subset \kappa$ such that

(1) $\forall s \in 2^{<\omega}\forall i \in 2[\mathrm{dom}(\sigma_s) = \gamma_s \wedge \sigma_{s^\frown\langle i\rangle} \supset \sigma_s{}^\frown\langle i\rangle]$,

(2) $\forall s \in 2^{<\omega}\forall i \in 2\forall\gamma < \mathrm{dom}(\sigma_s)[b_s \cap e^{1-\sigma_s(\gamma)}_{\sigma_s\restriction\gamma} \notin \mathfrak{I}^+(\mathcal{A}_\alpha) \wedge b_{s^\frown\langle i\rangle} = b_s \cap e^i_{\sigma_s}]$,

(3) $b_0 = b$ and $\forall s \in 2^{<\omega}[b_s \cap e^0_{\sigma_s} \in \mathfrak{I}^+(\mathcal{A}_\alpha) \wedge b_s \cap e^1_{\sigma_s} \in \mathfrak{I}^+(\mathcal{A}_\alpha)]$.

If $\mathfrak{T}^\alpha \subset \mathfrak{T}$ is any subtree of $2^{<\kappa}$ that is the union of fewer than $\mathfrak{c}$ chains, there is a $f \in 2^\omega$ such that $\tau = \bigcup_{n\in\omega}\sigma_{f\restriction n} \notin \mathfrak{T}$. Also, there is $c_0 \in [b]^\omega \cap \mathfrak{I}^+(\mathcal{A}_\alpha)$ such that $c_0 \subset^* b_{f\restriction n}$ for all $n \in \omega$. Note that if $\delta < \gamma = \sup\{\gamma_{f\restriction n} : n \in \omega\}$, then $\delta < \gamma_{f\restriction n}$ for some $n \in \omega$, and so by (2), $b_{f\restriction n} \cap e^{1-\tau(\delta)}_{\tau\restriction\delta} \notin \mathfrak{I}^+(\mathcal{A}_\alpha)$. But since $c_0 \subset^* b_{f\restriction n}$, $c_0 \cap e^{1-\tau(\delta)}_{\tau\restriction\delta} \notin \mathfrak{I}^+(\mathcal{A}_\alpha)$. Now proceed exactly as in the proof of Lemma 8 to find $c \in [c_0]^\omega$ which is a.d. from everything in $\mathcal{A}_\alpha$ and with the property that $\forall\delta < \gamma[c \subset^* e^{\tau(\delta)}_{\tau\restriction\delta}]$ (in the present situation $\mathrm{cf}(\kappa) \ne \omega$; so it is obvious that $\gamma < \kappa$).

Therefore, given $\{b_n : n \in \omega\} \subset \mathfrak{I}^+(\mathcal{A}_\alpha)$, proceed as in the proof of Theorem 5 to find $c_n \in [b_n]^\omega$ and $\tau_n \in 2^{<\kappa}$ so that each c_n is a.d. from $\mathcal{A}_\alpha$, $\tau_n \ne \tau_m$ and $c_n \cap c_m = 0$ whenever $n \ne m$, and $\forall\delta < \mathrm{dom}(\tau_n)[c_n \subset^* e^{\tau(\delta)}_{\tau_n\restriction\delta}]$. Put $\vec{E}_0(n) = c_n$ and use Lemma 20 to define sequences $\langle\sigma_s : s \in 2^{<\omega}\rangle \subset 2^{<\kappa}$, $\{\gamma_s : s \in 2^{<\omega}\} \subset \kappa$, $\langle\vec{E}_s : s \in 2^{<\omega}\rangle$, and $\langle\vec{C}_s : s \in 2^{<\omega}\rangle$ satisfying

(1) $\forall s \in 2^{<\omega}\forall i \in 2[\mathrm{dom}(\sigma_s) = \gamma_s \wedge \sigma_{s^\frown\langle i\rangle} \supset \sigma_s{}^\frown\langle i\rangle]$,

(2) $\forall s \in 2^{<\omega}[\vec{C}_s \in J_{\sigma_s} \wedge \vec{C}_s \prec \vec{E}_s]$,

(3) $\forall s \in \omega\left[\exists^\infty n \in \omega[|\vec{C}_s(n) \cap e^0_{\sigma_s}| = \omega] \wedge \exists^\infty n \in \omega[|\vec{C}_s(n) \cap e^1_{\sigma_s}| = \omega]\right]$,

(4) $\forall s \in 2^{<\omega}\forall i \in 2\forall n \in \omega[\vec{E}_{s^\frown\langle i\rangle}(n) = \vec{C}_s(k_n) \cap e^i_{\sigma_s}]$, where $\langle k_n : n \in \omega\rangle$ is a strictly increasing enumeration of $\{n \in \omega : |\vec{C}_s(n) \cap e^i_{\sigma_s}| = \omega\}$.

There exists $f \in 2^\omega$ so that $\tau = \bigcup_{n\in\omega}\sigma_{f\restriction n} \notin \mathcal{T}$, where

$$\mathcal{T} = \mathcal{T}^\alpha \cup \{\tau_n\restriction\delta : n < \omega \wedge \delta \leq \mathrm{dom}\,(\tau_n)\}.$$

Applying Lemma 11 (which is still true in the present context) to $\langle \sigma_{f\restriction n} : n \in \omega\rangle$, $\langle \gamma_{f\restriction n} : n \in \omega\rangle$, and $\langle \vec{C}_{f\restriction n} : n \in \omega\rangle$, find $\vec{E} \in J_\tau$ with $\vec{E} \prec \vec{C}_0 \prec \vec{E}_0$. The rest of the verification is exactly as in the proof of Theorem 5. ∎

4 Some Open Questions

Question 21 *Does $\mathfrak{s}_{\omega,\omega} = \mathfrak{s}$?*

If $\mathfrak{s}_{\omega,\omega} \neq \mathfrak{s}$, then, by Theorem 13, $\mathfrak{b} \leq \mathfrak{s}$. When $\mathfrak{s} = \mathfrak{b}$ and $P(\mathfrak{s})$ holds, note that the proof of Theorem 16 is producing a tree of height $\mathfrak{s}$ with the property that the sets along each cofinal branch behave like an $\mathfrak{s}_{\omega,\omega}$ family, though they may not constitute such a family. We conjecture that when $\mathfrak{s} \leq \mathfrak{b} < \aleph_\omega$, $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$.[2]

Shelah's construction works by comparing $\mathfrak{s}$ with $\mathfrak{a}$, while we have compared $\mathfrak{s}$ with $\mathfrak{b}$. We don't know if $\mathfrak{a}$ can replace $\mathfrak{b}$ in our construction, but we suspect not.

Question 22 *Is there a weakly tight family if $\mathfrak{s} \leq \mathfrak{a} < \aleph_\omega$?*

Though we have established the analogues of Shelah's cases 1 and 2 for weakly tight families, we have not been able to do this for his case 3. This would require showing that weakly tight families exist when $\mathfrak{b} < \mathfrak{s}$ provided that some suitable PCF type hypothesis holds, and would imply the existence of such families under $\mathfrak{c} < \aleph_\omega$. But we doubt whether this can be done even when $\mathfrak{c} = \aleph_2$.

Conjecture 23 *There is a model of $\aleph_1 = \mathfrak{b} < \mathfrak{s} = \aleph_2 = \mathfrak{c}$ in which there are no weakly tight families.*

Shelah [19] first established the consistency of $\mathfrak{b} < \mathfrak{s}$. The method is flexible enough to prove the consistency of both $\mathfrak{a} = \mathfrak{b} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$. The method for proving the consistency of $\mathfrak{a} = \mathfrak{b} < \mathfrak{s}$ can be modified to produce a model of $\mathfrak{b} < \mathfrak{s}$ where a weakly tight family exists. Assuming CH in the ground model, it is possible to construct a weakly tight family whose weak tightness is not destroyed by the relevant iteration. However, this weakly tight family will not have size $\mathfrak{c}$, and we don't know if there are any of size $\mathfrak{s}$ in this model. Later, Brendle [3] found a way to prove the consistency of $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$ via a c.c.c. iteration. We do not know whether weakly tight families exist in either Shelah's or Brendle's model for $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$.

Conjecture 24 *If $\mathfrak{s} \leq \mathfrak{b} < \aleph_\omega$, then there is a Sacks indestructible MAD family.*

As mentioned in Section 1, we may assume that $\mathfrak{a} = \mathfrak{c}$ for proving Conjecture 24. The difficulty seems to be in finding the right definition of $\mathcal{I}_\eta$. We need a definition of $\mathcal{I}_\eta$ which will allow us to do a fusion argument along a branch of cofinality ω, and hence get the analogue of Lemma 11.

[2] It has since been shown that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$.

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