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RIGHT CYCLICALLY ORDERED GROUPS[1]

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This paper presents a study of right cyclically ordered groups (RCO-groups) and their relation to right ordered groups (RO-groups). Cyclically ordered groups (CO-groups) and their connection with ordered groups (O-groups) have been studied by Rieger in [7] and by Swierczkowski in [8]. While some of the properties of RCO-groups are analogous to the corresponding ones for CO-groups, there are interesting exceptions. One of these is the existence of torsion-free RCO-groups that cannot be right ordered. Every torsion-free CO-groups is ordered—this follows form Theorem 21 of [3] using the fact that if $G \in 0$, then $G/Z(G) \in 0$. On the other extreme we show that every RCO-group can be obtained from some RO-group by the same construction that yields CO-groups from O-groups.

Recall that a group G is said to be cyclically ordered if for some triplets a, b, c of distinct elements of G a ternery relation (a, b, c) is defined satisfying the following properties:

I. Exactly one of (a, b, c) and (a, c, b) holds

II. $(a, b, c) \Rightarrow (b, c, a)$

III. (a, b, c) and $(a, c, d) \Rightarrow (a, b, d)$

IV. $(a, b, c) \Rightarrow (ax, bx, cx)$ for all $x \in G$

V. $(a, b, c) \Rightarrow (ya, yb, yc)$ for all $y \in G$.

The class RCO is obtained by deleting condition V from the above list. Note that every RO-group is also an RCO-group under the relations (a, b, c) holds if and only if either $a<b<c$ or $b<c<a$ or $c<a<b$ (cf. [9]). An alternative way to define an RCO-group G is to view it as relation preserving permutation group of some cyclically ordered set Λ. From this we can conclude that the class of left cyclically ordered groups (obtained by deleting condition IV from the above list) is the same as the class RCO.

A subgroup C of an O-group or an RO-group is called convex if for any

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$x \in G$, $c \in C$, $e < x < c \Rightarrow x \in C$. If $X \subseteq G$, then write $\{X\}_G$ to denote the intersection of all convex subgroups of G containing the subgroup $\langle X \rangle$. We call $\{X\}_G$ the convex subgroup generated by X. Notice that if $\langle X \rangle \subset Z(G)$, then for any $c \in \{X\}_G$, there exists $g_1, g_2 \in \langle X \rangle$ such that $g_1 < c < g_2$. The following main result on the structure of RCO-groups is due to S. D. Zeleva [9].

THEOREM A. *If $G \in$ RO with an element $z \in Z(G)$, $z > e$, such that $\{z\}_G = G$, then $G = G/\langle z \rangle$ can be right cyclically ordered by the rule: $(\bar{a}, \bar{b}, \bar{c})$ holds if and only if $\gamma_a < \gamma_b < \gamma_c$ or $\gamma_b < \gamma_c < \gamma_a$ or $\gamma_c < \gamma_a < \gamma_b$; where $\gamma_a, \gamma_b, \gamma_c$ are the unique coset representatives of $\bar{a}, \bar{b}, \bar{c}$ satisfying $e < \gamma_a, \gamma_b, \gamma_c < z$. Conversely, every* RCO-*group K can be obtained from a suitable* RO-*group G using the above construction.*

The following result generalizes Zeleva's Theorem 1 in [9].

THEOREM B. *Any periodic* RCO-*group is abelian, and hence locally cyclic.*

Observe that the infinite dihedral group can be right cyclically ordered (see also Zeleva [9]). For we can right order the group

$$G = \langle a, b; b^{-1}ab = a^{-1} \rangle$$

by taking $P = \{a^\alpha b^\beta; \beta > 0$, or $\beta = 0$ and $\alpha > 0\}$ to be the positive cone. Under this order $\{b^2\}_G = G$ and of course $b^2 \in Z(G)$ so that $G/\langle b^2 \rangle \in$ RCO. Zeleva uses this example to show that the periodic elements of RCO group need not form a subgroup. The following result gives a necessary and sufficient condition for periodic elements of RCO group to form a subgroup.

THEOREM C. *Let $G \in$ RO, $z \in Z(G)$ and $\{z\}_G = G$. Then the periodic elements of $G/\langle z \rangle$ form a subgroup if and only if the isolator J of $\langle z \rangle$ in G lies in $Z(G)$.*

Recall that a subgroup H of G is called isolated if $g^n \in H$ implies $g \in H$ for all $g \in G$, $n > 0$. The isolator in G of a subgroup K is the intersection of all isolated subgroups of G containing K.

THEOREM D. *There exist torsion-free (metabelian and polycyclic)* RCO-*groups that are not* RO-*groups.*

The following result and its proof are due to Prof. A. H. Rhemtulla. The author wishes to thank him for his permission to include it in this paper.

THEOREM E. *There exist torsion-free groups that are not* RCO-*groups.*

It would be interesting to know if one could use the concept of right cyclical order to find out if the integral group rings of torsion-free RCO-groups have no zero divisors.

Proof of Theorem B. Let K be a periodic RCO-group. Then K is order isomorphic to $G/\langle z \rangle$ for some RO-group G with $z \in Z(G)$, $z > e$ and $\{z\}_G = G$.

We write (z^m, a), $m \in Z$, $a \in K$, to denote the elements of G, in keeping with the notation established in the proof of Theorem 3, Zeleva [9]. Let (z^m, a), (z^n, b) be any two positive elements in G, and suppose that $(z^m, a) < (z^n, b)$ so that $0 < m < n$. If $m \neq 0$, then $(z^m, a)^{n+1} > (z^n, b)$. If $m = 0$, then for some integer $r < |a|$, $(e, a)^r = (z, a^r)$ and hence $(e, a)^{r(n+1)} > (z^n, b)$. Thus G is an archemedian RO-group and hence (Theorem 3.8, [2]) abelian. This completes the proof.

Proof of Theorem C. Let $G \in \mathrm{RO}$ with $z \in Z(G)$ and $\{z\}_G = G$. If the isolator J of $\langle z\rangle$ lies in $Z(G)$, then the periodic elements of $G/\langle z\rangle$ certainly lie in $Z(\bar{G})$, where $\bar{G} = G/\langle z\rangle$, and hence form a subgroup of $\bar{G}$. Conversely, let T denote the set of all periodic elements of $\bar{G}$ and assume that T forms a subgroup of $\bar{G}$. Since T is a periodic RCO-group, T is locally cyclic. Let $H = \{x \in G;\ x^n \in \langle z\rangle,\ 0 \neq n \in Z\}$. Then $H/\langle z\rangle \cong T$. H is abelian since it is locally cyclic extension of its centre. Clearly H is normal in G. If for some $x \in H$, $y \in G$, $x^y \neq x$, then $G_1 = \langle x, x^y\rangle$ is a torsion-free abelian group, and therefore the direct sum of infinite cyclic groups. But $G_1\langle z\rangle/\langle z\rangle$ is finite, hence $G_1 = \langle a\rangle$ for some $a \in G_1$, and $x = a^m$, $x^y = a^n$ for some integers m, n. Since $x^k \in \langle z\rangle$ for some $k \neq 0$, $a^{mk} = x^k = y^{-1}x^ky = (x^y)^k = a^{nk}$. Hence $m = n$, and $x^y = x$.

Proof of Theorem D. Let

$$G = \langle x, y;\ x^2y^{-1}x^2y = z = y^2x^{-1}y^2x,\ xz = zx,\ yz = zy\rangle.$$

It has a normal series $G = G_0 \supset G_1 \supset G_2 \supset G_3 \supset G_4 = \langle e\rangle$ with infinite cyclic factors where $G_1 = \langle x^2y^2z^{-1},\ y^4z^{-1},\ xy^{-1}\rangle$, $G_2 = \langle x^2y^2z^{-1}, y^4z^{-1}\rangle$, $G_3 = \langle x^2y^2z^{-1}\rangle$. We right order the group G by ordering the factors G_{i-1}/G_i. Let P_i be the positive cone of G_{i-1}/G_i, $i = 1, 2, 3, 4$. For any $g \in G_{i-1}/G_i$, make $g > e$ if $gG_i \in P_i$. This gives a right order on G under which G_i's become the convex subgroups. By choosing P_i appropriately, we can assume that $z > e$. Since $z \in G_1$ and G_0/G_1 is cyclic and therefore archemedian, $\{z\}_G = G$ and $G/\langle z\rangle \in$ RCO. The group $G/\langle z\rangle$ is torsion-free (p. 250, [4]) and cannot be right ordered (Theorem 1, [6]).

The group $\bar{G} = G/\langle t^9c^{-1}\rangle$ where $G = \langle a, b, t;\ [a, b] = c,\ ca = ac,\ cb = bc,\ a^t = b,\ b^t = (ab)^{-1}\rangle$ provides a basically different example to prove Theorem D. $\bar{G}$ is torsion-free (see [1]) and cannot be right ordered (Theorem 1, [6]).

$G \in \mathrm{RCO}$, because G can be right ordered as it is extension of $N = \langle a, b\rangle - a$ free nilpotent group (hence an O-group) by an infinite cyclic group. (See Conrad [2], Theorem 3.7, p. 271). The right ordering under reference can be described as follows:

Let P be a positive cone of N and P' that of G/N. We define the positive cone Q of G by

$$Q = \{e \neq g \in G:\quad \text{either} \quad g \in N \cap P \text{ or } \bar{g} \in P'\}$$

Now $t^3 \in Z(G)$ and hence $t^9c^{-1} = z^*(\text{say}) \in Z(G)$. Then it is easy to see that $\{z^*\}_G = G$ and $\bar{G} = G/\langle z^* \rangle \in \text{RCO}$ (Theorem A).

Proof of Theorem E. Let Λ be a set with $|\Lambda| > 2^{\aleph_0}$. For every $\lambda \in \Lambda$, let $G_\lambda = \bar{G}$ as in the proof of the Theorem D. Note that $G_\lambda \notin \text{RO}$, G_λ is torsion-free, and G_λ is nilpotent by finite. Let $D = \prod_{\lambda \in \Lambda} G_\lambda$ (the restricted direct product of groups G_λ).

Claim: $D \notin \text{RCO}$. If D were an RCO-group then there exists $B \in \text{RO}$ with $z \in Z(B)$, $\{z\}_B = B$ and $B/\langle z \rangle \cong D$ (Theorem A). Now B is locally nilpotent by finite. Thus $B \in C^*$ (see [5] Theorem 7.5.1). Let C be the largest convex subgroup of B such that $z \notin C$. Then $B/C \cong$ subgroup of the additive group of reals. (see [5] Theorem 7.4.1.).

Now $\langle z \rangle \cap C = \langle e \rangle$. Thus C is isomorphic to a subgroup D_1 of D. Also $C \in \text{RO}$, since $B \in \text{RO}$ and C is a subgroup of B. Thus $D_1 \in \text{RO}$.

Now $B/C\langle z \rangle \cong D/D_1$. Since $|B/C| \leq 2^{\aleph_0}$, in order to establish our claim it is sufficient to show that $|D/D_1| > 2^{\aleph_0}$.

Now $D = \prod_{\lambda \in \Lambda} G_\lambda$. None of G_λ is an RO group. But $D_1 \in \text{RO}$.

Let $\pi_\lambda : D \to G_\lambda$ be the projection. Now $\pi_\lambda(D_1) \in \text{RO}$ but $G_\lambda \notin \text{RO}$, therefore $\pi_\lambda(D_1) < G_\lambda$, $\lambda \in \Lambda$.

Let $D_2 = \prod_{\lambda \in \Lambda} (\pi_\lambda(D_1))$. Then $D_1 \leq D_2$ and $D/D_2 \cong \prod_{\lambda \in \Lambda} (G_\lambda/\pi_{\lambda(D_1)})$.

Since $G_\lambda/_{\pi\lambda(D_1)} > \{e\}$, $\lambda \in \Lambda$; $|D/D_1| \geq |D/D_2| > 2^{\aleph_0}$. This completes the proof of the Theorem E.

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