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RESEARCH ARTICLE

Generalized Ramsey–Turán density for cliques

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Abstract
We study the generalized Ramsey–Turán function $\mathrm{RT}(n, K_s, K_t, o(n))$, which is the maximum possible number of copies of K_s in an n-vertex K_t-free graph with independence number $o(n)$. The case when $s = 2$ was settled by Erdős, Sós, Bollobás, Hajnal, and Szemerédi in the 1980s. We combinatorially resolve the general case for all $s \geq 3$, showing that the (asymptotic) extremal graphs for this problem have simple (bounded) structures. In particular, it implies that the extremal structures follow a periodic pattern when t is much larger than s. Our results disprove a conjecture of Balogh, Liu, and Sharifzadeh and show that a relaxed version does hold.

Contents

1. Introduction

Ramsey theory, initially explored by Ramsey [32] in 1930, stands as a pivotal branch of combinatorics. It seeks to tackle a fundamental question: what is the minimum size required to guarantee the existence of a well-defined substructure within a larger, often chaotic, set or system? One of the most renowned results in Ramsey theory is Ramsey's theorem, which asserts that if n is large enough in terms of k, then no matter how one colors the edges of a complete graph of order n using two colors, there will always exist a monochromatic complete subgraph K_k.

In 1941, Turán [37] proposed and solved the following problem: what is the maximum number of edges that a graph G of order n can have without containing a complete graph K_k? He also proved that the value is attained only by the balanced complete $(k-1)$-partite graph, now known as the *Turán graph* $T_{k-1}(n)$. Subsequently, a new branch of extremal combinatorics named after him emerged: Turán-type problems. Formally, we define the generalized Turán function $\mathrm{ex}(n, H_1, H_2)$ as the maximum possible number of copies of H_1 in an n-vertex H_2-free graph. There has been extensive research on this function. When $H_1 \cong K_2$, Erdős, Stone, and Simonovits (see [14, 15]) gave an asymptotically satisfactory solution for all graphs H_2, and Erdős [13] additionally determined $\mathrm{ex}(n, K_s, K_t)$ for all $t > s \geq 3$. More recently, Alon and Shikhelman [1] systematically studied $\mathrm{ex}(n, H_1, H_2)$ for other graphs H_1, and there have been a number of results in this direction (see e.g., [9, 19, 20, 30, 31]).

In this paper, we study the following extremal quantity which mixes Ramsey theory with Turán-type problems. Define the *generalized Ramsey–Turán number* $\mathrm{RT}(n, H_1, H_2, \ell)$ to be the maximum number of copies of H_1 in an n-vertex H_2-free graph G with independence number $\alpha(G) < \ell$. We remark that the existence of such a graph G is controlled by the Ramsey number $R(H_2, K_\ell)$, which is defined to be the least N such that every graph G on N vertices contains either a subgraph isomorphic to H_2 or an independent set of size ℓ. This quantity is also inherently related to the generalized Turán function; indeed, we have $\mathrm{ex}(n, H_1, H_2) = \mathrm{RT}(n, H_1, H_2, n+1)$.

This beautiful way of combining Ramsey theory with Turán-type problems was first proposed in the late 1960s by Sós [34], who investigated $\mathrm{RT}(n, K_2, H, \ell)$. The most studied case is when the independence number is sublinear: $\ell = o(n)$. To eliminate minor fluctuations caused by small values of n, one usually considers the asymptotic behavior via the *Ramsey–Turán density* function,

$$\varrho_s(K_t) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\mathrm{RT}(n, K_s, K_t, \delta n)}{\binom{n}{s}}.$$

It is not hard to see that the above limits exist. Then define $\mathrm{RT}(n, K_s, K_t, o(n)) = \varrho_s(K_t)\binom{n}{s} + o(n^s)$. We say an n-vertex K_t-free graph G with $\alpha(G) = o(n)$ is an (asymptotic) extremal graph if its K_s-density attains $\varrho_s(K_t)$.

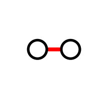
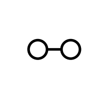
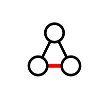
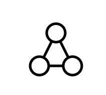
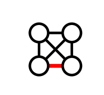
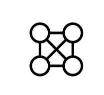
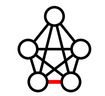
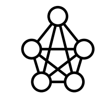
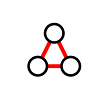
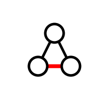
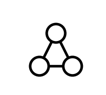
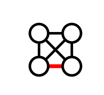
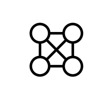
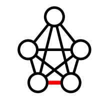
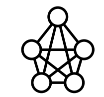
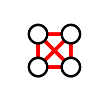
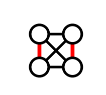
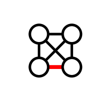
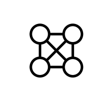
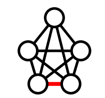
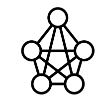
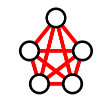
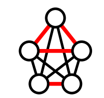
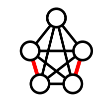
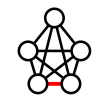
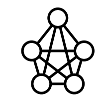
When $s = 2$, the Ramsey–Turán density has now been completely determined. It was, however, a bumpy road. In 1969, Erdős and Sós [17] showed that $\varrho_2(K_{2k+1}) = \frac{k-1}{k}$. The even cliques case became significantly more challenging. As a first application of the celebrated regularity lemma, Szemerédi [35] in 1972 proved that $\varrho_2(K_4) \leq \frac{1}{4}$, and in 1976 Bollobás and Erdős [10] obtained a matching lower bound $\varrho_2(K_4) \geq \frac{1}{4}$ via an astonishing geometric construction (now called the Bollobás–Erdős graph). Eventually, in 1983, Erdős, Hajnal, Sós and Szemerédi [16] completed the picture, showing that $\varrho_2(K_{2k}) = \frac{3k-5}{3k-2}$ for all $k \geq 2$. In fact, they proved a much stronger result showing that extremal graphs for $\varrho_2(K_t)$ exhibit the following periodic behavior:

($\star$) Let $t = 2p + r \geq 4$, where $r \in \{0, 1\}$. There is an extremal graph G for $\varrho_2(K_t)$ whose vertex set can be partitioned into $V_1 \cup \cdots \cup V_p$ satisfying (i) $G[V_1, V_2]$ has edge density $\frac{r+1}{2} - o(1)$; (ii) every other $G[V_i, V_j]$ has edge density $1 - o(1)$; and (iii) each $G[V_i]$ has edge density $o(1)$.

In other words, the extremal structure depends on the parity r of t and evolves as follows: the density of $G[V_1, V_2]$ increases as the parity r increases; and whenever t increases by 2, a new part is added and joined almost completely to all previous parts (depicted in the first row of Table 1). For more recent developments of the $s = 2$ case and related variations, we refer the interested reader to [2, 3, 4, 5, 6, 8, 11, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29].

Balogh, Liu, and Sharifzadeh [6] recently initiated the study of the general case $s \geq 3$, which turns out to be much more difficult and delicate than the $s = 2$ case. Note that $\varrho_s(K_{s+1}) = 0$: in any n-vertex

Table 1. Conjectured periodic extremal structure. Black edges have density 1 and red edges have density 1/2.

$s \backslash t$	4	5	6	7	8	9	10	11
2								
3	$\emptyset$							
4		$\emptyset$						
5			$\emptyset$					

K_{s+1}-free graph with independence number $o(n)$, each copy of K_{s-1} lies in $o(n)$ copies of K_s. Balogh, Liu, and Sharifzadeh [6] determined the first non-trivial cases $\varrho_3(K_t)$ and $\varrho_s(K_{s+2})$, and made a conjecture predicting the general case. We find it more convenient to work with the following definition, which helps reformulate their conjecture.

Definition 1.1. Given integers $b \geq a \geq 1$, a graph G *admits a (b, a)-partition* if its vertex set has a partition $V = V_1 \cup \cdots \cup V_a$ satisfying the following for $b_1, \ldots, b_a \in \{\lceil \frac{b}{a} \rceil, \lfloor \frac{b}{a} \rfloor\}$ with $\sum_{i=1}^{a} b_i = b$:

(1) For every distinct $i, j \in [a]$, $G[V_i, V_j]$ has edge density $1 - o(1)$; and
(2) For every $i \in [a]$, V_i admits an equipartition $V_i^1 \cup \cdots \cup V_i^{b_i}$ such that $G[V_i^j]$ has density $o(1)$ for all $j \in [b_i]$ and $G[V_i^j, V_i^k]$ has density $\frac{1}{2} - o(1)$ for all distinct $j, k \in [b_i]$.

For instance, if $a = b = p$ then each $b_i = 1$, so an n-vertex graph admits a (p, p)-partition if and only if it has edit-distance $o(n^2)$ to the Turán graph $T_p(n)$.

Conjecture 1.2 [6]. *Given integers $t - 2 \geq s \geq 3$, there is an extremal graph for $\varrho_s(K_t)$ which admits*

(i) *an $(s, t - 1 - s)$-partition if $s + 2 \leq t \leq 2s - 1$, or*
(ii) *a $(\lfloor \frac{t}{2} \rfloor, t - 1 - \lfloor \frac{t}{2} \rfloor)$-partition if $t \geq 2s$.*

The preceding conjecture can be better understood, using the language of Definition 1.1, as follows. For every $t = 2p + r \geq 4$ with $r \in \{0, 1\}$, the periodic behavior of $\varrho_2(K_t)$ in ($\star$) can be rephrased as '*an extremal graph for $\varrho_2(K_t)$ admits a $(p, p - 1 + r)$-partition*', which is precisely the statement of Conjecture 1.2(ii) when $s = 2$. Thus, Conjecture 1.2 speculates that similar periodic behavior occurs at the threshold $t \geq 2s$ for all $s \geq 3$ (see Table 1). Further supporting this prediction, it was proved in [6] that Conjecture 1.2 holds for $s = 3$.

We present infinitely many counterexamples showing that Conjecture 1.2 is false in general. The smallest counterexamples we observe are $s = 5$ and $t \in \{10, 11\}$ (see Figure 2). On the positive side, we prove that the predicted periodic behavior does eventually occur when $t \gg s$ for all $s \geq 3$.

Theorem 1.3. *Conjecture 1.2 is false when $2s \leq t \leq 2.08s$ for sufficiently large s. Given $t - 2 \geq s \geq 3$, Conjecture 1.2 is true if $t > s^2(s - 1)/2 + s$, $t = s + 2$, or $s = 3, 4$.*

Furthermore, our main result shows that a modified version of Conjecture 1.2 is true (which was the motivation for Definition 1.1). It reads as follows.

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Theorem 1.4. *For all integers $t-2 \geq s \geq 3$, there is a family of extremal graphs for $\varrho_s(K_t)$ admitting a (b,a)-partition for some parameters $1 \leq a \leq b$ satisfying $a+b=t-1$.*

Theorem 1.4 provides a detailed description of the extremal graphs for the generalized Ramsey–Turán problem for cliques, showing that they have simple and bounded structures. We remark that Theorem 1.4 resolves *combinatorially* the problem of determining the Ramsey–Turán density $\varrho_s(K_t)$. Indeed, given s and t, there is a bounded number of choices for a (because $a \leq t-1$). Once a is fixed, the structure of a graph admitting a (b,a)-partition is determined, so its K_s-density may be computed in terms of the fractions of vertices allocated to each part. Thus, Theorem 1.4 reduces determining $\varrho_s(K_t)$ to a bounded optimization problem (over $a+b=t-1$).

Organization. The rest of the paper is structured as follows. The proof of Theorem 1.4 consists of two parts. We first reduce it to a more tractable problem about clique densities in weighted graphs (see Theorem 2.4) in Section 2. Understanding this weighted problem is the bulk of the proof (see Theorem 3.1); we study it in Section 3. In Section 4, we give the proof of Theorem 1.3. Concluding remarks are given in Section 5.

Notation. We use $[n]$ to denote the finite set $\{1,2,\ldots,n\}$. For a vector $\boldsymbol{u}=(u_1,\ldots,u_k)\in\mathbb{R}^k$, we write $\|\boldsymbol{u}\|=\sqrt{\sum_{i=1}^k u_i^2}$ for its ℓ_2-norm. Let $G=(V(G),E(G))$ be a graph. For every $U,V\subseteq V(G)$, denote by $G[U,V]$ the induced bipartite subgraph of G on partite sets U and V, and by $G[U]$ the induced subgraph of G on set U. For convenience, we let $G-U=G[V(G)\setminus U]$.

2. Reduction to Weighted Graphs

The first step of the proof of Theorem 1.4 reduces understanding Ramsey–Turán density to a problem about clique density in weighted graphs. The aim of this section will be to prove Theorem 2.4, which demonstrates the equivalence between these two problems. However, before we can state Theorem 2.4, we need to define our notion of weighted graphs.

Definition 2.1. A *weighted graph* $R=(V,w)$ consists of a finite vertex set V together with a *weight function* $w: V\sqcup V^2\to[0,1]$ satisfying the following two properties. The vertex weights must sum to one (i.e., $\sum_{v\in V} w(v)=1$). Additionally, the edge weights must satisfy $w(v,v')=w(v',v)$ and $w(v,v)=0$ for any $v,v'\in V$. For $\alpha\in[0,1]$, denote by $R_{>\alpha}$ the spanning subgraph of R with all edges of weight larger than α.

Intuitively, a weighted graph may be thought of as a type of graph limit with a more discrete structure than a graphon. An r-vertex weighted graph can also be considered to represent a large r-partite graph G whose ith part V_i contains a $w(i)$-fraction of the vertex set, such that each induced bipartite subgraph $G[V_i,V_j]$ is a random graph of density $w(i,j)$.

With this perspective in mind, we define subgraph densities in a weighted graph.

Definition 2.2. Let H be a graph with vertex set $[s]$. The *H-density* of a weighted graph R is defined as

$$d_H(R)=\mathbb{E}_{\substack{v_1,\ldots,v_s\\ \in V(R)}}\left[\prod_{ij\in E(H)} w(v_i,v_j)\right]=\sum_{\sigma:[s]\to V(R)}\left(\prod_{i=1}^{s} w(\sigma(i))\right)\left(\prod_{ij\in E(H)} w(\sigma(i),\sigma(j))\right),$$

where vertices $v_1,\ldots,v_s\in V(R)$ are chosen independently at random according to the vertex weights of R.

We shall show that the Ramsey–Turán density $\varrho_s(K_t)$ is determined by the maximum possible K_s-density in a weighted graph avoiding the following forbidden configuration.

Definition 2.3. Let R be a weighted graph and $t\in\mathbb{N}$. The *weighted t-clique family* $\mathcal{K}_t$ consists of all pairs of subsets (S_1,S_2) with $S_2\subseteq S_1\subseteq V(R)$, $s_1=|S_1|$, $s_2=|S_2|\geq 1$ and $s_1+s_2=t$ such that S_1 induces a K_{s_1} in $R_{>0}$ and S_2 induces a K_{s_2} in $R_{>\frac{1}{2}}$. We say R is *$\mathcal{K}_t$-free* if R contains no such pair (S_1,S_2).

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We can then define the K_s-Turán density of $\mathcal{K}_t$ as

$$\pi_s(\mathcal{K}_t) = \sup\{d_{K_s}(R) : \ R \text{ is a } \mathcal{K}_t\text{-free weighted graph}\}.$$

At this point, we may state the main result of this section.

Theorem 2.4. *For $s, t \in \mathbb{N}$ with $2 \le s \le t-1$, we have $\varrho_s(K_t) = \pi_s(\mathcal{K}_t)$.*

The upper and lower bounds of Theorem 2.4 will be proven in the next two subsections.

2.1. Upper bound

Our proof of the upper bound on Theorem 2.4 relies on Szemerédi's regularity lemma. The regularity lemma states that any graph looks ε-close to a weighted graph whose number of vertices is bounded in terms of ε. We recall its statement here, beginning with some preliminary definitions.

Definition 2.5. Let G be a graph and let $X, Y \subseteq V(G)$. The *edge density* between X and Y, denoted by $d(X,Y)$, is the fraction of pairs $(x,y) \in X \times Y$ that are edges of G. Given $\varepsilon > 0$, we say the pair (X,Y) is *ε-regular* if, for all $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \ge \varepsilon|X|$ and $|Y'| \ge \varepsilon|Y|$, we have $|d(X',Y') - d(X,Y)| < \varepsilon$.

Definition 2.6. Let G be a graph. A vertex partition $V(G) = V_1 \cup \cdots \cup V_r \cup V_{r+1}$ is called *ε-regular* if $|V_1| = \cdots = |V_r|$ and $|V_{r+1}| \le \varepsilon n$, and additionally at most εr^2 pairs (V_i, V_j) with $i < j \le r$ are not ε-regular.

Theorem 2.7 (Regularity Lemma, [36]). *For every small constant $\varepsilon > 0$ and integer M_0, there exists an integer $M = M(\varepsilon, M_0)$ such that the following holds. Given any n-vertex graph G, there is an ε-regular partition of its vertices $V(G) = V_1 \cup \cdots \cup V_r \cup V_{r+1}$ such that $M_0 \le r \le M$.*

We also require the graph counting lemma. Intuitively, if a graph G looks like a weighted graph R, then this lemma implies that the K_s-density of G is approximately the K_s-density of R.

Lemma 2.8 (Graph Counting Lemma, [12]). *Let $\varepsilon > 0$, and let G be an s-partite graph with $V(G) = \bigcup_{i=1}^{s} V_s$. Suppose that the pair (V_i, V_j) is ε-regular for all $1 \le i < j \le s$. Then*

$$\left| N(K_s, G) \Big/ \left(\prod_{i=1}^{s} |V_i| \right) - \prod_{1 \le i < j \le s} d(V_i, V_j) \right| \le \sqrt{\varepsilon s^3},$$

where $N(K_s, G)$ is the number of copies of K_s in G.

We now prove the upper bound of Theorem 2.4.

Theorem 2.9. *Let s, t be integers with $2 \le s < t$. For any $\delta \in (0,1)$ there exists $\delta' \in (0,1)$ such that the following holds. Suppose G is a K_t-free graph with $\alpha(G) \le \delta'|V(G)|$. Then there is a $\mathcal{K}_t$-free weighted graph R such that $d_{K_s}(G) \le d_{K_s}(R) + 4s^2\delta$.*

Proof. Choose $\varepsilon < \frac{1}{2}$ small enough such that $(\delta - \varepsilon)^{t-1} > (t+1)\varepsilon$ and let $\delta' = \varepsilon/M$, where $M = M(\varepsilon, \frac{1}{\varepsilon})$ is the constant guaranteed by the regularity lemma (Theorem 2.7). Suppose G is a K_t-free graph on N vertices with $\alpha(G) \le \delta' N$.

Apply Theorem 2.7 with this value of ε to G. This yields an ε-regular partition $V(G) = V_1 \cup \cdots \cup V_r \cup V_{r+1}$ with $\frac{1}{\varepsilon} \le r \le M$. We show that a substructure similar to a weighted t-clique is forbidden among the edge densities $d(V_i, V_j)$.

Claim 2.10. *Suppose $S_2 \subseteq S_1 \subseteq [r]$ are sets of indices such that*

(i) *For any distinct $i, j \in S_1$, the pair (V_i, V_j) is ε-regular with density $d(V_i, V_j) > \delta$; and*
(ii) *For any distinct $i, j \in S_2$, the pair (V_i, V_j) has density $d(V_i, V_j) > \frac{1}{2} + \delta$.*

Then G contains a clique of order $|S_1| + |S_2|$. In particular, $|S_1| + |S_2| < t$.

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Proof of claim. Order the elements of S_1 as $a_1, \ldots, a_\ell$ with the elements of $S_1 \setminus S_2$ listed first. Set $k = |S_1| - |S_2|$. By Lemma 2.8, there exists a clique S of order ℓ in G such that $|S \cap V_{a_i}| = 1$ for each $i \in [\ell]$; as G is K_t-free, it follows that $\ell < t$. Our proof follows an ℓ-step process, where the ith step chooses one (if $i \le k$) or two (if $i > k$) vertices from V_{a_i} that are adjacent to all previously chosen vertices. For $0 \le i, j \le \ell$, write $W^{(i)}$ for the common neighborhood of those vertices chosen in the first i steps, and let $W_j^{(i)} = V_{a_j} \cap W^{(i)}$. In particular, $W^{(0)} = V(G)$. We will choose $2\ell - k$ vertices such that $|W_j^{(i)}| \ge (\delta - \varepsilon)|W_j^{(i-1)}| \ge (\delta - \varepsilon)^i |V_{a_j}|$ for all $0 \le i < j \le \ell$.

On the ith step with $1 \le i \le k$, choose one vertex $v_i \in W_i^{(i-1)}$ such that $W_j^{(i)} := N(v_i) \cap W_j^{(i-1)}$ has cardinality at least $(\delta - \varepsilon)|W_j^{(i-1)}|$ for each $j > i$. To show that such a vertex v_i exists, consider the sets

$$X_j = \left\{ v \in W_i^{(i-1)} : |N(v) \cap W_j^{(i-1)}| < (\delta - \varepsilon)|W_j^{(i-1)}| \right\}$$

for each $j > i$. Observe that $d\left(X_j, W_j^{(i-1)}\right) < \delta - \varepsilon < d(V_{a_i}, V_{a_j}) - \varepsilon$ by construction, and that $|W_j^{(i-1)}| \ge (\delta - \varepsilon)^{i-1}|V_{a_j}| \ge (\delta - \varepsilon)^t |V_{a_j}| \ge \varepsilon |V_{a_j}|$. Because the pair (V_{a_i}, V_{a_j}) is ε-regular, it follows that $|X_j| < \varepsilon |V_{a_i}|$. Thus,

$$\left| W_i^{(i-1)} - \bigcup_{j=i+1}^{\ell} X_j \right| > (\delta - \varepsilon)^{i-1}|V_{a_i}| - (\ell - i)\varepsilon |V_{a_i}| \ge ((\delta - \varepsilon)^{t-1} - (t-1)\varepsilon)|V_{a_i}| > 0.$$

It follows that there is a vertex $v_i \in W_i^{(i-1)}$ such that $|W_j^{(i)}| = |N(v_i) \cap W_j^{(i-1)}| \ge (\delta - \varepsilon)|W_j^{(i-1)}|$ for each $j > i$.

On the ith step with $k < i \le \ell$, choose two adjacent vertices $v_i, v_i' \in W_i^{(i-1)}$ such that $W_j^{(i)} := N(v_i) \cap N(v_i') \cap W_j^{(i-1)}$ has cardinality at least $2(\delta - \varepsilon)|W_j^{(i-1)}|$ for all $j > i$. To verify that such vertices exist, set

$$X_j = \left\{ v \in W_i^{(i-1)} : |N(v) \cap W_j^{(i-1)}| < \left(\frac{1}{2} + \delta - \varepsilon \right) |W_j^{(i-1)}| \right\}$$

for each $j > i$. The argument from the prior paragraph shows that

$$\left| W_i^{(i-1)} - \bigcup_{j=i+1}^{\ell} X_j \right| > \left((\delta - \varepsilon)^{t-1} - (t-1)\varepsilon \right) |V_{a_i}| > 2\varepsilon |V_{a_i}|.$$

Noting that $|V_{a_i}| = (N - |V_{r+1}|)/r > N/2r$, we have

$$\left| W_i^{(i-1)} - \bigcup_{j=i+1}^{\ell} X_j \right| > 2\varepsilon |V_{a_i}| > \frac{\varepsilon N}{r} \ge \delta' N \ge \alpha(G).$$

It follows that there are adjacent vertices $v_i, v_i' \in W_i^{(i-1)}$ such that $N(v_i) \cap W_j^{(i-1)}$ and $N(v_i') \cap W_j^{(i-1)}$ have cardinality at least $(1/2 + \delta - \varepsilon)|W_j^{(i-1)}|$ for each $j > i$. By the pigeonhole principle,

$$|W_j^{(i)}| = |N(v_i) \cap N(v_i') \cap W_j^{(i-1)}| \ge 2(\delta - \varepsilon)|W_j^{(i-1)}|$$

for each $j > i$, as desired.

After ℓ steps, this process results in $k + 2(\ell - k) = |S_1| + |S_2|$ vertices $v_1, \ldots, v_k, v_{k+1}, v_{k+1}', \ldots, v_\ell, v_\ell'$ that form a clique in G. It follows that $|S_1| + |S_2| < t$, because G is K_t-free. □

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Let R be the weighted graph on $[r]$ with vertex weights $w(i) = \frac{1}{r}$ for all $i \in [r]$ and edge weights

$$w(i,j) = \begin{cases} \max\{d(V_i,V_j) - \delta, 0\}, & \text{if } i \neq j \text{ and } (V_i,V_j) \text{ is } \varepsilon\text{-regular}, \\ 0, & \text{if } i = j \text{ or } (V_i,V_j) \text{ is not } \varepsilon\text{-regular}, \end{cases}$$

for all $i, j \in [r]$. We observe that R is $\mathcal{K}_t$-free as a direct consequence of Claim 2.10.

To conclude the proof, we bound the K_s-density of G. We have

$$d_{K_s}(G) = \frac{1}{N^s}\left(\sum_{a_1,\dots,a_s \in [r+1]} \#\{(v_1,\dots,v_s) \in V_{a_1} \times \cdots \times V_{a_s} \text{ that form a } K_s \text{ in } G\}\right).$$

If $a_1,\dots,a_s$ are distinct elements of $[r]$ and each pair (V_{a_i},V_{a_j}) is ε-regular, then we may simplify the summand using the graph-counting lemma. Indeed, Lemma 2.8 implies that the number of copies of K_s in $V_{a_1} \times \cdots \times V_{a_s}$ is at most

$$\left(\prod_{i=1}^{s} |V_{a_i}|\right)\left(\prod_{1 \le i < j \le s} d(V_{a_i},V_{a_j}) + \sqrt{\varepsilon s^3}\right) \le \frac{N^s}{r^s}\left(\prod_{1 \le i < j \le s} d(V_{a_i},V_{a_j}) + \sqrt{\varepsilon s^3}\right)$$

in this case. It remains to bound the contribution from terms where some a_i is $r+1$, the a_i are not distinct, or some pair (V_{a_i},V_{a_j}) is not ε-regular.

The terms where at least one index a_i equals $r+1$ contribute at most

$$\frac{s}{N^s}|V_{r+1}|N^{s-1} \le \varepsilon s$$

to the sum. The terms where $a_1,\dots,a_s$ are not all distinct contribute at most

$$\frac{1}{N^s}\sum_{i=1}^{r}\binom{s}{2}|V_i|^2 N^{s-2} \le r \times \binom{s}{2} \times \frac{1}{r^2} \le \varepsilon\binom{s}{2}$$

because $r \ge 1/\varepsilon$. The terms where a pair (V_{a_i},V_{a_j}) is not ε-regular contribute at most

$$\frac{1}{N^s}\sum_{\substack{1 \le i < j \le r, \\ (V_i,V_j) \text{ not} \\ \varepsilon\text{-regular}}} s(s-1)|V_i||V_j|N^{s-2} < \frac{s(s-1)}{r^2} \times (\#\varepsilon\text{-irregular pairs}) \le \varepsilon s(s-1).$$

Combining these estimates, we have

$$\begin{aligned} d_{K_s}(G) &\le \varepsilon\left(s + 3\binom{s}{2}\right) + r^{-s}\sum_{\substack{a_1,\dots,a_s \in [r] \\ \text{distinct}}}\left(\prod_{1 \le i < j \le s} d(V_{a_i},V_{a_j}) + \sqrt{\varepsilon s^3}\right) \\ &\le \varepsilon\left(s + 3\binom{s}{2}\right) + \sqrt{\varepsilon s^3} + r^{-s}\sum_{\substack{a_1,\dots,a_s \in [r] \\ \text{distinct}}}\left(\prod_{1 \le i < j \le s} d(V_{a_i},V_{a_j})\right). \end{aligned}$$

To compare this to $d_{K_s}(R)$, we observe the following inequality. If real numbers $x_1,\dots,x_k,y_1,\dots,y_k \in [0,1]$ satisfy $x_i \le y_i + \delta$ for each $i \in [k]$ then

$$\prod_{i=1}^{k} x_i - \prod_{i=1}^{k} y_i = (x_1 - y_1)x_2 \cdots x_k + y_1(x_2 - y_2)x_3 \cdots x_k + \cdots + y_1 \cdots y_{k-1}(x_k - y_k) \le k\delta.$$

Applying this inequality with $k = \binom{s}{2}$ to the real numbers $d(V_{a_i}, V_{a_j})$ and $w(a_i, a_j)$, it follows that

$$\begin{aligned} d_{K_s}(G) &\leq \varepsilon\left(s + 3\binom{s}{2}\right) + \sqrt{\varepsilon s^3} + r^{-s} \sum_{\substack{a_1,\ldots,a_s\in[r] \\ \text{distinct}}} \left(\prod_{1\leq i<j\leq s} w(a_i, a_j) + \delta\binom{s}{2}\right) \\ &\leq 2\varepsilon s^2 + \sqrt{\varepsilon s^3} + \delta\binom{s}{2} + d_{K_s}(R). \end{aligned}$$

Because $\varepsilon < (\delta - \varepsilon)^{t-1} < \delta^2$, it follows that $d_{K_s}(G) < d_{K_s}(R) + 4s^2\delta$, as desired. □

2.2. Lower bound

The lower bound construction for Theorem 2.4 hinges on a construction of Bollobás and Erdős [10] which achieves the tight lower bound $\varrho_2(K_4) = \frac{1}{4}$. We briefly describe this construction, following the notation used in [18].

Fix $0 < \varepsilon < 1$ and an integer $h \geq 16$, and set $\mu = \frac{\varepsilon}{\sqrt{h}}$. Let X and Y be sets of points on the unit sphere $\mathsf{S}^{h-1} \subset \mathbb{R}^h$. The Bollobás–Erdős graph $\mathrm{BE}(X, Y)$ is a graph on vertex set $X \cup Y$ constructed as follows.

(a) Join $\boldsymbol{x} \in X$ to $\boldsymbol{y} \in Y$ if $\|\boldsymbol{x} - \boldsymbol{y}\| < \sqrt{2} - \mu$.
(b) Join $\boldsymbol{x}, \boldsymbol{x}' \in X$ if $\|\boldsymbol{x} - \boldsymbol{x}'\| > 2 - \mu$. Similarly, join $\boldsymbol{y}, \boldsymbol{y}' \in Y$ if $\|\boldsymbol{y} - \boldsymbol{y}'\| > 2 - \mu$.

Bollobás and Erdős showed that this graph is K_4-free and that, if ε and h are tuned appropriately and X and Y are uniformly placed on S^{h-1}, it has independence number $o(|X| + |Y|)$ and edge density $\frac{1}{4} - o(1)$. Fox, Loh and Zhao [18] analyzed this construction in further detail, providing precise quantitative results on the independence number and minimum degree. We need some results from their work.

Lemma 2.11 [18]. *Let $0 < \varepsilon < 1$ and let $h \geq 16$ be an integer. Set $\mu = \varepsilon/\sqrt{h}$.*

(1) *If points $\boldsymbol{x}, \boldsymbol{y} \in S^h$ are chosen independently and uniformly at random then* $\Pr\left[\|\boldsymbol{x} - \boldsymbol{y}\| < \sqrt{2} - \mu\right] \geq \frac{1}{2} - \sqrt{2}\varepsilon$.

Now, fix $X, Y \subseteq S^{h-1}$, and let $G = \mathrm{BE}(X, Y)$ be the graph defined above with parameters ε and h.

(2) *The induced subgraphs $G[X]$ and $G[Y]$ are each K_3-free.*
(3) *G is K_4-free.*
(4) *If n is sufficiently large in terms of ε, h, there is a choice $X \subset S^h$ of size n such that the induced subgraph $G[X]$ has independence number at most $e^{-\varepsilon\sqrt{h}/4}n$.*

Using these preliminaries, we prove the lower bound of Theorem 2.4. It follows from Theorem 2.12 by choosing parameters (ε, h) such that $\varepsilon \to 0$ and $\varepsilon\sqrt{h} \to \infty$.

Theorem 2.12. *Suppose R is a weighted graph that is $\mathcal{K}_t$-free for some integer $t \geq 3$. Fix $\varepsilon > 0$ and an integer $h \geq 16$. For all sufficiently large N, there is a K_t-free graph G on N vertices with independence number $\alpha(G) \leq 3e^{-\varepsilon\sqrt{h}/4}N$ and K_s-density $d_{K_s}(G) \geq (1 - 2\sqrt{2}s^2\varepsilon)d_{K_s}(R)$.*

Proof. Suppose that $V(R) = [r]$ for some integer r. Increasing each edge weight to the next multiple of $\frac{1}{2}$ preserves the $\mathcal{K}_t$-freeness of R, so we may assume that all edge weights of R are 0, $\frac{1}{2}$, or 1. Set $\mu = \frac{\varepsilon}{\sqrt{h}}$ as in the Bollobás–Erdős construction.

Choose integers $n_i \geq \lfloor w(i)N \rfloor$ such that $N = n_1 + \cdots + n_r$. We construct an N-vertex graph G on vertex set $V_1 \cup \cdots \cup V_r$ as follows. Intuitively, $G[V_i, V_j]$ will be complete, empty, or a randomly rotated Bollobás–Erdős graph, depending on whether $w(i, j)$ is 1, 0, or $\frac{1}{2}$.

Suppose N is sufficiently large. For each i, we may choose a set V_i of n_i points on S^{h-1} satisfying Lemma 2.11(4). Connect vertices in $\bigcup_i V_i$ as follows. Within each part V_i, add an edge between $\boldsymbol{v}_i, \boldsymbol{v}_i' \in V_i$ if $\|\boldsymbol{v}_i - \boldsymbol{v}_i'\| > 2 - \mu$. Let $G[V_i, V_j]$ be complete bipartite if $w(i, j) = 1$ and empty if

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$w(i,j)=0$. If $w(i,j)=\frac{1}{2}$ for some $i<j$, then let $\rho_{ij}\in SO(h)$ be a rotation of S^{h-1} chosen uniformly at random. Connect $\boldsymbol{v}_i\in V_i$ and $\boldsymbol{v}_j\in V_j$ if $\|\rho_{ij}\boldsymbol{v}_i-\boldsymbol{v}_j\|<\sqrt{2}-\mu$.

Observe that each induced subgraph $G[V_i]$ is K_3-free with independence number $\alpha(G[V_i])\le e^{-\varepsilon\sqrt{h}/4}n_i$ by Lemma 2.11(2) and (4). It follows that $\alpha(G)\le e^{-\varepsilon\sqrt{h}/4}N$. Additionally, if $w(i,j)=\frac{1}{2}$, the induced subgraph $G[V_i\cup V_j]$ is the Bollobás–Erdős graph $\mathrm{BE}(\rho_{ij}(V_i),V_j)$, and is thus K_4-free by Lemma 2.11(3).

Using these properties, we verify that G is K_t-free. Suppose, for contradiction, that $G[W]$ is a clique, where W is some set of t vertices. Let $S_1=\{i\in[r]:|V_i\cap W|\ge 1\}$ and $S_2=\{i\in[r]:|V_i\cap W|\ge 2\}\subseteq S_1$. Because each induced subgraph $G[V_i]$ is K_3-free, it follows that W has at most two points in V_i, and thus that $|S_1|+|S_2|=|W|=t$. For any distinct $i,j\in S_1$, there is an edge between V_i and V_j, and it follows that $w(i,j)>0$. Additionally, for any distinct $i,j\in S_2$, there is a K_4 in $G[V_i\cup V_j]$. Because the Bollobás–Erdős graph is K_4-free, this implies that $w(i,j)>\frac{1}{2}$. We conclude that (S_1,S_2) form a weighted t-clique in R, which is a contradiction. It follows that G is K_t-free.

Lastly, we verify that G has large K_s-density in expectation, using the independence of the random rotations ρ_{ij}. By Lemma 2.11(1), if $w(i,j)=\frac{1}{2}$ then the expected edge density between V_i and V_j is at least $\frac{1}{2}-2\sqrt{2}\varepsilon$. Thus, if N is sufficiently large, we have

$$\begin{aligned}
\mathbb{E}[d_{K_s}(G)] &\ge \sum_{\substack{i_1,\dots,i_s\in[r]\\ \text{distinct}}}\left(\prod_{j=1}^{s}\frac{n_{i_j}}{N}\right)\left(\prod_{1\le j<k\le s}\begin{cases}0 & w(i_j,i_k)=0\\ 1/2-2\sqrt{2}\varepsilon & w(i_j,i_k)=1/2\\ 1 & w(i_j,i_k)=1\end{cases}\right)\\
&\ge \sum_{\substack{i_1,\dots,i_s\in[r]\\ \text{distinct}}}\left(\prod_{j=1}^{s}(1-\varepsilon)w(i_j)\right)\left(\prod_{1\le j<k\le s}(1-4\sqrt{2}\varepsilon)w(i_j,i_k)\right)\\
&=(1-\varepsilon)^s(1-4\sqrt{2}\varepsilon)^{s(s-1)/2}d_{K_s}(R)\ge(1-2\sqrt{2}s^2\varepsilon)d_{K_s}(R).
\end{aligned}$$

We conclude that there is a choice of the rotations ρ_{ij} such that $d_{K_s}(G)\ge(1-2\sqrt{2}s^2\varepsilon)d_{K_s}(R)$. □

3. Understanding the extremal weighted graphs

By Theorem 2.4, $\varrho_s(K_t)=\pi_s(\mathcal{K}_t)$ for all $3\le s\le t-2$. In this section, we show that the supremum $\pi_s(\mathcal{K}_t)$ is attained by a weighted graph on at most $t-1$ vertices, and characterize the structure of a minimum-size extremal weighted graph more precisely. Our results are summarized as follows; together with Theorem 2.4, they imply Theorem 1.4.

Theorem 3.1. *Fix integers s,t satisfying $3\le s\le t-2$. There is an extremal $\mathcal{K}_t$-free weighted graph R achieving K_s-density $\pi_s(\mathcal{K}_t)$ and satisfying the following properties.*

- **(A1)** *For any distinct $v,v'\in V(R)$, we have $w(v,v')\in\{\frac{1}{2},1\}$.*
- **(A2)** *There is a partition $V(R)=B_1\cup\cdots\cup B_a$ into nonempty parts such that vertices inside the same part B_i have the same weight, and an edge has weight $1/2$ if it lies within some B_i and weight 1 otherwise. Moreover, setting $b=\sum_{i\in[a]}|B_i|=|V(R)|$, we have $b\ge s$ and $a+b=t-1$.*
- **(A3)** *For any i and j, the cardinalities $|B_i|$ and $|B_j|$ differ by at most 1.*
- **(A4)** *If $|B_i|\ge|B_j|$ for any (possibly equal) $i,j\in[a]$, then $w(v_i)\le w(v_j)$ for any $v_i\in B_i$ and $v_j\in B_j$. In particular, if $|B_i|=|B_j|$ then all vertices in B_i and B_j have the same weight.*
- **(A5)** *Either $a=1$ and $|B_1|=s$ or $a\ge 2$ and $|B_i|\le s-1$ for each $i\in[a]$.*

Moreover, all extremal $\mathcal{K}_t$-free weighted graphs with minimum order satisfy (A1)–(A5).

Let t and s be integers such that $3\le s\le t-2$. We shall show in the following subsections that there exists a $\mathcal{K}_t$-free weighted graph R achieving K_s-density $\pi_s(\mathcal{K}_t)$ with $|V(R)|$ minimized and satisfying (A1)–(A5). Note that the weighted graph R might not be unique.

Before beginning the proof, we introduce some notation which will be used in this section. Let R be a weighted graph and let H be a graph on $[s]$. For a vertex set $S \subseteq V(R)$, the density of copies of H containing S is denoted by

$$d_H(R,S) = \sum_{\substack{\sigma:[s]\to V(R)\\ S\subseteq\{\sigma(1),\cdots,\sigma(s)\}}} \left(\prod_{i=1}^{s} w(\sigma(i))\right)\left(\prod_{ij\in E(H)} w(\sigma(i),\sigma(j))\right).$$

Similarly, write

$$d_H(R[S]) = \sum_{\sigma:[s]\to S} \left(\prod_{i=1}^{s} w(\sigma(i))\right)\left(\prod_{ij\in E(H)} w(\sigma(i),\sigma(j))\right) \quad \text{and}$$
$$d_H(R-S) = \sum_{\sigma:[s]\to V(R)\setminus S} \left(\prod_{i=1}^{s} w(\sigma(i))\right)\left(\prod_{ij\in E(H)} w(\sigma(i),\sigma(j))\right)$$

for the density of copies of H within S and avoiding S, respectively. For convenience, we write $d_H(R,v)$ instead of $d_H(R,\{v\})$ and $d_H(R-v)$ instead of $d_H(R-\{v\})$. Additionally, when $H = K_0$, we define all densities to be 1.

We shall also require the following well-known inequality regarding symmetric functions. This is a special case of Maclaurin's inequality.

Lemma 3.2. *Let $x_1,\ldots,x_n$ be positive real numbers and let $x = (x_1+\cdots+x_n)/n$ be their average. For any integer $1\le k\le n$, we have*

$$\sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k} \le \binom{n}{k}x^k,$$

with equality if and only if all the x_i are equal.

3.1. Proof of (A1)

For each integer $n \ge s$, let R_n be an n-vertex $\mathcal{K}_t$-free weighted graph of maximum K_s-density. Such a weighted graph R_n exists because the space of n-vertex weighted graphs, which is parametrized by possible choices of the vertex and edge weights, is compact.

We claim that $d_{K_s}(R_{n-1}) \ge d_{K_s}(R_n)$ if R_n contains an edge of weight 0. Suppose that $w(v_1,v_2)=0$ for two distinct vertices $v_1,v_2\in V(R_n)$. For $i\in[2]$, let R_i' be the $(n-1)$-vertex weighted graph obtained from R_n by deleting v_{3-i} and increasing the weight of v_i to $w(v_1)+w(v_2)$. Clearly, both R_1' and R_2' are $\mathcal{K}_t$-free. Moreover, writing $\alpha_i = \frac{w(v_i)}{w(v_1)+w(v_2)}$ for $i\in[2]$, we have

$$\begin{aligned}\alpha_1\cdot d_{K_s}(R_1') + \alpha_2\cdot d_{K_s}(R_2') &= d_{K_s}(R_n-\{v_1,v_2\}) + d_{K_s}(R_n-v_2,v_1) + d_{K_s}(R_n-v_1,v_2)\\ &= d_{K_s}(R_n). \end{aligned} \tag{3.1}$$

This implies $d_{K_s}(R_{n-1}) \ge \max_{i\in[2]} d_{K_s}(R_i') \ge d_{K_s}(R_n)$. In particular, we have $d_{K_s}(R_{n-1}) \ge d_{K_s}(R_n)$ for all $n\ge t$, as any $\mathcal{K}_t$-free weighted graph on at least t vertices must contain an edge of weight 0.

It follows that $\pi_s(\mathcal{K}_t) = \sup_{n\ge s} d_{K_s}(R_n)$ is attained by a weighted graph R_n on at most $t-1$ vertices. Moreover, any minimal-order weighted graph attaining the K_s-density $\pi_s(\mathcal{K}_t)$ must have strictly positive edge weights.

We conclude the proof of (A1) by observing that if R is a $\mathcal{K}_t$-free weighted graph with maximum K_s-density and strictly positive edge weights, then all edge weights of R must be either $\frac{1}{2}$ or 1, as

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increasing any edge weight to the next multiple of $\frac{1}{2}$ will preserve the $\mathcal{K}_t$-freeness of R while increasing its K_s-density.

In the remaining subsections, we will show that any extremal $\mathcal{K}_t$-free weighted graph satisfying (A1) — and in particular, all such graphs of minimum order — also satisfy (A2)–(A5).

3.2. *Proof of (A2)*

Let R be an extremal $\mathcal{K}_t$-free weighted graph satisfying (A1). We observe that R must have at least s vertices, as $d_{K_s}(R)$ would be 0 if $|V(R)| < s$. Moreover, if R has exactly s vertices, say $v_1, \dots, v_s$, then

$$d_{K_s}(R) = s!\left(\prod_{i=1}^{s} w(v_i)\right)\left(\prod_{1\le i<j\le s} w(v_i, v_j)\right)$$

is maximized when the vertex weights are equal and the number of edges of weight 1 is maximized subject to the constraint that $R_{>1/2}$ is K_{t-s}-free. The latter condition holds if and only if $R_{>1/2}$, viewed as an unweighted graph, is the Turán graph $T_{t-s-1}(s)$. Equivalently, $V(R)$ must admit a partition $V(R) = B_1 \cup \cdots \cup B_a$ with $a \le t-s-1$ such that edges have weight $1/2$ if they lie within some part B_i and weight 1 otherwise.

We now show that R admits such a partition if $|V(R)| = b \ge s+1$. It suffices to show that any two vertices $v_1, v_2 \in V(R)$ with $w(v_1, v_2) = \frac{1}{2}$ must be identical [i.e., $w(v_1) = w(v_2)$ and $e(v_1, u) = e(v_2, u)$ for any third vertex $u \in V(R)$]. For $i \in [2]$, let R_i be the graph obtained from R by changing the edge weight of (v_{3-i}, u) to $w(v_i, u)$ for all $u \in V(R) \setminus \{v_1, v_2\}$, and changing the vertex weights of both v_1 and v_2 to $\frac{w(v_1)+w(v_2)}{2}$. We claim that R_1 and R_2 are $\mathcal{K}_t$-free. Indeed, suppose that R_1 contains a weighted t-clique (S_1, S_2) with $S_2 \subseteq S_1 \subseteq V(R_1)$ and $|S_1| + |S_2| = t$. Because $w(v_1, v_2) = \frac{1}{2}$ in R_1, the set S_2 cannot contain both v_1 and v_2; as these vertices are indistinguishable in R_i, we may assume that S_2 does not contain v_2. Hence, R_1 and R have the same edge weights between vertices of S_2. Furthermore, all edge weights of R (and in particular, all edge weights between vertices of S_1) are positive because R satisfies (A1). It follows that S_1 and S_2 form a weighted t-clique in R, a contradiction. The proof that R_2 is $\mathcal{K}_t$-free is analogous.

Write $\alpha_i = \frac{w(v_i)}{w(v_1)+w(v_2)}$ for $i \in [2]$ as in (3.1). We see that

$$\sum_{i\in[2]} \alpha_i \cdot \left(d_{K_s}(R_i) - d_{K_s}(R_i, \{v_1, v_2\})\right) = d_{K_s}(R) - d_{K_s}(R, \{v_1, v_2\}).$$

To compare $d_{K_s}(R, \{v_1, v_2\})$ and $d_{K_s}(R_i, \{v_1, v_2\})$, let $S = \{v_1, \dots, v_s\} \subseteq V(R)$ be any set of s vertices containing v_1 and v_2. We observe that

$$\begin{aligned} d_{K_s}(R, S) &= s! \cdot w(v_1)w(v_2)w(v_1, v_2)\left(\prod_{i=3}^{s} w(v_i)\right)\left(\prod_{i=3}^{s} w(v_1, v_i)\right)\left(\prod_{i=3}^{s} w(v_2, v_i)\right)\left(\prod_{s\ge i>j\ge 3} w(v_i, v_j)\right) \\ &= \frac{s!}{2} w(v_1)w(v_2)W_1W_2W_3, \end{aligned}$$

where $W_1 := \prod_{i=3}^{s} w(v_1, v_i)$, $W_2 := \prod_{i=3}^{s} w(v_2, v_i)$, and $W_3 := \left(\prod_{i=3}^{s} w(i)\right)\left(\prod_{s\ge i>j\ge 3} w(v_i, v_j)\right)$. Furthermore, by the AM-GM inequality, we have

$$\sum_{i\in[2]} \alpha_i \cdot d_{K_s}(R_i, S) = \sum_{i\in[2]} \alpha_i \cdot \frac{s!}{2}\left(\frac{w(v_1)+w(v_2)}{2}\right)^2 W_i^2 W_3 \ge \frac{s!}{2} w(v_1)w(v_2)W_1W_2W_3 = d_{K_s}(R, S),$$

with equality if and only if $w(v_1) = w(v_2)$ and $W_1 = W_2$.

Summing over all such sets S, we have

$$\sum_{i\in[2]} \alpha_i \cdot d_{K_s}(R_i,\{v_1,v_2\}) = \sum_{i\in[2]} \sum_{\substack{S\supseteq\{v_1,v_2\},\\ |S|=s}} \alpha_i \cdot d_{K_s}(R_i,S) \geq \sum_{\substack{S\supseteq\{v_1,v_2\},\\ |S|=s}} d_{K_S}(R,S) = d_{K_s}(R,\{v_1,v_2\}).$$

Moreover, equality holds if and only if $w(v_1) = w(v_2)$ and $W_1 = W_2$ for all sets S. We claim that this condition implies that $w(v_1,u) = w(v_2,u)$ for each $u \in V(R) - \{v_1,v_2\}$. Indeed, because all edge weights are either $1/2$ or 1, it follows that

$$\sum_{i=3}^{s} w(v_1,v_i) = \sum_{i=3}^{s} w(v_2,v_i)$$

for any $s-2$ distinct vertices $v_3,\ldots,v_s \in V(R) - \{v_1,v_2\}$. Letting $\mathbf{w}^{(1)},\mathbf{w}^{(2)} \in \{\frac{1}{2},1\}^{b-2}$ be vectors defined as $\mathbf{w}^{(i)}_u = w(v_i,u)$ for $u \in V(R) - \{v_1,v_2\}$, this yields a linear relation $\mathbf{v}\cdot\mathbf{w}^{(1)} = \mathbf{v}\cdot\mathbf{w}^{(2)}$, where $\mathbf{v} \in \mathbb{R}^{b-2}$ is the indicator vector of $\{v_3,\ldots,v_s\}$. When $b > s$, the vectors $\mathbf{v}$ span $\mathbb{R}^{b-2}$, and it follows that $\mathbf{w}^{(1)} = \mathbf{w}^{(2)}$.

We conclude that $\alpha_1 \cdot d_{K_s}(R_1) + \alpha_2 \cdot d_{K_s}(R_2) \geq d_{K_s}(R)$, with equality only if $w(v_1) = w(v_2)$ and $w(v_1,u) = w(v_2,u)$ for any third vertex u. Because R is extremal, it follows that any $v_1,v_2 \in V(R)$ with $w(v_1,v_2) = 1/2$ must satisfy these conditions. This implies that $V(R)$ may be partitioned into $B_1 \cup \cdots \cup B_a$ such that vertices within each part have the same weights, and edges have weight $1/2$ if they lie within some part B_i and weight 1 otherwise.

Lastly, we show that if R is extremal and $V(R)$ admits such a partition $B_1 \cup \cdots \cup B_a$ then $a+b = t-1$, where $b = |V(R)| \geq s$. If $a+b \geq t$ then we may form a weighted t-clique (S_1,S_2) by setting $S_1 = V(R)$ and letting S_2 contain one vertex from each of $B_1,\ldots,B_{t-b}$. If $a+b \leq t-2$ then we claim that R is not extremal. Choose a vertex $v \in B_1$ and let R' be the weighted graph obtained from R by replacing v with two vertices v_1,v_2 of weight $\frac{w(v)}{2}$, setting $w(v_1,v_2) = 1/2$ and $w(v_i,u) = w(v,u)$ for $u \in V(R) - \{v\}$ and $i \in [2]$. We note that R' is $\mathcal{K}_t$-free: $R'_{>1/2}$ is K_{a+1}-free, so for any weighted clique (S_1,S_2) in R', we have $|S_1| + |S_2| \leq |V(R')| + a = b+1+a \leq t-1$. Moreover, it is clear that $d_{K_s}(R') > d_{K_s}(R)$, contradicting the extremality of R. It follows that $a+b = t-1$, as desired.

3.3. *Proof of (A3) and (A4) for* $a = 2$

We first prove (A3) and (A4) for weighted graphs R satisfying (A2) with $a = 2$ parts. For convenience, we introduce the following notation. Given positive integers P,Q and real numbers $p,q \in (0,1)$ satisfying $pP+qQ = 1$, let $R(p,P;q,Q)$ denote the $(P+Q)$-vertex weighted graph satisfying (A2) with parameters $a = 2$, $|B_1| = P$, and $|B_2| = Q$, such that $w(v_1) = p$ and $w(v_2) = q$ for any vertex $v_1 \in B_1$ or $v_2 \in B_2$.

In Lemmas 3.3, 3.5 and 3.8 below, we show that if $R = R(p,P;q,Q)$ does not satisfy (A3) or (A4) then there is another weighted graph R' on $P+Q$ vertices such that $R'_{>\frac{1}{2}}$ is also bipartite and $d_{K_m}(R') > d_{K_m}(R)$ for all m in the range $2 \leq m \leq P+Q$. This is a slightly stronger statement than necessary to handle the $a = 2$ case, but it will prove necessary when we consider $a \geq 3$ in the next subsection.

Lemma 3.3. *Let P,Q be positive integers and $p,q \in (0,1)$ real numbers such that $pP+qQ = 1$. If $P \geq Q+1$ and $(P-1)p > Qq$, then there exists a weighted graph R' with $P+Q$ vertices such that $R'_{>\frac{1}{2}}$ is bipartite and $d_{K_m}(R(p,P;q,Q)) < d_{K_m}(R')$ for all $2 \leq m \leq P+Q$.*

Proof. Set $R = R(p,P;q,Q)$ and let u be a vertex in R with weight p. Let R' be the graph obtained from R by changing the weights of all edges incident to u: if (u,v) has edge weight $w \in \{\frac{1}{2},1\}$ in R, then we assign it the weight $\frac{3}{2} - w$ in R'. It is clear that $R'_{>\frac{1}{2}}$ is a complete bipartite graph with parts of size $P-1$ and $Q+1$.

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Fix m with $2 \le m \le P+Q$. We verify that $d_{K_m}(R,u) < d_{K_m}(R',u)$. We have that

$$d_{K_m}(R,u) = m! \sum_{x+y=m-1} \binom{P-1}{x}\binom{Q}{y} p^{x+1} q^y \left(\frac{1}{2}\right)^{\binom{x}{2}+\binom{y}{2}+x},$$

$$d_{K_m}(R',u) = m! \sum_{x+y=m-1} \binom{P-1}{x}\binom{Q}{y} p^{x+1} q^y \left(\frac{1}{2}\right)^{\binom{x}{2}+\binom{y}{2}+y}.$$

Let $M(x,y) = \binom{P-1}{x}\binom{Q}{y} p^{x+1} q^y$. If $x \ge y$ then

$$\frac{M(x,y)}{M(y,x)} = \frac{p^{x-y}}{q^{x-y}} \prod_{i=1}^{x-y} \frac{P-y-i}{Q+1-y-i} \ge \frac{p^{x-y}}{q^{x-y}} \left(\frac{P-1}{Q}\right)^{x-y} \ge 1,$$

with equality only if $x = y$. This in turn implies that

$$M(x,y)\left(\frac{1}{2}\right)^x + M(y,x)\left(\frac{1}{2}\right)^y \le M(x,y)\left(\frac{1}{2}\right)^y + M(y,x)\left(\frac{1}{2}\right)^x$$

for any x, y, again with equality only if $x = y$. Thus,

$$\begin{aligned} 2d_{K_m}(R,u) &= m! \sum_{x+y=m-1} \left(\frac{1}{2}\right)^{\binom{x}{2}+\binom{y}{2}} \left(M(x,y)\left(\frac{1}{2}\right)^x + M(y,x)\left(\frac{1}{2}\right)^y\right) \\ &< m! \sum_{x+y=m-1} \left(\frac{1}{2}\right)^{\binom{x}{2}+\binom{y}{2}} \left(M(x,y)\left(\frac{1}{2}\right)^y + M(y,x)\left(\frac{1}{2}\right)^x\right) = 2d_{K_m}(R',u). \end{aligned}$$

To conclude the proof, we observe that $d_{K_m}(R-u) = d_{K_m}(R'-u)$, and thus

$$d_{K_m}(R) = d_{K_m}(R,u) + d_{K_m}(R-u) < d_{K_m}(R',u) + d_{K_m}(R'-u) = d_{K_m}(R'). \qquad \square$$

Next, we handle (A3) in the case that $(P-1)p \le Qq$. To help us bound $d_{K_m}(R(p,P;q,Q))$ in this case, we shall write it in terms of the following quantities. Given integers m, r with $0 \le r \le m/2$, define

$$\begin{aligned} N_{m,r}(p,P;q,Q) &= r! \cdot \binom{P}{r} p^r \cdot r! \cdot \binom{Q}{r} q^r \sum_{\substack{x+y=m,\\ x,y \ge r}} \binom{P-r}{x-r} p^{x-r} \binom{Q-r}{y-r} q^{y-r} \\ &= \sum_{\substack{x+y=m,\\ x,y \ge r}} \binom{P}{x} p^x \binom{Q}{y} q^y \frac{x!y!}{(x-r)!(y-r)!}. \end{aligned}$$

Intuitively, $N_{m,r}$ should be thought of as counting copies of K_m in $R(p,P;q,Q)$ with r labeled vertices in each part. We now show that $d_{K_m}(R(p,P;q,Q))$ is a positive linear combination of these quantities.

Lemma 3.4. *Fix a positive integer m. There exist constants $c_r > 0$ for $0 \le r \le \lfloor m/2 \rfloor$ such that*

$$d_{K_m}(R(p,P;q,Q)) = \sum_{r=0}^{\lfloor m/2 \rfloor} c_r N_{m,r}(p,P;q,Q)$$

for any P, Q, p, q.

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Proof. Observe that each $N_{m,r}$ is a linear combination of the $\lfloor \frac{m}{2} \rfloor + 1$ polynomials

$$\left\{ \binom{P}{x}\binom{Q}{m-x} p^x q^{m-x} + \binom{P}{m-x}\binom{Q}{x} p^{m-x} q^x : 0 \leq x \leq \frac{m}{2} \right\},$$

with nonzero coefficient if and only if $x \geq r$. Thus, $\{N_{m,r} : 0 \leq r \leq m/2\}$ is a basis for the space of all linear combinations of these polynomials, which includes the K_m-density

$$d_{K_m}(R(p,P;q,Q)) = m! \cdot \sum_{x+y=m} \binom{P}{x}\binom{Q}{y} p^x q^y 2^{xy-\binom{m}{2}} = \frac{m!}{2^{\binom{m}{2}}} \cdot \sum_{x+y=m} \binom{P}{x}\binom{Q}{y} p^x q^y 2^{xy}.$$

It follows that there exist real numbers c_r such that

$$d_{K_m}(R(p,P;q,Q)) = \sum_{r=0}^{\lfloor m/2 \rfloor} c_r N_{m,r}(p,P;q,Q).$$

Moreover, by setting the coefficients of the rth term equal to each other, we conclude that

$$\frac{m!}{2^{\binom{m}{2}}} \cdot 2^{r(m-r)} = \sum_{i=0}^{r} \frac{r!(m-r)!}{(r-i)!(m-r-i)!} c_i \tag{3.2}$$

for each $r \leq m/2$.

We show that the coefficients c_r are positive by induction on r. Clearly, $c_0 > 0$ by (3.2) when $r = 0$. Now, suppose $c_i > 0$ for all $i < r$. By (3.2), we have

$$\begin{aligned}\sum_{i=0}^{r} \frac{r!(m-r)!}{(r-i)!(m-r-i)!} c_i &= 2^{r(m-r)} \cdot \frac{m!}{2^{\binom{m}{2}}} = \frac{m!}{2^{\binom{m}{2}}} 2^{m-2r+1} 2^{(r-1)(m-r+1)} \\ &= 2^{m-2r+1} \sum_{i=0}^{r-1} \frac{(r-1)!(m-r+1)!}{(r-1-i)!(m-r+1-i)!} c_i.\end{aligned}$$

We claim that

$$2^{m-2r+1} \frac{(r-1)!(m-r+1)!}{(r-1-i)!(m-r+1-i)!} > \frac{r!(m-r)!}{(r-i)!(m-r-i)!}$$

for each $i < r$. Indeed, it suffices to show that

$$2^{m-2r+1} \frac{r-i}{m-r+1-i} \geq 1 > \frac{r}{m-r+1},$$

which, recalling that $r \leq m/2$, follows from the inequalities $2^{m-2r+1} \geq m - 2r + 2$ and $\frac{r-i}{m-r+1-i} = \frac{r-i}{m-2r+1+(r-i)} \geq \frac{1}{m-2r+2}$. Hence,

$$\sum_{i=0}^{r} \frac{r!(m-r)!}{(r-i)!(m-r-i)!} c_i = 2^{m-2r+1} \sum_{i=0}^{r-1} \frac{(r-1)!(m-r+1)!}{(r-1-i)!(m-r+1-i)!} c_i > \sum_{i=0}^{r-1} \frac{r!(m-r)!}{(r-i)!(m-r-i)!} c_i,$$

which implies $c_r > 0$. □

Using Lemma 3.4, we handle (A3) in the case that $(P-1)p \leq Qq$.

Lemma 3.5. *Let P, Q be positive integers with $P \geq Q + 2$ and let $p, q \in (0, 1)$ be real numbers such that $Pp + Qq = 1$. Set $P' = P - 1$, $Q' = Q + 1$, and choose $p', q' \in (0, 1)$ such that $P'p' = Pp$ and $Q'q' = Qq$. If $(P-1)p \leq Qq$ then $d_{K_m}(R(p,P;q,Q)) < d_{K_m}(R(p',P';q',Q'))$ for all $2 \leq m \leq P+Q$.*

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Proof. Set $R = R(p, P; q, Q)$ and $R' = R(p', P'; q', Q')$, and fix m with $2 \leq m \leq P + Q$. For convenience, we write $N_{m,r}(R) = N_{m,r}(p, P; q, Q)$ and $N_{m,r}(R') = N_{m,r}(p', P'; q', Q')$.

By Lemma 3.4, it suffices to compare $N_{m,r}(R)$ and $N_{m,r}(R')$. We begin with some preliminary inequalities. Let $p_- = \min\{p', q'\}$ and $p_+ = \max\{p', q'\}$.

Claim 3.6. *We have the following inequalities.*

(1) *For any* $0 < r \leq |Q|$, *we have* $(1 - \frac{r}{P})(1 - \frac{r}{Q}) < (1 - \frac{r}{P'})(1 - \frac{r}{Q'})$.
(2) $p \leq p_- \leq p_+ < q$.
(3) *For any* $0 < r \leq |Q|$, *we have* $(P - r)p + (Q - r)q < (P' - r)p' + (Q' - r)q'$.

Proof of claim.
(1) Because $P \geq Q + 2$, we have

$$\begin{aligned}\left(1 - \frac{r}{P'}\right)\left(1 - \frac{r}{Q'}\right) - \left(1 - \frac{r}{P}\right)\left(1 - \frac{r}{Q}\right) &= \left(1 + \frac{r^2 - (P+Q)r}{(P-1)(Q+1)}\right) - \left(1 + \frac{r^2 - (P+Q)r}{PQ}\right)\\ &= \left(r^2 - (P+Q)r\right)\left(\frac{1}{PQ + P - 1 - Q} - \frac{1}{PQ}\right) > 0.\end{aligned}$$

(2) Using the relation $(P - 1)p \leq Qq$, it follows that $p < \frac{P}{P-1}p = p' \leq \frac{PQ}{(P-1)^2}q < q$ and that $q > \frac{Qq}{Q+1} = q' \geq \frac{(P-1)p}{Q+1} \geq p$.
(3) First, observe that

$$p + q - p' - q' = p + q - \frac{P}{P-1}p - \frac{Q}{Q+1}q = \frac{q}{Q+1} - \frac{p}{P-1} > 0$$

because $p \leq Qq/(P - 1) < q$. Thus,

$$(P' - r)p' + (Q' - r)q' - (P - r)p - (Q - r)q = (P'p' + Q'q' - Pp - Qq) + r(p + q - p' - q') > 0$$

if $r > 0$. $\square$

We now compare $N_{m,r}(R)$ and $N_{m,r}(R')$.

Claim 3.7. *We have* $N_{m,r}(R) \leq N_{m,r}(R')$ *for any* $0 \leq r \leq \lfloor m/2 \rfloor$. *Moreover, the inequality is strict when* $r > 0$.

Proof of claim. By Claim 3.6(1),

$$\begin{aligned}r! \cdot \binom{P}{r}p^r \cdot r! \cdot \binom{Q}{r}q^r &= (Pp)^r(Qq)^r \prod_{i=0}^{r-1}\left(1 - \frac{i}{P}\right)\left(1 - \frac{i}{Q}\right)\\ &\leq (P'p')^r(Q'q')^r \prod_{i=0}^{r-1}\left(1 - \frac{i}{P'}\right)\left(1 - \frac{i}{Q'}\right) = r! \cdot \binom{P'}{r}(p')^r \cdot r! \cdot \binom{Q'}{r}(q')^r,\end{aligned}$$

with equality only if $r = 0$. It remains to show that

$$\sum_{\substack{x+y=m-2r\\x,y\geq 0}} \binom{P-r}{x}p^x\binom{Q-r}{y}q^y \leq \sum_{\substack{x+y=m-2r\\x,y\geq 0}} \binom{P'-r}{x}(p')^x\binom{Q'-r}{y}(q')^y. \tag{3.3}$$

Set $X = (\underbrace{p, p, \ldots, p}_{P-r}, \underbrace{q, q, \ldots, q}_{Q-r})$ and $X' = (\underbrace{p', p', \ldots, p'}_{P-r-1}, \underbrace{q', q', \ldots, q'}_{Q-r+1})$. Letting σ be the $(m - 2r)$th elementary symmetric function

$$\sigma(y_1, \ldots, y_{P+Q-2r}) := \sum_{1 \le i_1 < \cdots < i_{m-2r} \le P+Q-2r} y_{i_1} \cdots y_{i_{m-2r}},$$

we may rewrite (3.3) as $\sigma(X) \le \sigma(X')$.

Let Y be the $(P+Q-2r)$-tuple obtained via the following process. Set $Y = X$ initially and repeat the following transformation, which will never decrease $\sigma(Y)$.

(∗) If there exist indices i, j such that $Y_i < p_-$ and $Y_j > p_+$, set $\varepsilon = \min\{p_- - Y_i,\ Y_j - p_+\}$, and replace Y_i and Y_j with $Y_i + \varepsilon$ and $Y_j - \varepsilon$, respectively.

Each iteration increases the number of coordinates equal to p_- or p_+, so the process will terminate in at most $P+Q-2r$ steps. Moreover, recalling that $p \le p_- \le p_+ < q$, we observe that the final tuple Y either takes the form

$$Y = (Y_1, \ldots, Y_{P-r}, p_+, \ldots, p_+) \text{ with } Y_i \le p_- \text{ for all } i \le P-r, \text{ or} \tag{Case 1}$$
$$Y = (p_-, \ldots, p_-, Y_{P-r+1}, \ldots, Y_{P+Q-2r}) \text{ with } Y_i \ge p_+ \text{ for all } i > P-r. \tag{Case 2}$$

Let $X'' = (p_-, \ldots, p_-, p_+, \ldots, p_+)$ be the result of sorting X' in increasing order, and let $k \in \{P-r-1, Q-r+1\}$ be the number of occurrences of p_-. In case 1, we have $Y_i \le X''_i$ for each i, implying $\sigma(Y) \le \sigma(X'')$. In case 2, let y be the average value of $\{Y_{k+1}, \ldots, Y_{P+Q-2r}\}$ and let $Y' = (p_-, \ldots, p_-, y, \ldots, y)$ be the result of replacing all but the first k terms of Y with y. By Lemma 3.2, $\sigma(Y) \le \sigma(Y')$. Moreover, $\mathrm{sum}(Y') = \mathrm{sum}(X) < \mathrm{sum}(X'')$ by Claim 3.6(3). Because X'' is obtained from Y' by replacing each y with p_+, it follows that $y < p_+$ and $\sigma(Y') < \sigma(X'')$. Thus, we have $\sigma(X) \le \sigma(Y) \le \sigma(X'') = \sigma(X')$ in both cases, completing the proof that $N_{m,r}(R) \le N_{m,r}(R')$. □

Combining Lemma 3.4 and Claim 3.7, we have that

$$d_{K_m}(R) = \sum_{r=0}^{\lfloor m/2 \rfloor} c_r N_{m,r}(R) < \sum_{r=0}^{\lfloor m/2 \rfloor} c_r N_{m,r}(R') = d_{K_m}(R'). \qquad \square$$

Lastly, we turn our attention to (A4). If $P > Q$, this is an immediate consequence of Lemma 3.3; it remains to handle the $P = Q$ case.

Lemma 3.8. *Let P be a positive integer and $p, q \in (0,1)$ real numbers such that $pP + qP = 1$. If $p \ne q$, then there exists a weighted graph R' with $2P$ vertices such that $R'_{>\frac{1}{2}}$ is bipartite and $d_{K_m}(R(p,P;q,P)) < d_{K_m}(R')$ for all $2 \le m \le 2P$.*

Proof. Set $R = R(p,P;q,P)$ and let u and v be vertices in R with weights p and q, respectively. Let R' be the graph obtained from R by setting the weights of both u and v to $\frac{p+q}{2} = \frac{1}{2P}$.

Observe that $d_{K_m}(R - \{u,v\}) = d_{K_m}(R' - \{u,v\})$. Set

$$f(x,y) = m! \binom{P-1}{x} \binom{P-1}{y} \left(\frac{1}{2}\right)^{\binom{x}{2} + \binom{y}{2}}.$$

We have that

$$\begin{aligned} &d_{K_m}((R'-v),u) + d_{K_m}((R'-u),v) - d_{K_m}((R-v),u) - d_{K_m}((R-u),v) \\ &= \sum_{x+y=m-1} f(x,y) p^x q^y \left[\frac{p+q}{2} \left(\left(\frac{1}{2}\right)^x + \left(\frac{1}{2}\right)^y \right) - \left(p \left(\frac{1}{2}\right)^x + q \left(\frac{1}{2}\right)^y \right) \right] \end{aligned}$$

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$$= \sum_{x+y=m-1} f(x,y)p^x q^y \left(\frac{q-p}{2}\right)\left(\left(\frac{1}{2}\right)^x - \left(\frac{1}{2}\right)^y\right)$$

$$= \sum_{\substack{x+y=m-1,\\ x<y}} f(x,y)p^x q^x (q^{y-x} - p^{y-x})\left(\frac{q-p}{2}\right)\left(\left(\frac{1}{2}\right)^x - \left(\frac{1}{2}\right)^y\right) > 0$$

because $q-p$ and $q^{y-x} - p^{y-x}$ both have the same sign. Additionally,

$$d_{K_m}(R', \{u,v\}) - d_{K_m}(R, \{u,v\}) = \sum_{x+y=m-2} f(x,y)p^x q^y \left(\frac{1}{2}\right)^{x+y} \left[\left(\frac{p+q}{2}\right)^2 - pq\right] > 0.$$

Combining the preceding inequalities, we conclude that

$$\begin{aligned} d_{K_m}(R) &= d_{K_m}(R - \{u,v\}) + d_{K_m}((R-v),u) + d_{K_m}((R-u),v) + d_{K_m}(R,\{u,v\}) \\ &< d_{K_m}(R' - \{u,v\}) + d_{K_m}((R'-v),u) + d_{K_m}((R'-u),v) + d_{K_m}(R',\{u,v\}) = d_{K_m}(R') \end{aligned}$$

for any integer $2 \le m \le 2P$. □

3.4. Proof of (A3) and (A4) for all a

Using the results of the prior subsection, we prove (A3) and (A4) for all a. Suppose R is an extremal $\mathcal{K}_t$-free weighted graph satisfying (A1) and (A2) with parts $B_1, \ldots, B_a$. Given indices $i \neq j$, we may regard $R[B_i \cup B_j]$ as a scaled-down version of some weighted graph $R_0 = R(p, |B_i|; q, |B_j|)$, where the weights of vertices in B_i and B_j are αp and αq respectively, for some $\alpha \le 1$.

Without loss of generality, suppose $|B_i| \ge |B_j|$. We claim that $|B_i| \le |B_j| + 1$ and that $p \le q$. Indeed, if $|B_i| \ge |B_j| + 2$ then Lemma 3.3 or Lemma 3.5 yields a weighted graph R_0' on $|B_i| + |B_j|$ vertices such that $(R_0')_{>\frac{1}{2}}$ is bipartite and $d_{K_m}(R_0') > d_{K_m}(R_0)$ for all $2 \le m \le |B_i| + |B_j|$. Note that $d_{K_m}(R_0) = d_{K_m}(R_0')$ for all other m: this value is 1 if $m = 1$ and 0 if $m > |B_i| + |B_j|$. If $p > q$, then Lemma 3.3 (if $|B_i| > |B_j|$) or Lemma 3.8 (if $|B_i| = |B_j|$) provides a weighted graph R_0' with the same properties.

Let R' be the weighted graph obtained from R by replacing $R[B_i \cup B_j]$ with a scaled-down copy of R_0'. That is, the vertex weights of $R'[B_i \cup B_j]$ are those of R_0' multiplied by α, and the edge weights of $R'[B_i \cup B_j]$ are exactly those of R_0'. We verify that R' is $\mathcal{K}_t$-free. Suppose (S_1, S_2) is a weighted clique in R'. Because $(R_0')_{>1/2}$ is bipartite, S_2 contains at most two vertices of $B_i \cup B_j$; additionally, S_2 contains at most one vertex from each other part B_k. Hence $|S_1| + |S_2| \le |V(R)| + a = b + a = t - 1$, so R' is $\mathcal{K}_t$-free. However, because

$$d_{K_m}(R[B_i \cup B_j]) = \alpha^m d_{K_m}(R_0) \quad \text{and} \quad d_{K_m}(R'[B_i \cup B_j]) = \alpha^m d_{K_m}(R_0'),$$

we have that

$$\begin{aligned} d_{K_s}(R) &= \sum_{m=0}^{s} \binom{s}{m} \alpha^m d_{K_m}(R_0) \cdot d_{K_{s-m}}(R - (B_i \cup B_j)) \\ &< \sum_{m=0}^{s} \binom{s}{m} \alpha^m d_{K_m}(R_0') \cdot d_{K_{s-m}}(R - (B_i \cup B_j)) = d_{K_s}(R'). \end{aligned}$$

Thus, if the parts (B_i, B_j) do not satisfy (A3) or (A4) then $d_{K_s}(R) < d_{K_s}(R')$, contradicting the extremality of R.

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3.5. Proof of (A5)

Suppose that R is an extremal $\mathcal{K}_t$-free weighted graph satisfying (A1) and thus (A2)–(A4) with parts $B_1, \ldots, B_a$. Without loss of generality, suppose B_1 has maximal cardinality among all parts B_i. We assume that $r = |B_1|$ satisfies $r \geq s + 1$ (if $a = 1$) or $r \geq s$ (if $a \geq 2$) and derive a contradiction.

Let p be the weight of each vertex in B_1; by (A4), it follows that all vertices of R have weight at least p. Let R' be the graph obtained from R by replacing two vertices $u, v \in B_1$ with a new vertex v' of weight $2p$ and setting $w(v', u') = 1$ for any $u' \in V(R') \setminus \{v'\}$. We observe that R' is $\mathcal{K}_t$-free. Indeed, if (S_1, S_2) is a weighted clique configuration in R' then S_2 may contain at most one vertex from each of the sets $\{v'\}, B_1 - \{u, v\}, B_2, \ldots, B_a$. It follows that $|S_1| + |S_2| \leq |V(R')| + (a+1) = |V(R)| + a$, which is $t - 1$ by (A2). To conclude the proof of (A5), we show that $d_{K_s}(R) < d_{K_s}(R')$ if R does not satisfy (A5), contradicting the extremality of R.

Set $B' = (B_1 - \{u, v\}) \cup \{v'\} \subseteq V(R')$ and for each $0 \leq m \leq s$ set

$$f_m = \frac{d_{K_m}(R[B_1])}{m!}, \quad g_m = \frac{d_{K_m}(R'[B'])}{m!}, \quad \text{and} \quad h_m = \frac{d_{K_m}(R - B_1)}{m!} = \frac{d_{K_m}(R' - B')}{m!}.$$

Recalling that $|B_1| = r \geq s \geq 3$, we have

$$\begin{aligned} f_m &= \binom{r}{m} p^m \left(\frac{1}{2}\right)^{\binom{m}{2}}, \text{ and} \\ g_m &= \binom{r-2}{m} p^m \left(\frac{1}{2}\right)^{\binom{m}{2}} + \binom{r-2}{m-1} (2p) p^{m-1} \left(\frac{1}{2}\right)^{\binom{m-1}{2}} \\ &= \left[\binom{r-2}{m} + 2^m \binom{r-2}{m-1}\right] p^m \left(\frac{1}{2}\right)^{\binom{m}{2}}. \end{aligned}$$

For any $1 \leq m \leq r-1$, observe that

$$\begin{aligned} \binom{r-2}{m} + 2^m \binom{r-2}{m-1} - \binom{r}{m} &= \binom{r-1}{m} + (2^m - 1)\binom{r-2}{m-1} - \left(\binom{r-1}{m} + \binom{r-1}{m-1}\right) \\ &= \left(2^m - 1 - \frac{r-1}{r-m}\right)\binom{r-2}{m-1} \geq (2^m - 1 - m)\binom{r-2}{m-1} \geq 0, \end{aligned}$$

with equality if and only if $m = 1$. Recalling that $f_0 = g_0 = 1$, we conclude that $f_m \leq g_m$ for all $0 \leq m \leq r-1$ with equality if and only if $m < 2$. If $r \geq s+1$, then this inequality holds for all $2 \leq m \leq s \leq r-1$, and we conclude that

$$\begin{aligned} d_{K_s}(R) &= \sum_{m=0}^{s} \binom{s}{m} d_{K_m}(R[B_1]) d_{K_{s-m}}(R - B_1) = \sum_{m=0}^{s} s! f_m h_{s-m} \\ &< \sum_{m=0}^{s} s! g_m h_{s-m} = \sum_{m=0}^{s} \binom{s}{m} d_{K_m}(R'[B']) d_{K_{s-m}}(R' - B') = d_{K_s}(R'). \end{aligned}$$

Now, suppose $r = s$ and $a \geq 2$. We observe that $h_1 = \sum_{v \in V(R) - B_1} w(v) \geq |V(R) - B_1| p \geq p$, so

$$\begin{aligned} f_{s-1} h_1 + f_s h_0 &= s p^{s-1} \left(\frac{1}{2}\right)^{\binom{s-1}{2}} h_1 + p^s \left(\frac{1}{2}\right)^{\binom{s}{2}} \leq \left(s \cdot \left(\frac{1}{2}\right)^{s-2} + \left(\frac{1}{2}\right)^{2s-3}\right) p^{s-1} \left(\frac{1}{2}\right)^{\binom{s-2}{2}} h_1 \\ &< (2p) p^{s-2} \left(\frac{1}{2}\right)^{\binom{s-2}{2}} h_1 = g_{s-1} h_1 \leq g_{s-1} h_1 + g_s h_0. \end{aligned}$$

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Once again, we conclude

$$d_{K_s}(R) = \sum_{m=0}^{s} s!f_m h_{s-m} < \sum_{m=0}^{s} s!g_m h_{s-m} = d_{K_s}(R').$$

Thus, if R does not satisfy (A5) then $d_{K_s}(R') > d_{K_s}(R)$, contradicting the extremality of R. This completes the proof of Theorem 3.1.

4. Eventual periodicity and counterexamples to Conjecture 1.2

In this section, we prove Theorem 1.3, providing conditions under which Conjecture 1.2 does and does not hold. Applying Theorem 2.4, we may reduce this to a problem about weighted graphs.

Say a weighted graph R *admits a* (b,a)*-partition* if it has b vertices and satisfies the five conditions (A1)–(A5) of Theorem 3.1, with a being the number of parts in (A2). Theorem 1.4 shows that $\varrho_s(K_t)$ is the maximum K_s-density of a weighted graph R admitting a (b,a)-partition with $a = t-1-b$ parts for some $b < t$; such R are inherently $\mathcal{K}_t$-free. From (A2), we have $b \geq s$; additionally, because R cannot have fewer vertices than parts, we have $b \geq \lfloor t/2 \rfloor$. Conjecture 1.2 hypothesizes that the optimal density is attained when b matches one of these lower bounds.

Conjecture 4.1 (Conjecture 1.2, rephrased in terms of weighted graphs). *Fix integers* s,t *with* $3 \leq s \leq t-2$*. The maximum* K_s*-density of a weighted graph admitting a* $(b, t-1-b)$*-partition is attained when* $b = \max\{s, \lfloor t/2 \rfloor\}$.

For $t \geq 2s$, Conjecture 4.1 hypothesizes that the extremal $\mathcal{K}_t$-free weighted graphs follow an alternating pattern as depicted in Table 1. For odd t, the conjectural extremal construction is the *complete balanced weighted graph* $K^w_{\lfloor t/2 \rfloor}$, which has $b = \lfloor t/2 \rfloor$ vertices with weight $1/b$ each and has all $\binom{b}{2}$ edge weights equal to 1. For even t, the conjectural extremal construction has $b = t/2$ vertices divided into one part of size 2 and $b-2$ parts of size 1. That is, all edges have weight 1 except for one edge of weight $1/2$ within the part of size 2.

The proof of Theorem 1.3 is divided into three subsections. In Section 4.1, we show that Conjecture 4.1 holds if $t > s^2(s-1)/2 + s + 1$, implying that the extremal constructions do eventually follow the aforementioned alternating pattern. In Section 4.2, we show that Conjecture 4.1 holds if $s = 3, 4$ or if $t = s+2$. Lastly, in Section 4.3, we provide counterexamples to Conjecture 4.1 for $s = 5$ as well as for any sufficiently large s.

4.1. Eventual Periodicity

We first show that Conjecture 4.1 holds when t is sufficiently large as a function of s.

Lemma 4.2. *Fix integers* s,t *with* $3 \leq s \leq t-2$*. Suppose* R *is a weighted graph admitting a* (b,a)*-partition into* $B_1 \cup \cdots \cup B_a$ *for some* a, b *satisfying* $a + b = t - 1$.

(1) *Suppose* t *is odd with* $t > s^2(s-1)/2$*. Set* $r = (t-1)/2$ *and let* K^w_r *be the complete balanced weighted graph on* r *vertices. We have* $d_{K_s}(R) \leq d_{K_s}(K^w_r)$ *with equality if and only if* $R = K^w_r$.
(2) *Suppose* t *is even with* $t > s^2(s-1)/2 + s$*. If* $d_{K_s}(R) = \pi_s(\mathcal{K}_t)$ *then* $a-1$ *of the parts* B_i *must have cardinality 1, and the last part must have cardinality 2.*

Proof. We first prove (1). Suppose $t > s^2(s-1)/2$ is odd. Let R_0 be the weighted graph obtained from R by changing all edges weights of $1/2$ into 0 and let R_1 be the weighted graph obtained from R by changing all edge weights of $1/2$ into 1. We claim that

$$d_{K_s}(R) \leq \frac{d_{K_s}(R_0) + d_{K_s}(R_1)}{2}.$$

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Indeed, if vertices $v_1, \ldots, v_s \in V(R)$ induce $m \geq 1$ edges of weight $1/2$ in R, then the product of their edge weights is $\left(\frac{1}{2}\right)^m \leq \frac{1}{2}$ in R and is 1 in R_1.

We have $d_{K_s}(K_r^w) = s!\binom{r}{s}\frac{1}{r^s}$. Write $w(B_i) = \sum_{v \in B_i} w(v)$ for the total weight of vertices in B_i. By Lemma 3.2, we have

$$d_{K_s}(R_0) = s! \sum_{1 \leq i_1 < \cdots < i_s \leq a} w(B_{i_1}) \cdots w(B_{i_s}) \leq s!\binom{a}{s}\frac{1}{a^s}, \quad \text{and}$$

$$d_{K_s}(R_1) = \sum_{\substack{v_1, \ldots, v_s \in V(R) \\ \text{distinct}}} w(v_1) \cdots w(v_s) \leq s!\binom{b}{s}\frac{1}{b^s}.$$

Thus, setting $f(x) = s!\binom{x}{s}x^{-s} = (1 - \frac{1}{x}) \cdots (1 - \frac{s-1}{x})$, it suffices to show that $\frac{f(a)+f(b)}{2} \leq f(\frac{a+b}{2}) = d_{K_s}(K_r^w)$.

First, observe that

$$f'(x) = \sum_{i=1}^{s-1} \frac{i}{x^2} \prod_{\substack{1 \leq j < s, \\ j \neq i}} \left(1 - \frac{j}{x}\right)$$

and that

$$\begin{aligned} f''(x) &= \sum_{i=1}^{s-1} \left(-\frac{2i}{x^3} \prod_{\substack{1 \leq j < s, \\ j \neq i}} \left(1 - \frac{j}{x}\right) + \sum_{\substack{1 \leq j < s, \\ j \neq i}} \frac{ij}{x^4} \cdot \prod_{\substack{1 \leq k < s, \\ k \neq i,j}} \left(1 - \frac{k}{x}\right) \right) \\ &= \sum_{i=1}^{s-1} \left(-\frac{2i}{x^3} + \sum_{\substack{1 \leq j < s, \\ j \neq i}} \frac{ij}{x^4} \left(\frac{x}{x - j}\right) \right) \left(\prod_{\substack{1 \leq j < s, \\ j \neq i}} \left(1 - \frac{j}{x}\right) \right) \\ &= \sum_{i=1}^{s-1} \frac{2i}{x^3} \left(-1 + \sum_{\substack{1 \leq j < s, \\ j \neq i}} \frac{j}{2(x - j)} \right) \left(\prod_{\substack{1 \leq j < s, \\ j \neq i}} \left(1 - \frac{j}{x}\right) \right). \end{aligned}$$

It follows that $f''(x) < 0$ when $x \geq \binom{s}{2}$: for $s = 3$, this can be checked manually, and for $s \geq 4$, this follows from the inequality

$$\sum_{j=1}^{s-1} \frac{j}{2(x - j)} \leq \sum_{j=1}^{s-1} \frac{j}{x} = \binom{s}{2}\frac{1}{x} \leq 1.$$

Hence, if $a \geq \binom{s}{2}$, Jensen's inequality implies

$$d_{K_s}(R) \leq \frac{f(a) + f(b)}{2} \leq f\left(\frac{a + b}{2}\right) = d_{K_s}(K_r^w)$$

with equality if and only if $a = b = r$, (i.e., $R = K_r^w$). To see that $a \geq \binom{s}{2}$, note that (A5) implies $b \leq (s - 1)a$, yielding $sa \geq a + b = t - 1 \geq s\binom{s}{2}$ as desired.

We now prove (2) using (1). Suppose $t > s^2(s-1)/2+s$ is even. Because $t-1 = a+b = \sum_{i=1}^{a}(|B_i|+1)$ is odd, there exists some k such that $|B_k| + 1$ is odd. Scaling $R - B_k$ up by a factor of $(1 - w(B_k))^{-1}$ yields a weighted graph R_0 admitting a (b', a')-partition, where $a' = a - 1$ and $b' = b - |B_k|$.

Set $t' = a' + b' + 1 = t - |B_k| - 1$, which is odd, and set $r = (t' - 1)/2$. Let R' be the weighted graph obtained from R by replacing $V(R) - B_k$ with a set B' of r vertices of weight $(1 - w(B_k))/r$

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each, and setting $w(v', v) = 1$ for each $v' \in B'$ and $v \in V(R') \setminus \{v\}$. That is, $R'[B']$ is a copy of K_r^w scaled down by a factor of $1 - w(B_k)$. We note that R' is $\mathcal{K}_t$-free. Indeed, if (S_1, S_2) is a weighted clique in R', then S_2 contains at most one vertex from B_k, so $|S_2| \leq r + 1$. Hence, $|S_1| + |S_2| \leq |V(R')| + r + 1 = |B_k| + 2r + 1 = t - 1$.

Observe that $|B_k| \leq s - 1$ by (A5), so $t' > s^2(s-1)/2$. By part (1), we have

$$d_{K_m}(R - B_k) = (1 - w(B_k))^m d_{K_m}(R_0) \leq (1 - w(B_k))^m d_{K_m}(K_r^w) = d_{K_m}(R'[B'])$$

for all $m \leq s$. Equality holds if and only if $R_0 = K_r^w$, (i.e., if and only if $|B_i| = 1$ for each $i \neq k$). Hence, if $|B_j| > 1$ for some $j \neq k$, we have

$$d_{K_s}(R) = \sum_{m=0}^{s} \binom{s}{m} d_{K_m}(R - B_k) d_{K_{s-m}} R[B_k] < \sum_{m=0}^{s} \binom{s}{m} d_{K_m}(R'[B']) d_{K_{s-m}}(R'[B_k]) = d_{K_s}(R'),$$

contradicting the assumption that $d_{K_s}(R) = \pi_s(\mathcal{K}_t)$. To complete the proof of (2), we recall that $|B_k|$ is even and $|B_k| \leq |B_i| + 1$ for any $i \neq k$ by (A3). Hence, if $|B_i| = 1$ for all $i \neq k$ then $|B_k| = 2$. □

4.2. The Remaining Positive Results

We now prove Conjecture 4.1 in the remaining cases described in Theorem 1.3. We remark that the cases $t = s + 2$ and $s = 3$ were proven in [6]. However, the proofs of these cases are very straightforward when framed in terms of weighted graphs, so we present them for completeness.

Lemma 4.3. *Fix $s \geq 3$ and let $t = s + 2$. If R is a $\mathcal{K}_t$-free weighted graph R with $d_{K_s}(R) = \pi_s(\mathcal{K}_t)$ of minimum cardinality, it admits an $(s, 1)$-partition.*

Proof. From Theorem 3.1(A2), R admits a (b, a) partition for parameters $a \geq 1$ and $b \geq s$ satisfying $a + b = t - 1 = s + 1$. It is immediate that $a = 1$ and $b = s$. □

In the next two lemmas, we prove Conjecture 4.1 for $t \geq s + 3$ and $s = 3, 4$.

Lemma 4.4. *Set $s = 3$ and fix $t \geq s + 3$. Suppose R is a $\mathcal{K}_t$-free weighted graph with $d_{K_s}(R) = \pi_s(\mathcal{K}_t)$ of minimum cardinality. Then R admits a (b, a)-partition with $b = \max\{s, \lfloor t/2 \rfloor\}$.*

Proof. By Theorem 3.1, R admits a (b, a)-partition $B_1 \cup \cdots \cup B_a$ such that $a + b = t - 1$. Moreover, combining (A5) with the inequality $t \geq 6$, we conclude $a \geq 2$ and $|B_i| \leq 2$ for each i.

To show that $b = \lfloor t/2 \rfloor = \max\{s, \lfloor t/2 \rfloor\}$ when $t \geq 6$, it suffices to show that all but at most one of the parts B_i have cardinality 1. Equivalently, we must show that R does not contain disjoint edges uv and xy with $w(u, v) = w(x, y) = 1/2$.

Suppose the contrary. We may assume without loss of generality that $d_{K_3}(R, \{x, y\}) \geq d_{K_3}(R, \{u, v\})$. Let R' be the graph obtained from R by setting $w(u, v) = 0$ and $w(x, y) = 1$. We observe that $d_{K_3}(R', \{x, y\}) = 2d_{K_3}(R, \{x, y\})$ and $d_{K_3}(R', \{u, v\}) = 0$, so

$$\begin{aligned} d_{K_3}(R') - d_{K_3}(R) &= d_{K_3}(R', \{x, y\}) + d_{K_3}(R', \{u, v\}) - d_{K_3}(R, \{x, y\}) - d_{K_3}(R, \{u, v\}) \\ &= d_{K_3}(R, \{x, y\}) - d_{K_3}(R, \{u, v\}) \geq 0. \end{aligned}$$

Moreover, R' is $\mathcal{K}_t$-free, as the clique numbers of $R'_{>\frac{1}{2}}$ and $R'_{>0}$ satisfy

$$\omega(R'_{>\frac{1}{2}}) + \omega(R'_{>0}) \leq \left(\omega(R_{>\frac{1}{2}}) + 1\right) + (|V(R)| - 1) = (a + 1) + (b - 1) < t.$$

It follows that R' is also an extremal $\mathcal{K}_t$-free weighted graph of minimal cardinality. However, this contradicts Theorem 3.1(A1), because R' contains an edge of weight 0. We conclude that R cannot contain disjoint edges uv and xy of weight $1/2$. This implies that all but at most one of the parts B_i have cardinality 1, which is equivalent to showing that $b = \lfloor t/2 \rfloor$. □

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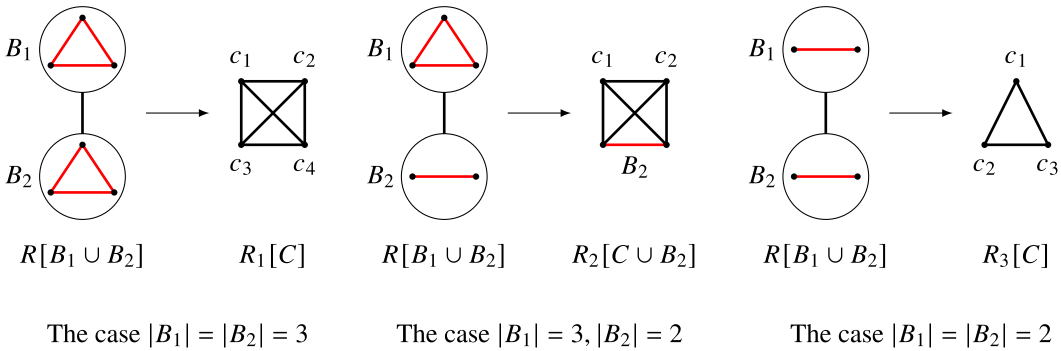

Figure 1. *The optimizations in the proof of $s = 4$. Red edges have weight $1/2$ and black edges have weight 1.*

Lemma 4.5. *Set $s = 4$ and fix $t \geq s + 3$. Suppose R is a $\mathcal{K}_t$-free weighted graph with $d_{K_s}(R) = \pi_s(\mathcal{K}_t)$ of minimum cardinality. Then R admits a (b, a)-partition with $b = \max\{s, \lfloor t/2 \rfloor\}$.*

Proof. By Theorem 3.1, R admits a (b, a)-partition $B_1 \cup \cdots \cup B_a$ such that $a + b = t - 1$. Combining (A5) with the inequality $t \geq 7$, we conclude that $a \geq 2$ and that each part has cardinality at most 3. If $t = 7$, then we must have $a = 2$ and $b = 4$, because (A2) implies $b \geq 4$. Henceforth, we assume $t \geq 8$ and show that $b = \lfloor t/2 \rfloor$. Equivalently, we must show that at most one of the parts $B_1, \ldots, B_a$ has cardinality greater than 1.

Order the parts such that $3 \geq |B_1| \geq |B_2| \geq \cdots \geq |B_a| \geq |B_1| - 1$; the last inequality is a consequence of (A3). Suppose for the sake of contradiction that $|B_2| \geq 2$. We split our proof into three cases, based on $|B_1|$ and $|B_2|$. In each case, we derive a contradiction by constructing a $\mathcal{K}_t$-free weighted graph with larger K_s-density than R; these constructions are given in Figure 1.

Case 1: $|B_1| = |B_2| = 3$. In this case, the six vertices in B_1 and B_2 have the same weight p. Let R_1 be the weighted graph obtained from R by replacing $B_1 \cup B_2$ with a set $C = \{c_1, c_2, c_3, c_4\}$ of four vertices with weight $3p/2$ each such that $w(c_i, c_j) = w(c_i, v) = 1$ for any $i, j \in [4]$ and $v \in B_3 \cup \cdots \cup B_a$. We note that $(R_1)_{>\frac{1}{2}}$ and $(R_1)_{>0}$ have clique numbers satisfying

$$\omega((R_1)_{>\frac{1}{2}}) + \omega((R_1)_{>0}) \leq \left(\omega(R_{>\frac{1}{2}}) + 2\right) + (|V(R)| - 2) = (a + 2) + (b - 2) < t,$$

so R_1 is $\mathcal{K}_t$-free. Additionally, one computes that

$$d_{K_0}(R[B_1 \cup B_2]) = d_{K_0}(R_1[C]) = 1, \qquad d_{K_1}(R[B_1 \cup B_2]) = d_{K_1}(R_1[C]) = 6p,$$

$$d_{K_2}(R[B_1 \cup B_2]) = p^2\left(18 + 12 \cdot \frac{1}{2}\right) < \left(\frac{3p}{2}\right)^2 \cdot 12 = d_{K_2}(R_1[C]),$$

$$d_{K_3}(R[B_1 \cup B_2]) = p^3\left(18 \cdot \frac{1}{2} + 2 \cdot \left(\frac{1}{2}\right)^3\right) \cdot 3! < \left(\frac{3p}{2}\right)^3 \cdot 24 = d_{K_3}(R_1[C]),$$

$$d_{K_4}(R[B_1 \cup B_2]) = p^4\left(9 \cdot \left(\frac{1}{2}\right)^2 + 6 \cdot \left(\frac{1}{2}\right)^3\right) \cdot 4! < \left(\frac{3p}{2}\right)^4 \cdot 24 = d_{K_4}(R_1[C]).$$

We conclude that

$$d_{K_4}(R_1) - d_{K_4}(R) = \sum_{m=0}^{4} \binom{4}{m} \left(d_{K_m}(R_1[C]) - d_{K_m}(R[B_1 \cup B_2])\right) d_{K_{4-m}}(R - (B_1 \cup B_2)) > 0,$$

contradicting the extremality of R.

Case 2: $|B_1| = 3$ and $|B_2| = 2$. Let p and q be the weights of vertices in B_1 and B_2 respectively; by (A4), we have $p \leq q$. Let R_2 be the weighted graph obtained from R by replacing B_1 with a set $C = \{c_1, c_2\}$ of two vertices with weight $3p/2$ each such that $w(c_1, c_2) = w(c_i, v) = 1$ for any $i \in [2]$ and $v \in B_2 \cup \cdots \cup B_a$. We note that $(R_2)_{>\frac{1}{2}}$ and $(R_2)_{>0}$ have clique numbers satisfying

$$\omega((R_2)_{>\frac{1}{2}}) + \omega((R_2)_{>0}) \leq \left(\omega(R_{>\frac{1}{2}}) + 1\right) + (|V(R)| - 1) = (a+1) + (b-1) < t,$$

so R_2 is $\mathcal{K}_t$-free. One computes that

$$\begin{aligned}
&d_{K_0}(R[B_1]) = d_{K_0}(R_2[C]) = d_{K_0}(R[B_2]) = 1, \\
&d_{K_1}(R[B_1]) = d_{K_1}(R_2[C]) = 3p, && d_{K_1}(R[B_2]) = 2q, \\
&d_{K_2}(R[B_1]) = p^2 \cdot 6 \cdot \frac{1}{2} < \left(\frac{3p}{2}\right)^2 \cdot 2 = d_{K_2}(R_2[C]), && d_{K_2}(R[B_2]) = q^2 \cdot 2 \cdot \frac{1}{2}, \\
&d_{K_3}(R[B_1]) = p^3 \cdot 6 \cdot \left(\frac{1}{2}\right)^3.
\end{aligned}$$

We now claim that

$$d_{K_m}(R_2[C \cup B_2]) - d_{K_m}(R[B_1 \cup B_2]) = \sum_{r=0}^{m} \binom{m}{r} (d_{K_r}(R_2[C]) - d_{K_r}(R[B_1])) d_{K_{m-r}}(R[B_2])$$

is positive for $2 \leq m \leq 4$. This is immediate for $m = 2$. For $m = 3, 4$, only the $r = 3$ term is negative, and one may check that the sum of the $r = 2$ and $r = 3$ terms is positive via the relations

$$d_{K_3}(R[B_1]) - d_{K_3}(R_2[C]) = \frac{p}{2}\left(d_{K_2}(R_2[C]) - d_{K_2}(R[B_1])\right)$$

and $d_{K_0}(R[B_2]) \leq \frac{2}{p} d_{K_1}(R[B_2]) \leq \left(\frac{2}{p}\right)^2 d_{K_2}(R[B_2])$. Thus, $d_{K_m}(R_2[C \cup B_2]) > d_{K_m}(R[B_1 \cup B_2])$ for $2 \leq m \leq 4$. It follows that

$$d_{K_4}(R_2) - d_{K_4}(R) = \sum_{m=0}^{4} \binom{4}{m} \left(d_{K_m}(R_2[C \cup B_2]) - d_{K_m}(R[B_1 \cup B_2])\right) d_{K_{4-m}}(R - (B_1 \cup B_2)) > 0,$$

contradicting the extremality of R.

Case 3: $|B_1| = |B_2| = 2$. In this case, the four vertices in B_1 and B_2 have the same weight p. Moreover, we have $a \geq 3$ because $t \geq 8$. Let R_3 be the weighted graph obtained from R by replacing $B_1 \cup B_2$ with a set $C = \{c_1, c_2, c_3\}$ of three vertices with weight $4p/3$ each such that $w(c_i, c_j) = w(c_i, v) = 1$ for any $i, j \in [3]$ and $v \in B_3 \cup \cdots \cup B_a$. We note that $(R_3)_{>\frac{1}{2}}$ and $(R_3)_{>0}$ have clique numbers satisfying

$$\omega((R_3)_{>\frac{1}{2}}) + \omega((R_3)_{>0}) \leq \left(\omega(R_{>\frac{1}{2}}) + 1\right) + (|V(R)| - 1) = (a+1) + (b-1) < t,$$

so R_3 is $\mathcal{K}_t$-free. One computes that

$$\begin{aligned}
&d_{K_0}(R[B_1 \cup B_2]) = d_{K_0}(R_3[C]) = 1, && d_{K_1}(R[B_1 \cup B_2]) = d_{K_1}(R_3[C]) = 4p, \\
&d_{K_2}(R[B_1 \cup B_2]) = p^2\left(8 + 4 \cdot \frac{1}{2}\right) < \left(\frac{4p}{3}\right)^2 \cdot 6 = d_{K_2}(R_3[C]),
\end{aligned}$$

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$$d_{K_3}(R[B_1 \cup B_2]) = p^3 \cdot 24 \cdot \frac{1}{2} < \left(\frac{4p}{3}\right)^3 \cdot 6 = d_{K_3}(R_3[C]),$$
$$d_{K_4}(R[B_1 \cup B_2]) = p^4 \cdot 24 \cdot \left(\frac{1}{2}\right)^2.$$

We now claim that

$$d_{K_m}(R_3[C \cup B_3]) - d_{K_m}(R[B_1 \cup B_2 \cup B_3]) = \sum_{r=0}^{m} \binom{m}{r} (d_{K_r}(R_3[C]) - d_{K_r}(R[B_1 \cup B_2]))d_{K_{m-r}}(R[B_3])$$

is positive for $2 \leq m \leq 4$. This is immediate for $m = 2, 3$. For $m = 4$, only the $r = 4$ term is negative. By (A4), we have $d_{K_1}(R[B_3]) = \sum_{v \in B_3} w(v) \geq p$, so

$$\begin{aligned}\binom{4}{3}(d_{K_3}(R_3[C]) - d_{K_3}(R[B_1 \cup B_2]))d_{K_1}(R[B_3]) &\geq 4 \cdot \left(\frac{10}{27p} d_{K_4}(R[B_1 \cup B_2])\right) \cdot p \\ &> \binom{4}{4} d_{K_4}(R[B_1 \cup B_2]) d_{K_0}(R[B_3]),\end{aligned}$$

and thus $d_{K_m}(R_3[C \cup B_3]) > d_{K_m}(R[B_1 \cup B_2 \cup B_3])$ for $2 \leq m \leq 4$. Therefore, setting $B = B_1 \cup B_2 \cup B_3$, we have

$$d_{K_4}(R_3) - d_{K_4}(R) = \sum_{m=0}^{4} \binom{4}{m} \big(d_{K_m}(R_3[C \cup B_3]) - d_{K_m}(R[B])\big) d_{K_{4-m}}(R - B) > 0,$$

contradicting the extremality of R. □

4.3. Counterexamples to Conjecture 1.2

We conclude this section by presenting some counterexamples to Conjecture 1.2 when t is slightly larger than $2s$. We begin with two counterexamples in the $s = 5$ case, then use the same ideas to derive a family of counterexamples for all sufficiently large s and any t with $2s \leq t \leq 2.08s$.

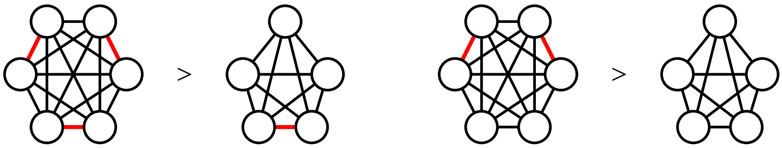
We first observe that Conjecture 1.2 is not true for $s = 5$ and $t \in \{10, 11\}$, via the counterexamples pictured in Figure 2.

For the case $s = 5$ and $t = 10$, Conjecture 1.2 hypothesizes that $\pi_s(\mathcal{K}_t)$ is attained by a weighted graph R_1 of order 5 with exactly one edge of weight $1/2$, such that the two vertices incident to the edge of weight $1/2$ have the same weight p and the remaining three vertices have the same weight q. Let R_2 be a weighted graph of order 6 in which every vertex has weight $1/6$, three disjoint edges have weight $1/2$, and all remaining edges have weight 1. It is straightforward to check that R_2 is $\mathcal{K}_{10}$-free and that

$$\frac{d_{K_5}(R_2)}{d_{K_5}(R_1)} \geq \frac{\binom{6}{5}\left(\frac{1}{6}\right)^5 \frac{1}{4}}{\max\left\{\frac{p^2q^3}{2} : 2p + 3q = 1\right\}} > 1.$$

Thus Conjecture 1.2 is not true in the case $s = 5$, $t = 10$.

For the case $s = 5$ and $t = 11$, Conjecture 1.2 hypothesizes that $\pi_s(\mathcal{K}_t)$ is attained by the complete balanced weighted graph K_5^w, which has 5 vertices of weight $1/5$ each and has all edge weights equal to 1. Let R' be a weighted graph of order 6 with two disjoint edges of weight $1/2$ and all other edge weights equal to 1, such that the four vertices incident to the weight-$1/2$ edges have weight p and the



The case $s = 5, t = 10$ The case $s = 5, t = 11$

Figure 2. *Counterexamples to Conjecture 1.2 for $s = 5$ and $t \in \{10, 11\}$. Red edges have weight $1/2$ and black edges have weight 1.*

remaining two vertices have weight q. It is straightforward to check that R' is $\mathcal{K}_{11}$-free. Moreover, if the parameters p, q are optimized, we have

$$\frac{d_{K_s}(R')}{d_{K_s}(K_5^w)} = \frac{\max\left\{\frac{4p^3q^2}{2} + \frac{2p^4q}{4} : 4p + 2q = 1\right\}}{(\frac{1}{5})^5} > 1.$$

Notably, the inequality holds with $p = 0.16$ and $q = 0.18$. Thus Conjecture 1.2 is also false in the case $s = 5, t = 11$.

We now show that a similar construction works if s is sufficiently large.

Lemma 4.6. *Conjecture 1.2 is false for all sufficiently large s and any t satisfying $2s \leq t \leq 2.08s$.*

Proof. First suppose t is odd (i.e., $t = 2r + 1$ for some integer $r \geq s$). Conjecture 1.2 hypothesizes that $\pi_s(\mathcal{K}_t)$ is attained by the complete balanced weighted graph K_r^w, which has r vertices of weight $1/r$ each and has all edge weights equal to 1. Let R' be a weighted graph on $(r+1)$ vertices with two disjoint edges of weight $1/2$ and all other edge weights equal to 1, such that the four vertices incident to weight-$1/2$ edges have weight $3/4r$ and the remaining $r - 3$ vertices have weight $1/r$. It is straightforward to check that R' is $\mathcal{K}_t$-free. Moreover, because $s \leq r \leq 1.04s$, we have

$$\frac{d_{K_s}(R')}{d_{K_s}(K_r^w)} \geq \frac{\binom{r+1}{s} \cdot (\frac{1}{r})^{s-4}(\frac{3}{4r})^4\frac{1}{4}}{\binom{r}{s}(\frac{1}{r})^s} = \frac{r+1}{r+1-s} \cdot \frac{3^4}{4^5} \geq \frac{r+1}{0.04r+1} \cdot 0.079 > 1$$

if $s \leq r$ is sufficiently large.

Next, suppose t is even (i.e., $t = 2r$ for some integer $r \geq s$). Conjecture 1.2 hypothesizes that $\pi_s(\mathcal{K}_t)$ is attained by a weighted graph R_1 on r vertices with exactly one edge of weight $1/2$. By Lemma 3.2, we have that

$$d_{K_s}(R_1) \leq \sum_{\substack{v_1,\ldots,v_s \in V(R) \\ \text{distinct}}} w(v_1)\cdots w(v_s) \leq s!\binom{r}{s}\left(\frac{1}{r}\right)^s = d_{K_s}(K_r^w).$$

Let R_2 be a weighted graph on $(r + 1)$ vertices with three disjoint edges of weight $1/2$ and all other edge weights equal to 1, such that the six vertices incident to weight-$1/2$ edges have weight $5/6r$ and the remaining $r - 5$ vertices have weight $1/r$. It is straightforward to check that R_2 is $\mathcal{K}_t$-free. Moreover, because $s \leq r \leq 1.04s$, we have

$$\frac{d_{K_s}(R_2)}{d_{K_s}(R_1)} \geq \frac{d_{K_s}(R_2)}{d_{K_s}(K_r^w)} \geq \frac{\binom{r+1}{s} \cdot (\frac{1}{r})^{s-6}(\frac{5}{6r})^6\frac{1}{8}}{\binom{r}{s}(\frac{1}{r})^s} = \frac{r+1}{r+1-s} \cdot \frac{5^6}{6^6} \cdot \frac{1}{8} \geq \frac{r+1}{0.04r+1} \cdot 0.041 > 1$$

if $s \leq r$ is sufficiently large. □

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5. Concluding Remarks

In this paper, we combinatorially resolve the generalized Ramsey–Turán problem for cliques, reducing its determination to a bounded optimization problem about finding the optimal (b, a)-partition, which remains an intriguing problem.

Problem 5.1. Given integers $t - 2 \geq s \geq 3$, which (b, a)-partition with $a + b = t - 1$ achieves the Ramsey–Turán density $\varrho_s(K_t)$?

An easier, yet still interesting, problem is the following. By Theorem 1.3, the threshold value of t for the extremal periodic behavior lies somewhere between $2.08s$ and s^3. Which bound is closer to the truth?

For general graphs, an Erdős–Stone–Simonovits type result is still out of reach. For example, we do not know whether $\varrho_2(K_{2,2,2}) > 0$ [16, 33]. In light of this, we wonder the following.

Problem 5.2. Decide if $\varrho_3(K_{2,2,2,2}) > 0$ or not.

Another natural future direction is to study $\mathrm{RT}(n, K_s, K_t, f(n))$ for smaller independence numbers (e.g., when $f(n) = n^{1-\varepsilon}$ or when it is the inverse function of the Ramsey function, say $f(n) = \sqrt{n \log n}$).

Note. After this paper was written, we learned that Balogh, Magnan and Palmer [7] independently proved some related results.

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Competing interest. The authors have no competing interests to declare.

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