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RESEARCH ARTICLE

Invariant Ideal Axiom

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Abstract

We introduce and prove the consistency of a new set theoretic axiom we call the *Invariant Ideal Axiom*. The axiom enables us to provide (consistently) a full topological classification of countable sequential groups, as well as fully characterize the behavior of their finite products.

We also construct examples that demonstrate the optimality of the conditions in IIA and list a number of open questions.

Contents

Introduction

In this paper, we introduce a new set theoretic principle we call the *Invariant Ideal Axiom* (IIA for short) and prove its consistency with the usual axioms of ZFC. As the main application of IIA we show that it implies that all countable sequential groups are either metrizable or k_ω and, in particular, every countable sequential group has a definable (in fact $F_{\sigma\delta}$) topology, thus concluding the project initiated in the 1970s of determining the structural theory of countable Fréchet and sequential groups (see [6, 8, 23, 34, 42, 43, 46, 47, 49, 84] for some early papers on the subject). This line of research is considered central in topological algebra (see [11, 21, 38, 62, 64, 65, 67, 79, 82, 81] and several others) with some of the early questions answered only recently (see [19, 39, 66, 68]). For a more comprehensive overview of the field, including the history, the fundamental results and the open problems, the reader may wish to consult several excellent surveys available on the subject, as well as a number of books on topological algebra: [9, 24, 40, 45, 61].

The axiom is fully accessible to mathematicians working in topology or algebra and does not require any knowledge of modern set theory. Aside from giving the ultimate structural result for countable sequential groups, the axiom has a profound impact on product properties of sequential groups.

Our hope and expectation is that the axiom IIA provides both a canonical environment and a test model for future study of convergence in topological algebra.

To better relate our results to the existing body of research, one may recall that arguments in analysis and topology often depend on establishing the extent of various special classes of topological spaces. Fusing algebraic and topological properties proved to be among the most fruitful techniques. Classical examples of such results are the implication $T_1 \Rightarrow T_{3\frac{1}{2}}$ in general topological groups (or even $T_3 \Rightarrow T_{3\frac{1}{2}}$ in *para*topological groups; see [10]) and the *Birkhoff-Kakutani theorem* (see [9] for these and other facts about topological groups) on the metrizability of first countable T_1 topological groups. Metrizability theorems in particular drew a lot of attention, stimulating a search for the weakest set of conditions that jointly imply that a given topology is generated by a metric.

In the class of topological groups, compactness and countable tightness together imply metrizability (see [9]) so it is natural to look for a similar yet less restrictive set of conditions that may yield the same result. Reasoning along these lines led V. Malykhin to ask about the existence of countable (equivalently, separable) Fréchet non-metrizable groups (see [6]).

Malykhin's problem generated a large body of research that illustrates another important quality shared by several results in this area. Namely, the effect of a particular set of restrictions is greatly influenced by set theory. As a case in point, Malykhin's problem has an affirmative answer in a variety of set theoretic universes, including models of MA. The conclusive result, establishing the independence of the answer to Malykhin's problem of the axioms of ZFC was obtained by the first author and U. A. Ramos-García in [39] using a forcing construction. The same paper also contains a construction of a countable Fréchet non-metrizable group under a very weak set theoretic assumption $\diamond(2^{=})$.

Malykhin asked (see [58]) a related question about the class of countable sequential abelian groups (see below for all the appropriate definitions). This question was fully solved in [58] by E. Zelenyuk and V. Protasov, who established (in ZFC) the existence of countable sequential group topologies that are not Fréchet on any infinite countable abelian group. The existence of such topology on *nonabelian* countable groups (specifically, the free group) was well known (see [55]).

The investigation into the class of sequential groups prompted P. Niykos (see [46]) to look at their *sequential order*, which can be roughly thought of as the ordinal measure of the complexity of the closure operator in sequential spaces. The existence of sequential groups (of any size) of sequential order strictly between 1 and ω_1 turned out to be independent of the axioms of ZFC, as well (see [65], [68] and [71]).

A thorough review of existing ZFC constructions of sequential non-Fréchet groups (see [40], [76], [58], [21]) reveals a structure common to all such examples. Their topology is determined by a countable family of (countably) compact subspaces (i.e., is k_ω; see [32] and the definition below). Perhaps the most widely known family of k_ω spaces is the class of countable CW-complexes (see [16]). Various results in algebraic topology (such as the homotopy equivalence for filtered spaces theorem of J. Milnor; see [15]) heavily depend on the k_ω property. The class of k_ω spaces is well behaved; in particular, it is finitely productive, and every countable k_ω space is sequential and analytic (see below for the definitions and further discussion).

In [69], answering a question of S. Todorčević and C. Uzcátegui, the second author showed that at least in the definable case (more specifically, in the class of countable *analytic* groups), the only sequential examples of countable groups are k_ω or metrizable. This naturally brought about the question (posed in [68]) whether it is consistent with ZFC that all countable sequential groups are either metrizable or k_ω (equivalently, whether all countable sequential groups are analytic).

The main tool introduced in this paper, *the Invariant Ideal Axiom*, or IIA, is used to answer this question in the affirmative. As important corollaries, we show that IIA generates a complete classification of sequential group topologies on countable groups, as well as allows for a transparent description of products of such groups.

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To support our claim of the optimal nature of IIA in the study of convergence in countable groups, we show that its natural generalization fails to stay consistent. We also construct an example demonstrating the differences between the case of countable sequential groups and their separable counterparts. We conclude by listing a few open questions we believe will lead to greater insight about this field of research.

1. Preliminaries

All topological spaces and groups considered are T_1 and completely regular. To see more about topological groups, consult [6, 9, 22, 23, 24, 61, 77].

Recall that a topological space X is *Fréchet* if for any $x \in \overline{A} \subseteq X$, there is a sequence $S \subseteq A$ such that $S \to x$. A space X is *sequential* if for every $A \subseteq X$ that is not closed, there is a $C \subseteq A$ such that $C \to x \notin A$. The term 'Fréchet space' appears to have been coined by Arkhangel'skii in [3], while the term 'sequential' appears for the first time in Franklin's [30], where the following notion is defined:

Given $A \subseteq X$, define the *sequential closure* of A as

$$[A]' = \{\, x \in X : C \to x \text{ for some } C \subseteq A \,\}, \text{ and then recursively}$$

$$[A]_0 = A \text{ and } [A]_\alpha = \cup\{\, [[A]_\beta]' : \beta < \alpha \,\} \text{ for } \alpha \leq \omega_1.$$

Then X is sequential if and only if $\overline{A} = [A]_{\omega_1}$ for every $A \subseteq X$, and the *sequential order* of X is defined as

$$\mathfrak{so}(X) = \min\{\, \alpha \leq \omega_1 : [A]_\alpha = \overline{A} \text{ for every } A \subseteq X \,\}.$$

Fréchet spaces are easily seen to be exactly those sequential spaces X for which $\mathfrak{so}(X) \leq 1$. The following definition plays a central role in our investigation.

Definition 1. A topological space X is called a k_ω-space (c_ω-space) if there exists a countable family $\mathcal{K}$ of (countably) compact subspaces of X such that a $U \subseteq X$ is open in X if and only if $U \cap K$ is relatively open in K for every $K \in \mathcal{K}$.

Countable k_ω spaces are always sequential, and the class of k_ω-spaces is productive. Such spaces are definable objects and have $F_{\sigma\delta}$ topologies.

What follows is a short discussion of *test spaces*:

- **Arens space** ([2]): $S_2 = [\omega]^{\leq 2}$, where $U \subseteq S_2$ is open if and only if for every $s \in U$ such that $|s| < 2$ the set $\{s \cup \{n\} \in S_2 : s \cup \{n\} \notin U\}$ is finite,
- **sequential fan** ([1]): the quotient $S(\omega) = S_2/[\omega]^{\leq 1}$, and finally
- **convergent sequence of discrete sets** ([25]): $D(\omega) = \omega \times \omega \cup \{(\omega, \omega)\} \subseteq (\omega + 1)^2$ in the natural product topology.

The sequential fan $S(\omega)$ and the convergent sequence of discrete sets $D(\omega)$ are both Fréchet spaces; $D(\omega)$ is metrizable while $S(\omega)$ has character $\mathfrak{d}$. The space S_2 is sequential, and $\mathfrak{so}(S_2) = 2$. Both $S(\omega)$ and S_2 are k_ω-spaces, while $D(\omega)$ is not.

Proposition 2.

1. *(Y. Tanaka [75]) A sequential space contains a copy of $S(\omega)$ if and only if it contains a closed copy of $S(\omega)$.*
2. *(Y. Tanaka [75]) A countable sequential topological group is Fréchet if and only if it does not contain a closed copy of $S(\omega)$.*
3. *(T. Banakh and L. Zdomskyĭ [11]) If a topological group $\mathbb{G}$ contains closed copies of $S(\omega)$ and $D(\omega)$, it also contains a subset D such that D is not closed in $\mathbb{G}$ and is almost disjoint from every convergent sequence in $\mathbb{G}$.*

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Recall that the *Cantor-Bendixson* derivative A' of a topological space A is defined by $A' = A \setminus \{x : x \text{ is an isolated point of } A\}$. The Cantor-Bendixson derivative can be iterated—recursively define $(A)_\alpha$ for any ordinal α by putting $(A)_0 = A$, $(A)_{\alpha+1} = (A)'_\alpha$ and $(A)^\lambda = \bigcap_{\alpha<\lambda}(A)_\alpha$ for λ limit.

The *full Cantor-Bendixson derivative* (also called the *Cantor-Bendixson* (or *perfect*) *kernel*) of a space A is $(A)_\alpha$, where α is an ordinal such that $(A)_\alpha = (A)_\beta$ for any $\beta \geq \alpha$.

A topological space A is *scattered* if every subset of A contains an isolated (in the subset) point, equivalently if its full Cantor-Bendixson derivative is empty. Every scattered space is thus naturally stratified into levels, $x \in A$ belonging to the *α-th level* (denoted by $\operatorname{scl}(x, A) = \alpha$) if and only if α is the unique ordinal such that $x \in (A)_\alpha \setminus (A)_{\alpha+1}$. The *height* of A ($\operatorname{scl}(A)$) is the smallest ordinal α such that $(A)_\alpha = \varnothing$.

Throughout the paper, $\mathbf{csc}(X)$ denotes the ideal generated by closed scattered subsets of X, $\mathbf{nwd}(X)$ is the ideal of nowhere dense subsets of X and $\mathbf{cpt}(X)$ stands for the ideal generated by all the compact subsets of X.

2. The Invariant Ideal Axiom

2.1. Introducing IIA

Analyzing the proofs of

Theorem 3 ([39]). *It is consistent that every countable Fréchet group is metrizable.*

Theorem 4 ([68]). *It is consistent that every separable sequential group is either metrizable or has sequential order ω_1.*

we have isolated the Invariant Ideal Axiom IIA which we shall present next.

First let us introduce the relevant notation. Recall that an *ideal* is a family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{G})$ closed under taking subsets and finite unions, and it is *invariant* if both $g \cdot I = \{g \cdot h : h \in I\}$ and $I \cdot g = \{h \cdot g : h \in I\}$, as well as $I^{-1} = \{h^{-1} : h \in I\}$ are in $\mathcal{I}$ for every $I \in \mathcal{I}$ and $g \in \mathbb{G}$. We shall assume throughout the paper that all ideals contain all finite subsets of $\mathbb{G}$. Recall also that $\mathcal{I}^+ = \mathcal{P}(\mathbb{G}) \setminus \mathcal{I}$. Given a point x in a topological space (or a topological group), we denote by

$$\mathcal{I}_x = \{A \subseteq X : x \notin \overline{A}\}$$

the dual ideal to the filter of neighbourhoods of x. An ideal $\mathcal{I}$ on a set X is *ω-hitting* if for every countable family $\mathcal{Y}$ of infinite subsets of X, there is an $I \in \mathcal{I}$ such that $Y \cap I$ is infinite for every $Y \in \mathcal{Y}$.

We call an ideal $\mathcal{I}$ *tame* if for every $Y \in \mathcal{I}^+$ every $f : Y \to \omega$ and every ω-hitting ideal $\mathcal{J}$ on ω, there is a $J \in \mathcal{J}$ such that $f^{-1}[J] \notin \mathcal{I}$: that is, if no ideal *Katětov-below* a restriction of $\mathcal{I}$ to a positive set is ω-hitting (see, e.g., [37] for more on Katětov order and ω-hitting ideals). Finally, we call an ideal $\mathcal{I}$ of subsets of a topological group $\mathbb{G}$ *weakly closed* if for every set $A \subseteq \mathbb{G}$ and every sequence $C \subseteq \mathbb{G}$ convergent to $1_{\mathbb{G}}$,

$$A \in \mathcal{I} \text{ if and only if } A \cup \{x : C \cdot x \subseteq^* A\} \in \mathcal{I}.$$

It is immediate from the definition that every ideal generated by (sequentially) closed subsets of $\mathbb{G}$ is weakly closed, in particular the ideals $\mathbf{csc}(\mathbb{G})$, $\mathbf{nwd}(\mathbb{G})$ and $\mathbf{cpt}(\mathbb{G})$ are all invariant weakly closed ideals in any topological group.

We call a subset Y of a topological space X *entangled* if $\mathcal{I}_x \restriction Y$ is ω-hitting for every $x \in X$. We shall call a topological space X *groomed* if it does not contain a dense entangled set.

The class of groomed spaces includes all non-discrete Fréchet and sequential spaces, as well as all non-discrete *subsequential* spaces (i.e., subspaces of sequential spaces).

Lemma 5. *Every non-discrete subsequential space is groomed.*

Proof. Let X be a dense subspace of a sequential space Y. Let $D \subseteq X$ be dense, and let $x \in X$ be a point that is not isolated. As Y is sequential, there are countably many disjoint infinite subsets $\{C_n : n \in \omega\}$ of D such that

1. Each C_n converges to some point $x_n \in Y$ (not necessarily distinct), and
2. For every neighbourhood U of x there are infinitely many $n \in \omega$ such that $x_n \in U$.

This is easily proved by induction on the sequential order of x in D:

If there is a sequence $C \subseteq D \setminus \{x\}$ convergent to x let $\{C_n : n \in \omega\}$ be any collection of infinite pairwise disjoint subsets of C.

For the inductive step, assume that $x \in Y$ is the limit of a convergent sequence $\{x_n : n \in \omega\}$ such that for each x_n (by the inductive hypothesis) exist pairwise disjoint sequences $\{C^n_m : m \in \omega\}$ of elements of D convergent each to a point x^n_m such that every open set U containing x_n contains infinitely many of the x^n_m. Let $\{D^n_m : n, m \in \omega\}$ be a disjoint refinement of $\{C^n_m : n, m \in \omega\}$. Then it is a collection of pairwise disjoint sequences convergent in Y, and every neighbourhood U of x will contain all but finitely many of the $\{x_n : n \in \omega\}$, and, consequently, infinitely many of the $\{x^n_m : n, m \in \omega\}$.

Then, however, D is not entangled, as $x \in \overline{Z}$ for every Z such that $|Z \cap C_n| = \omega$ for every $n \in \omega$. □

Examples of spaces that are not groomed are discrete spaces, ω^* and $2^{\mathfrak{c}}$.

We are now ready to introduce the *Invariant Ideal Axiom*:

IIA: *For every countable groomed topological group $\mathbb{G}$ and every tame, weakly closed invariant ideal $\mathcal{I} \subseteq 2^{\mathbb{G}}$, one of the following holds:*

1. there is a countable $\mathcal{S} \subseteq \mathcal{I}$ such that for every infinite sequence C convergent in $\mathbb{G}$, there is an $I \in \mathcal{S}$ such that $C \cap I$ is infinite,
2. there is a countable $\mathcal{H} \subseteq \mathcal{I}^+$ such that for every non-empty open $U \subseteq \mathbb{G}$, there is an $H \in \mathcal{H}$ such that $H \setminus U \in \mathcal{I}$.

We refer to the $\mathcal{S}$ from the first alternative as a *sequence capturing* set, and the set $\mathcal{H}$ from the second alternative as an *almost π-network*.

To see the relevance of the axiom, let us deduce the solution to the Malykhin problem from it. We first recall the following simple lemma from the literature (we include the short proofs for the sake of completeness):

Lemma 6. *Let X be a countable Fréchet space without isolated points.*

1. *([12]) If $x \in X$ and $\mathcal{X}$ is a countable collection of nowhere dense subsets of X, then there is a $C \subseteq X$ convergent to x such that $X \cap N$ is finite for every $N \in \mathcal{X}$.*
2. *([39]) The ideal* $\mathbf{nwd}(X)$ *of nowhere dense subsets of X is tame.*

Proof. Ad (1): Enumerate $\mathcal{X}$ as $\langle M_n : n \in \omega \rangle$. As X is Fréchet without isolated points, there is a one-to-one sequence $\langle x_n : n \in \omega \rangle \subseteq X \setminus \{x\}$ converging to x. For each $n \in \omega$, the set $X_n = X \setminus (\{x\} \cup \bigcup_{i<n} M_i)$ is dense in X, so using the Fréchet property again, there is for each $n \in \omega$ a sequence $\langle y^n_i : i \in \omega \rangle \subseteq X_n$ converging to x_n. Then $x \in \overline{\{y^n_i : i, n \in \omega\}}$, hence there is a sequence $C \subseteq \{y^n_i : i, n \in \omega\}$ converging to x. Now, $C \cap M_n \subseteq \{y^m_i : i \in \omega,\ m \le n\}$, and as each sequence $\langle y^n_i : i \in \omega \rangle \subseteq X_n$ converges to $x_n \neq x$, $C \cap M_n$ is finite for every $n \in \omega$.

To see (2), let $Y \in \mathsf{nwd}^+(X)$, let $f : Y \to \omega$, and let an ω-hitting ideal $\mathcal{J}$ on ω be given. Put $Z = \mathrm{Int}(\overline{Y}) \cap Y$. Then either

(a) there is an $n \in \omega$ such that $f^{-1}(n) \in \mathsf{nwd}^+(X)$, or
(b) for every $x \in Z$, there is a sequence $C_x \subseteq Z \setminus \{x\}$ converging to x such that $f|_{C_x}$ is finite-to-one.

If there is an $n \in \omega$ such that $f^{-1}(n) \in \mathsf{nwd}^+(X)$, let $J = \{n\} \in \mathcal{J}$. If, on the other hand, $f^{-1}(n) \in \mathsf{nwd}(X)$ for all $n \in \omega$, apply (1) to Z and every $x \in Z$ with $\mathcal{X} = \{f^{-1}(n) \cap Z : n \in \omega\}$ to get $\{C_x : x \in Z\}$ as in (b). The family$\{f[C_x] : x \in Z\}$ is then a countable collection of infinite subsets of ω, hence there is a $J \in \mathcal{J}$ such that $J \cap f[C_x]$ is infinite for every $x \in Z$. Then $f^{-1}[J]$ is dense in Z, hence in either case J is an element of $\mathcal{J}$ such that $f^{-1}[J] \in \mathsf{nwd}^+(X)$. □

Theorem 7. *Assuming* IIA, *every separable Fréchet group is metrizable.*

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Proof. Let $\mathbb{H}$ be a separable Fréchet group, and let $\mathbb{G} \subseteq \mathbb{H}$ be a dense countable subgroup. Apply IIA to $\mathbb{G}$ and $\mathbf{nwd}(\mathbb{G})$. Alternative (1) fails by Lemma 6, so there is a countable family $\mathcal{X}$ of somewhere dense subsets of $\mathbb{G}$, hence also somewhere dense in $\mathbb{H}$ such that every open set contains mod $\mathbf{nwd}$ an element of $\mathcal{X}$. Then

$$\{\mathrm{int}(\overline{X}) : X \in \mathcal{X}\},$$

where the interior and closure are taken in $\mathbb{H}$, form a countable π-base, and as π-weight and weight coincide in topological groups, the group $\mathbb{H}$ is second countable and hence metrizable. □

Corollary 8. *Assuming IIA,* $\mathfrak{p} = \omega_1$ *and* $\mathfrak{b} > \omega_1$.

Proof. It is well known that if either $\mathfrak{p} > \omega_1$ or $\mathfrak{b} = \omega_1$, then there is a separable non-metrizable Fréchet group; see, for example, [46, 47, 56]. □

The next remark we would like to make is that the assumption that the group is groomed cannot be dropped from the statement of IIA, as is shown in the proposition below.

Proposition 9. *There is a countable topological group* $\mathbb{G}$ *and a tame, weakly closed, invariant ideal* $\mathcal{I} \subseteq \mathcal{P}(\omega)$ *such that IIA fails for* $\mathbb{G}$ *and* $\mathcal{I}$.

Proof. Without loss of generality, we can assume IIA as its failure provides an example. So, in particular, we can assume that $\mathfrak{b} > \omega_1$.

Let $\{A_\alpha, B_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$ be a *Hausdorff gap*: that is,

1. $A_\alpha \subseteq^* A_\beta \subseteq^* B_\beta \subseteq^* B_\alpha$ for $\alpha < \beta < \omega_1$, and
2. there is no X such that $A_\alpha \subseteq^* X \subseteq^* B_\alpha$ for every $\alpha < \omega_1$.

Topologize the group $\mathbb{G} = [\omega]^{<\omega}$ by declaring the sets (in fact, subgroups) $[F]^{<\omega}$ open neighbourhoods of $\emptyset$, where F is such that there is an $\alpha < \omega_1$ with $B_\alpha \subseteq^* F$, and let

$$\mathcal{I} = \{A \subseteq \mathbb{G} : \ \forall \alpha < \omega_1 \bigcup A \subseteq^* B_\alpha\}.$$

Now, the fact that $\mathcal{I}$ is tame follows easily by noting that no restriction of $\mathcal{I}$ to a positive set is tall, and $\mathcal{I}$ is invariant as $\bigcup a \triangle I =^* \bigcup I$ for every $I \in \mathcal{I}$ and $a \in \mathbb{G}$. The fact that $\mathcal{I}$ is weakly closed follows immediately from the fact that every set of the form $[C]^{<\omega}$ is closed in the topology of $\mathbb{G}$, and every set in $\mathcal{I}$ is contained in en element of $\mathcal{I}$ of this form ($C \in \mathcal{I}$ if and only if $[\bigcup C]^{<\omega} \in \mathcal{I}$).

To see that the alternative (1) of IIA fails for $\mathcal{I}$, note first that $C \to 0$ if and only if C is a point-finite family of finite sets and $C \in \mathcal{I}$. The fact that there cannot be a countable family of elements of $\mathcal{I}$ intersecting infinitely every convergent sequence follows directly from the fact that we started with a Hausdorff gap (hence there cannot be a single such element of $\mathcal{I}$) and the fact that $\mathfrak{b} > \omega_1$, hence $\{B_\alpha : \alpha < \omega_1\}$ cannot form the upper half of an (ω, ω_1)-gap by the Theorem of Rothberger.

Alternative (2) fails as $X \in \mathcal{I}^+$ iff $\bigcup X \setminus B_\alpha$ is infinite for some $\alpha < \omega_1$, and having countably many such X, there is an α that is a witness for all of them, hence none of them is mod $\mathcal{I}$ contained in $[B_\alpha]^{<\omega}$. □

We shall return to further discussion of the consequences of IIA later on, but first we shall see that the axiom is consistent.

2.2. *Consistency of* IIA

All the tools to prove the consistency of the *Invariant Ideal Axiom* have been presented in [39] and [19], although the models constructed there are not models of IIA. We shall first recall all the relevant lemmata from the above-mentioned papers. Those we can quote directly we do not prove. Those that require some (very) minor changes we do prove, although in all cases, the changes are mere technicalities.

Recall first that the *Laver-Mathias-Prikry forcing* $\mathbb{L}_{\mathcal{F}}$ associated to a filter $\mathcal{F}$ on ω is defined as the set of those trees $T \subseteq \omega^{<\omega}$ with stem s_T such that for all $s \in T$ extending s_T, the set $\mathrm{succ}_T(s) = \{n \in \omega \colon s^\frown n \in T\}$ belongs to $\mathcal{F}$. The set $\mathbb{L}_{\mathcal{F}}$ is ordered by inclusion.

The forcing $\mathbb{L}_{\mathcal{F}}$ is σ-centered and adds generically a dominating real $\dot{\ell}_{\mathcal{F}} \colon \omega \to \omega$ (The function $\dot{\ell}_{\mathcal{F}}$ is the unique branch through $\omega^{<\omega}$ that belongs to all trees in the generic filter, and it eventually dominates all ground model reals). Its range $\dot{A}_{gen} = \mathrm{ran}(\dot{\ell}_{\mathcal{F}})$ *separates* the filter $\mathcal{F}$ (that is, the set $\dot{A}_{gen}$ is almost contained in all members of $\mathcal{F}$ and has infinite intersection with every $\mathcal{F}$-positive set).

Names for reals in forcings of the type $\mathbb{L}_{\mathcal{F}}$ can be analysed using *ranks*, as introduced by Baumgartner and Dordal in [13] and further developed by Brendle [17, 18]. Given a formula φ in the forcing language and $s \in \omega^{<\omega}$, we say that *s favors* φ if there is no condition $T \in \mathbb{L}_{\mathcal{F}}$ with stem s such that $T \Vdash$ '$\neg\varphi$', or equivalently, every condition $T \in \mathbb{L}_{\mathcal{F}}$ with stem s has an extension T' such that $T' \Vdash$ 'φ'.

Recall also that a forcing notion $\mathbb{P}$ *strongly preserves* ω-hitting if for every sequence $\langle \dot{A}_n \colon n \in \omega\rangle$ of $\mathbb{P}$-names for infinite subsets of ω, there is a sequence $\langle B_n \colon n \in \omega\rangle$ of infinite subsets of ω such that for any $B \in [\omega]^\omega$, if $B \cap B_n$ is infinite for all n, then $\Vdash_{\mathbb{P}}$ '$B \cap \dot{A}_n$ is infinite for all n'.

In our terminology, one of the lemmas of [19] becomes:

Lemma 10 ([19]). *Let $\mathcal{I}$ be an ideal on ω, and let $\mathcal{F} = \mathcal{I}^*$ be the dual filter. Then the following are equivalent:*

1. *$\mathcal{I}$ is tame,*
2. *$\mathbb{L}_{\mathcal{F}}$ strongly preserves ω-hitting,*
3. *$\mathbb{L}_{\mathcal{F}}$ preserves ω-hitting,*

and the standard preservation under finite support iteration argument gives

Lemma 11 ([19]). *Finite support iteration of ccc forcings strongly preserving ω-hitting strongly preserves ω-hitting.*

Recall also that given an ideal $\mathcal{I}$, a forcing notion $\mathbb{P}$ and a $\mathbb{P}$-name $\dot{A}$ for a subset of ω, we say that $\mathbb{P}$ *seals the ideal $\mathcal{I}$ via $\dot{A}$* if $\Vdash_{\mathbb{P}}$ '$\dot{A} \in \mathcal{I}^+ \wedge \mathcal{I} \restriction \dot{A}$ is ω-hitting'.

Following [39], we say that an ideal $\mathcal{I}$ is *ω-hitting mod* filter $\mathcal{F}$ if $\mathcal{I} \cap \mathcal{F} = \emptyset$ and for every countable family $\mathcal{H} \subset \mathcal{F}^+$, there is an $I \in \mathcal{I}$ such that $H \cap I \in \mathcal{F}^+$ for all $H \in \mathcal{H}$.

Lemma 12 ([39]). *The forcing $\mathbb{L}_{\mathcal{F}}$ seals an ideal $\mathcal{I}$ via $\dot{A}_{gen}$ if and only if $\mathcal{I}$ is ω-hitting mod $\mathcal{F}$.*

The following corollary is the main tool for the consistency of IIA.

Corollary 13 ([39]). *Let $\mathbb{G}$ be a countable topological group and $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be a tame invariant ideal such that both alternatives of IIA fail. Then:*

1. *$\mathbb{L}_{\mathcal{I}^*}$ forces $\mathbb{G}$ is not groomed, and*
2. *$\mathbb{L}_{\mathcal{I}^*}$ strongly preserves ω-hitting.*

Proof. To see (1), one essentially only needs to translate from one language to another; to force that $\mathbb{G}$ is not groomed—that is, contains a dense entangled set—means (in the language of sealing) adding a dense subset $\dot{A}$ of $\mathbb{G}$ such that $\mathbb{L}_{\mathcal{I}^*}$ seals the ideal $\mathcal{I}_g$ for every $g \in \mathbb{G}$. Now, by Lemma 12, it is enough to show that the ideal $\mathcal{I}_g$ is ω-hitting mod $\mathcal{I}^*$, and hence, $\mathbb{L}_{\mathcal{I}^*}$ seals the ideal $\mathcal{I}_g$ via $\dot{A}_{gen}$ for every $g \in \mathbb{G}$, which is simply the negation of the existence of a countable almost π-network of $\mathcal{I}$-positive sets (alternative (2) of IIA). The failure of capturing convergent sequences (alternative (1) of IIA) guarantees that every element of $\mathcal{I}^*$ is dense in $\mathbb{G}$, hence also $\mathbb{L}_{\mathcal{I}^*}$ forces $\dot{A}_{gen}$ to be dense in $\mathbb{G}$.

(2) follows directly from Lemma 10. □

The last part of the forcing argument, which deals with the preservation of dense entangled sets, really uses algebra.

Let $(\mathbb{G}, \cdot)$ be an abstract group, and let $A \subseteq \mathbb{G} \setminus \{1_{\mathbb{G}}\}$. A subset Y of $\mathbb{G}$ is called A-large if for every $a \in A$ and $b \in \mathbb{G}$, either $b \in Y$ or $a \cdot b^{-1} \in Y$. By A-large, we will denote the collection of all subsets of

$\mathbb{G}$ that are A-large. A family $\mathcal{C}$ of subsets of $\mathbb{G}$ is ω-hitting with respect to A if given $\langle Y_n : n \in \omega\rangle \subset A$-large, there is a $C \in \mathcal{C}$ such that $C \cap Y_n$ is infinite for all n. Finally, We say that a relation $R \subseteq \mathbb{G} \times \mathbb{G}$ is *large* if for every $a, b \in \mathbb{G}$, either $\langle a, b\rangle \in R$ or $\langle a, a \cdot b^{-1}\rangle \in R$.

Lemma 14 ([39]). *Let $\mathbb{G}$ be a countable group, and let $\mathcal{I}$ be a weakly closed invariant ideal on $\mathbb{G}$ for which (1) of the IIA fails, and let $\langle R_n : n \in \omega\rangle$ be a sequence of large relations. Then there is a sequence C convergent to $1_{\mathbb{G}}$ such that $R_n^{-1}[C \setminus F] \in \mathcal{I}^+$ for every $n \in \omega$ and $F \in [\mathbb{G}]^{<\omega}$.*

Proof. For every $n \in \omega$, let $B_n = \{b \in \mathbb{G} : R_n^{-1}(b) \in \mathcal{I}\}$, and put

$$\mathcal{S} = \{R_n^{-1}(b) \cdot b^{-1} : b \in B_n, n \in \omega\} \cup \{B_n : B_n \in \mathcal{I}\}.$$

As (1) of IIA fails, there is a sequence C converging to $1_{\mathbb{G}}$ such that $C \cap S$ is finite for every $S \in \mathcal{S}$. We claim that $R_n^{-1}[C \setminus F] \in \mathcal{I}^+$ for every $n \in \omega$ and $F \in [\mathbb{G}]^{<\omega}$. To see this, let $n \in \omega$ and $F \in [\mathbb{G}]^{<\omega}$ be given. Consider two cases.

Case 1. $B_n \in \mathcal{I}$.

Then there is an $b \in C \setminus F$ such that $R_n^{-1}(b) \in \mathcal{I}^+$.

Case 2. $B_n \in \mathcal{I}^+$.

Fix $b \in B_n$. Then $a \notin R_n^{-1}(b) \cdot b^{-1}$ (or, equivalently, $a \cdot b \notin R_n^{-1}(b)$) for all but finitely many $a \in C$. Since R_n is a large relation, $\langle a \cdot b, a\rangle \in R_n$ for all but finitely many $a \in C$. In particular, $\{a \cdot b : a \in C\} \subseteq^* R_n^{-1}[C \setminus F]$ and $\{a \cdot b : a \in C\}$ converges to b. Thus,

$$B_n \subseteq R_n^{-1}[C \setminus F] \cup \{b \in \mathbb{G} : C \cdot b \subseteq^* R_n^{-1}[C \setminus F]\},$$

hence, as $\mathcal{I}$ is weakly closed, also $R_n^{-1}[C \setminus F] \in \mathcal{I}^+$. □

Lemma 15 ([39]). *Let $\mathbb{G}$ be a countable topological group and $\mathcal{I}$ an invariant ideal on $\mathbb{G}$ for which (1) of the IIA fails. Then*

$$\Vdash_{\mathbb{L}_{\mathcal{I}^*}} \text{‘}\mathcal{C} \textit{ is } \omega\textit{-hitting with respect to } \dot{A}_{gen}\text{’},$$

where $\mathcal{C} = \mathcal{I}^{\perp}_{1_{\mathbb{G}}}$ is the ideal consisting of sequences converging to $1_{\mathbb{G}}$.

Proof. Aiming for a contradiction, assume that there are a sequence $\langle \dot{B}_n : n \in \omega\rangle$ of $\mathbb{L}_{\mathcal{I}^*}$-names and a condition $T^* \in \mathbb{L}_{\mathcal{I}^*}$ such that $T^* \Vdash$ ‘$\forall n \in \omega\, (\dot{B}_n \in \dot{A}_{gen}$-large)’, and for every $C \in \mathcal{C}$, there are a condition $T_C \in \mathbb{L}_{\mathcal{I}^*}$ with $T_C \leqslant T^*$, a natural number n_C and a finite subset F_C of $\mathbb{G}$ such that

$$T_C \Vdash \text{‘}C \cap \dot{B}_{n_C} \subseteq F_C\text{’}. \qquad (\star)$$

For each $s \in T^*$ with $s \supseteq s_{T^*}$ and each natural number n, put

$$R_{s,n} = \{\langle a, b\rangle : a \in \mathrm{succ}_{T^*}(s) \Rightarrow s^\frown a \text{ favors } b \in \dot{B}_n\}.$$

Claim. The relation $R_{s,n}$ is large.

Let a and b be two elements of $\mathbb{G}$. Assume that $\langle a, b\rangle \notin R_{s,n}$. Assuming $a \in \mathrm{succ}_{T^*}(s)$, we have to show that $\langle a, a \cdot b^{-1}\rangle \in R_{s,n}$. There is a condition $T' \leq T^*$ with $s_{T'} = s^\frown a$ such that $T' \Vdash$ ‘$b \notin \dot{B}_n$’. Then $T' \Vdash$ ‘$a \in \dot{A}_{gen}$ and $b \notin \dot{B}_n$’, but also $T' \Vdash$ ‘$\dot{B}_n \in \dot{A}_{gen}$-large’ so $T' \Vdash$ ‘$a \cdot b^{-1} \in \dot{B}_n$’. This finishes the proof of the claim.

By Lemma 14, there is a $C \in \mathcal{C}$ such that $R_{s,n}^{-1}[C \setminus F] \in \mathcal{I}^+$ for every $s \in T^*$ with $s \supseteq s_{T^*}$, $n \in \omega$ and $F \in [\mathbb{G}]^{<\omega}$. In particular, $R_{s_C,n_C}^{-1}[C \setminus F_C] \in \mathcal{I}^+$, where $s_C = s_{T_C}$. Pick an $a \in \mathrm{succ}_{T_C}(s_C) \cap R_{s_C,n_C}^{-1}[C \setminus F_C]$. Then there is a $b \in C \setminus F_C$ such that $s_C^\frown a$ favors $b \in \dot{B}_{n_C}$, and hence there is a condition $T \leqslant T_C$ whose stem extends $s_C^\frown a$ such that $T \Vdash$ ‘$b \in \dot{B}_{n_C}$’, a contradiction to the initial assumption $(\star)$. □

We say that a forcing notion $\mathbb{P}$ *strongly preserves* ω-hitting with respect to A if for every $\mathbb{P}$-name $\dot{Y}$ for an A-large subset of a group $\mathbb{G}$, there is a sequence $\langle Y_n : n \in \omega\rangle \subset A$-large such that for any $C \subseteq \mathbb{G}$,

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if $C \cap Y_n$ is infinite for all n, then $\Vdash_{\mathbb{P}}$ '$C \cap \dot{Y}$ is infinite'. Clearly, every forcing notion that strongly preserves ω-hitting with respect to A preserves ω-hitting with respect to A.

Lemma 16 ([39]). *Let $\mathbb{P}$ be a σ-centered forcing notion. Then $\mathbb{P}$ strongly preserves ω-hitting with respect to A.*

Lemma 17 ([39]). *Finite support iteration of ccc forcings strongly preserving ω-hitting with respect to A strongly preserves ω-hitting with respect to A.*

Now we are in position to state and prove the main theorem of this section:

Theorem 18. *The Invariant Ideal Axiom* IIA *together with the Martin's Axiom* MA*(σ-centered strongly ω-hitting preserving) is consistent with ZFC.*

Proof. Assume that the ground model $\mathbf{V}$ satisfies CH, split the set S^2_1—the stationary subset of ω_2 consisting of ordinals of cofinality ω_1—into two disjoint stationary sets S_0 and S_1 and suppose $\langle A_\alpha : \alpha \in S^2_1 \rangle$ witnesses that both $\diamond(S_0)$ and $\diamond(S_1)$ hold.[1]

Construct a finite support iteration $\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ so that at a stage $\alpha \in S_0$, if A_α codes a $\mathbb{P}_\alpha$-name for a σ-centered forcing $\hat{\mathbb{Q}}$ that strongly preserves ω-hitting families, then let $\mathbb{Q}_\alpha = \hat{\mathbb{Q}}$; otherwise, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathbb{L}_{\mathbf{nwd}^*(\mathbb{Q})}$, where $\mathbb{Q}$ are the rational numbers; at a stage $\alpha \in S_1$, if A_α codes a group operation $\circ$ on ω, a $\mathbb{P}_\alpha$-name for a regular group topology τ with no isolated points on $(\omega, \circ)$ and a $\circ$-invariant tame ideal such that neither (1) nor (2) of the IIA hold, we let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathbb{L}_{\mathcal{I}^*}$. If α is not of this form, let $\dot{\mathbb{Q}}_\alpha$ be again $\mathbb{P}_\alpha$-name for $\mathbb{L}_{\mathbf{nwd}^*(\mathbb{Q})}$, where $\mathbb{Q}$ are the rational numbers. Let G_{ω_2} be a $\mathbb{P}_{\omega_2}$-generic over $\mathbf{V}$. A standard argument shows that MA for σ-centered partial orders strongly preserving ω-hitting families holds in $\mathbf{V}[G_{\omega_2}]$.

We shall show that, in $\mathbf{V}[G_{\omega_2}]$, IIA holds.

Aiming toward a contradiction, assume that in $\mathbf{V}[G_{\omega_2}]$, there is a countable groomed group $\mathbb{G}$ with a group topology τ and a tame invariant ideal $\mathcal{I}$ on $\mathbb{G}$ satisfying neither (1) nor (2) of IIA.

Now, by a standard closing off argument, there is a set $E \subset S^2_1$ that is a club relative to S^2_1 such that for all $\alpha \in E$,

1. $\mathbf{V}[G_\alpha] \models \tau_\alpha$ is groomed, where $\tau_\alpha = \tau \cap \mathbf{V}[G_\alpha]$,
2. $\mathbf{V}[G_\alpha] \models \mathcal{I}_\alpha = \mathcal{I} \cap \mathbf{V}[G_\alpha]$ is a $\mathbb{G}$-invariant tame ideal satisfying neither (1) nor (2) of IIA, and
3. every sequence in $\mathbf{V}[G_\alpha]$ that is τ_α-convergent is forced to be τ-convergent in $\mathbf{V}[G_{\omega_2}]$.

Therefore, at some stage $\alpha \in S_1$, we would have added a set A_{gen} such that $\mathbf{V}[G_{\alpha+1}] \models A_{gen}$ is a dense entangled subset of $\mathbb{G}$: that is, the ideal $\mathcal{I}_g(\tau_\alpha)|_{A_{gen}}$ is ω-hitting for every $g \in \mathbb{G}$ (Proposition 13 (1)).

We claim that A_{gen} is also dense in $\mathbf{V}[G_{\omega_2}]$: that is, $A_{gen} \in \mathcal{I}^+_g(\tau)$ for every $g \in \mathbb{G}$. As $\mathbb{G}$ is a group, it suffices to show this at $0_{\mathbb{G}}$. If it were not true, in $\mathbf{V}[G_{\omega_2}]$, there is a τ-open neighbourhood U of 0 disjoint from A_{gen} such that $U \cdot U \cap A_{gen} = \varnothing$. Then $Y = \mathbb{G} \setminus U$ is A_{gen}-large. By Lemma 15, in $\mathbf{V}[G_{\alpha+1}]$, the ideal $\mathcal{I}^\perp_0(\tau_\alpha) = \mathcal{I}^\perp_0(\tau) \cap \mathbf{V}[G_\alpha]$ is ω-hitting with respect to A_{gen}, and by Lemmata 16 and 17, it follows that the ideal $\mathcal{I}^\perp_0(\tau_\alpha)$ is also ω-hitting with respect to A_{gen} in $\mathbf{V}[G_{\omega_2}]$. In particular, there is a $C \in \mathcal{I}^\perp_0(\tau_\alpha)$ such that $C \cap Y$ is infinite: that is, $C \in \mathbf{V}[G_\alpha]$ is a sequence τ_α-converging to 0, which intersects Y infinitely often. However, by the item 2, $\mathcal{I}^\perp_0(\tau_\alpha) \subset \mathcal{I}^\perp_0(\tau)$, therefore C is also τ-converging to 0. This however leads to a contradiction as both $Y \cap U = \varnothing$ and $C \setminus U$ is finite.

By Proposition 13 (2) and Lemma 11, in $\mathbf{V}[G_{\omega_2}]$ the ideal $\mathcal{I}_g(\tau_\alpha)|_{A_{gen}}$ is ω-hitting for every $g \in \mathbb{G}$, contradicting that $\mathbb{G}$ was groomed in $\mathbf{V}[G_{\omega_2}]$. □

3. Countable sequential groups under IIA

In this section, we prove the main result of the paper, which confirms a conjecture of the second author [68] by proving a common extension of Theorems 3 and 4 that provides the ultimate (consistent) classification for the topologies of countable sequential topological groups, namely:

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[1]Given a stationary subset S of ω_2, the principle $\diamond(S)$ asserts the existence of a sequence $\langle A_\alpha : \alpha \in S \rangle$ of subsets of ω_2 such that for any $A \subset \omega_2$, the set $\{\alpha \in S : A_\alpha = A \cap \alpha\}$ is stationary.

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Theorem 19. *Assuming IIA, every countable sequential group is either metrizable or k_ω.*

Countable k_ω groups are completely classified by their *compact scatteredness rank* defined as the supremum of the Cantor-Bendixson index of their compact subspaces by the theorem of Zelenyuk:

Theorem 20 (E. Zelenyuk [84]). *Countable k_ω groups of the same compact scatteredness rank are homeomorphic.*

To give a more concrete feel for how strong Theorem 19 actually is, let us introduce the following notation: Given an indecomposable ordinal[2] $\alpha < \omega_1$, let $\mathcal{K}_\alpha$ be a fixed countable family of compact subsets of the rationals $\mathbb{Q}$ closed under translations, inverse and algebraic sums such that $\alpha = \sup\{\operatorname{rank}_{CB}(K) : K \in \mathcal{K}_\alpha\}$, and let

$$\tau_\alpha = \{U \subseteq \mathbb{Q} : \ \forall K \in \mathcal{K}_\alpha : U \cap K \text{ is open in } K\}.$$

Then τ_α is a k_ω sequential group topology on $\mathbb{Q}$, and we denote $\mathbb{Q}_\alpha = (\mathbb{Q}, \tau_\alpha)$. Note, in particular, that if $\alpha = 0$, then τ_0 is the discrete topology on $\mathbb{Q}$, and that the usual topology on $\mathbb{Q}$ is similarly determined by taking into account *all* of its compact subsets, so it makes sense to denote it as $\mathbb{Q}_{\omega_1}$. Hence Theorem 19 can be reformulated as:

Theorem 21. *Assuming IIA, for every infinite countable sequential group $\mathbb{G}$, there is exactly one $\alpha \leq \omega_1$ such that $\mathbb{G}$ is homeomorphic to $\mathbb{Q}_{\omega^\alpha}$.*

Note that in the argument above, we may have started with an arbitrary countable *topologizable* (i.e., admitting a nondiscrete Hausdorff group topology) group $\mathbb{G}$ instead of $\mathbb{Q}$ by possibly choosing a coarser first countable topology on $\mathbb{G}$ first. Thus *every* countable topologizable group admits every possible k_ω group topology, showing that in a model of IIA, the algebraic structure of the group has almost no influence on the kind of sequential topology the group can admit. Indeed, in such models, the number of nonisomorphic topologizable countable groups ($\mathfrak{c}$) is greater than the number of nonhomeomorphic sequential group topologies (ω_1).

We shall prove Theorem 19 in a sequence of lemmata.

Lemma 22. *Let $\mathbb{G}$ be a countable nondiscrete sequential group. Suppose $\mathcal{P} \subseteq \mathbf{csc}(\mathbb{G})$ is a countable family such that for every $S \to 1_{\mathbb{G}}$, there exists a $P \in \mathcal{P}$ such that $|S \cap P| = \omega$. Let $\mathcal{D} \subseteq [\mathbb{G}]^\omega$ be a countable family of closed discrete subsets of $\mathbb{G}$. Then for every $g \in \mathbb{G}$, there exists an open $U \ni g$ such that $U \cap D$ is finite for every $D \in \mathcal{D}$.*

Proof. Let $\mathcal{D} = \{ D_n : n \in \omega \} \subseteq 2^{\mathbb{G}}$ be a collection of closed discrete subsets of $\mathbb{G}$. For brevity, call a point $g \in \mathbb{G}$ a *vD-point* of $\mathcal{D}$ if for every open $U \ni g$ there is a $D \in \mathcal{D}$ with the property $|U \cap D| = \omega$. The statement of the lemma is equivalent to claiming that there are no vD-points of $\mathcal{D}$. Suppose $g \in \mathbb{G}$ is a vD-point of $\mathcal{D}$. By translating each $D \in \mathcal{D}$ if necessary, we may assume that $g = 1_{\mathbb{G}}$.

Let $\mathcal{P} = \{ P_n : n \in \omega \}$ be a collection of closed scattered subsets of $\mathbb{G}$ such that for any $S \to 1_{\mathbb{G}}$, there is a $P \in \mathcal{P}$ such that $|S \cap P| = \omega$. By requiring $\mathcal{P}$ to be closed under finite unions, we may assume that $S \subseteq^* P$.

Pick a family $\{ O_n : n \in \omega \}$ of open neighbourhoods of $1_{\mathbb{G}}$ such that $\overline{O_{n+1}} \subseteq O_n$ and $\bigcap_{n\in\omega} O_n = \{1_{\mathbb{G}}\}$. Put $P = \bigcup_{n\in\omega} P_n \cap \overline{O_n}$. One may verify that P is closed and scattered, and for any $S \to 1_{\mathbb{G}}$, $S \subseteq^* P$. Changing P if necessary, we may further require that $\alpha_P = \operatorname{scl}(1_{\mathbb{G}}, P)$ is the smallest one among all P with such properties.

Let $P' = \{ p \in P \setminus \{1_{\mathbb{G}}\} : \operatorname{scl}(p, P) \geq \alpha_P \}$. Note that $1_{\mathbb{G}} \notin \overline{P'}$, and we may therefore assume (by taking an appropriate subset of P, if necessary) that $\operatorname{scl}(1_{\mathbb{G}}, P) = \operatorname{scl}(P) > \operatorname{scl}(p, P)$ for any $p \in P$ such that $p \neq 1_{\mathbb{G}}$. Now

[2]An ordinal number α is *indecomposable* if it cannot be written as an ordinal sum of two strictly smaller ordinals, equivalently, there is a $\beta \leq \alpha$ such that $\alpha = \omega^\beta$, where ω^β denotes ordinal exponentiation.

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(1) P is a closed scattered subset of $\mathbb{G}$ such that $S \subseteq^* P$ for every $S \to 1_{\mathbb{G}}$; moreover, $\alpha_P = \mathrm{scl}(P) = \mathrm{scl}(1_{\mathbb{G}}, P) > \mathrm{scl}(p, P)$ for any $p \in P \setminus \{1_{\mathbb{G}}\}$, and α_P is the smallest possible.

Let $g \in \mathbb{G}$, and consider the family $\mathcal{D}_g = \{ D_n \cap (g \cdot P) : n \in \omega \}$. Suppose g is not a vD-point of $\mathcal{D}_g$ for any $g \in \mathbb{G}$. For each $g_i \in \mathbb{G}$, pick an open neighbourhood $U_i \ni g_i$ such that $U_i \cap (g_i \cdot P) \cap D_n = F_i^n$ is finite for every $n \in \omega$. Put $D'_n = D_n \setminus \bigcup_{i \le n} F_i^n$. Then $1_{\mathbb{G}}$ is a vD-point of $\mathcal{D}' = \{ D'_n : n \in \omega \}$, and therefore, $1_{\mathbb{G}} \in \overline{\bigcup \mathcal{D}'}$.

Let $S \subseteq \bigcup \mathcal{D}'$ be an infinite sequence such that $S \to g_i$ for some $g_i \in \mathbb{G}$. Then by (1), $S \subseteq^* U_i \cap (g_i \cdot P)$. Since every $D \in \mathcal{D}'$ is closed discrete, we may assume that $S = \langle s_n : n \in \omega \rangle$ is such that $s_n \in D'_{m(n)}$ for some $m(n) \ge n$. Let $n > i$ be such that $s_n \in U_i \cap (g_i \cdot P)$. Then $s_n \in F_i^{m(n)}$, $i < n < m(n)$, contradicting $s_n \in D'_{m(n)} = D_{m(n)} \setminus \bigcup_{i \le m(n)} F_i^{m(n)}$. Thus no such S exists, making $\bigcup \mathcal{D}'$ almost disjoint from every convergent sequence in $\mathbb{G}$, contradicting $1_{\mathbb{G}} \in \overline{\bigcup \mathcal{D}'}$ and the sequentiality of $\mathbb{G}$.

We may therefore assume that $D \subseteq P$ for every $D \in \mathcal{D}$ and some $g \in \mathbb{G}$ is a vD-point of $\mathcal{D}$. Note that $g \in P$, since P is closed. Let $p \in P$ be a vD-point of $\mathcal{D}$ such that $\mathrm{scl}(p, P)$ is the smallest. By picking a neighbourhood $U \ni p$ relatively open in P such that $\mathrm{scl}(x, P) < \mathrm{scl}(p, P)$ for any $x \in \overline{U} \setminus \{p\}$ and restricting $\mathcal{D}$ to U, if necessary, we may assume that p is the only vD-point of $\mathcal{D}$. Using an argument similar to the one in the previous paragraph, by possibly removing a finite subset from each $D \in \mathcal{D}$, we may assume that $\mathbb{D} = \bigcup \mathcal{D} \cup \{p\}$ is closed, p is the only nonisolated point of $\mathbb{D}$ and p is a vD-point of $\mathcal{D}$.

Consider the translation $\mathcal{D}'' = \{ D \cdot p^{-1} : D \in \mathcal{D} \}$ of $\mathcal{D}$. Suppose $1_{\mathbb{G}} \in \overline{\bigcup \mathcal{D}'' \setminus P}$. By the closedness of $\mathbb{D}$ and property (1), the set $\bigcup \mathcal{D}'' \setminus P$ contains no infinite converging sequence, contradicting the sequentiality of $\mathbb{G}$.

(2) There exists a countable family $\mathcal{D}$ of closed discrete subsets of $\mathbb{G}$ such that $\bigcup \mathcal{D} \subseteq P$, $1_{\mathbb{G}}$ is the only nonisolated point of $\bigcup \mathcal{D} \cup \{1_{\mathbb{G}}\}$, which is closed in $\mathbb{G}$, and $1_{\mathbb{G}}$ is a vD-point of $\mathcal{D}$.

Suppose there exists an $S \to 1_{\mathbb{G}}$ such that $(S \cdot p) \setminus P$ is infinite for every $p \in P \setminus \{1_{\mathbb{G}}\}$. We may then pick an infinite sequence $S' \subseteq S \cdot S$ so that $S' \to 1_{\mathbb{G}}$ and $S' \subseteq \mathbb{G} \setminus P$, contradicting (1). Thus for every sequence $S \to 1_{\mathbb{G}}$, there exists a $p \in P \setminus \{1_{\mathbb{G}}\}$ such that $(S \cdot p) \subseteq^* P$.

Suppose $A \subseteq \mathbb{G}$ is such that for some ordinal β, $\mathrm{scl}(a, P) \le \beta < \alpha_P$ for every $a \in A \cap P$. Then there exists a sequence $S \to 1_{\mathbb{G}}$ such that $S \setminus \bigcup_{a \in F} (P \cdot a^{-1})$ is infinite for every $F \in [A]^{<\omega}$.

Indeed, suppose no such S exists, and let $A = \{ p_n : n \in \omega \}$ list all the points in A. For each $n \in \omega$, find a neighbourhood $U_n \ni p_n$ relatively open in P so that $\mathrm{scl}(\overline{U_n} \cap P) \le \beta$ if $p_n \in P$, and put $P'_n = (\overline{U_n} \cap P) \cdot p_n^{-1}$. If $p_n \notin P$, put $P'_n = \emptyset$. Note that $\mathcal{P}' = \{ P'_n : n \in \omega. \}$ is a collection of closed scattered subsets of $\mathbb{G}$ with the property that $\mathrm{scl}(P_n) \le \beta$ for every $n \in \omega$, and for every $S \to 1_{\mathbb{G}}$, there exists an $F \in [\omega]^{<\omega}$ such that $S \subseteq^* \bigcup_{n \in F} P'_n$.

Repeating the construction used to build P out of P_n at the beginning of this argument, we may construct a closed scattered $P' \subseteq \mathbb{G}$ such that $\mathrm{scl}(P') \le \beta$ and $S \subseteq^* P'$ for every $S \to 1_{\mathbb{G}}$, contradicting the minimality of α_P in (1).

Let $P \setminus \{1_{\mathbb{G}}\} = \{ p_n : n \in \omega \}$ list all the points in P other than $1_{\mathbb{G}}$. For each $n \in \omega$, pick an open $O_n \ni 1_{\mathbb{G}}$ such that $\beta = \mathrm{scl}(p_n, P) = \mathrm{scl}(\overline{O_n \cdot p_n} \cap P) < \alpha_P$, $\overline{O_{n+1}} \subseteq O_n$, and $\bigcap_{n \in \omega} O_n = \{1_{\mathbb{G}}\}$.

Restricting $\mathcal{D}$ to O_0 if necessary, assume that $\bigcup \mathcal{D} \subseteq O_0$. By induction, pick disjoint closed discrete $D'_n \subseteq \cup \mathcal{D}$ so that $D'_n \subseteq O_n$ and each D_n is covered by finitely many D'_n. To see that this is possible, put $D'_n = (D_n \cap O_n) \cup ((O_n \setminus O_{n+1}) \cap (\bigcup \mathcal{D} \setminus \bigcup_{i<n} D'_i))$ and observe that the intersection inside the second pair of parentheses is a closed and discrete subspace of $\mathbb{G}$, since $1_{\mathbb{G}}$ is the only nonisolated point of $\mathbb{D}$. Put $\mathcal{D}' = \{ D'_n : n \in \omega \}$. Note that $1_{\mathbb{G}}$ is a vD-point of $\mathcal{D}'$. To simplify notation, we will assume that $D_n \subseteq O_n$ in what follows.

Let $n \in \omega$. By the choice of O_n, $D_n \subseteq O_n \subseteq O_i$, so $D_n \cdot p_i \subseteq O_i \cdot p_i$, whenever $i \le n$. Thus there is a $\beta < \alpha_P$ such that $\mathrm{scl}(a, P) \le \beta$ for every $a \in A_n = \bigcup_{i \le n} D_n \cdot p_i$. Find a sequence $S_n \to 1_{\mathbb{G}}$ such that $S_n \setminus \bigcup_{a \in F} (P \cdot a^{-1})$ is infinite for every $F \in [A_n]^{<\omega}$. Let $D_n = \{ d_i : i \in \omega \}$ and $S_n = \{ s_i : i \in \omega \}$ be 1-1 listings of D_n and S_n. For each $i \in \omega$, pick an $m(i) > n$ so that $m(i)$ is strictly increasing, $s_{m(i)} \cdot d_i \cdot p_j \notin P$ for every $i \in \omega$ and $j \le n$, and $e_i^n = s_{m(i)} \cdot d_i \in O_n$. Note that the latter is possible, since $S \to 1_{\mathbb{G}}$ and $d_i \in D_n \subseteq O_n$. Put $B_n = \{ e^n_i : i \in \omega \}$ and $B = \bigcup_{n\in\omega} B_n$, and note that each B_n is a closed and discrete subspace of $\mathbb{G}$.

Now, $1_{\mathbb{G}} \in \overline{B}$. Indeed, let $U \ni 1_{\mathbb{G}}$ be any open neighbourhood of $1_{\mathbb{G}}$. Find an open $V \ni 1_{\mathbb{G}}$ such that $V \cdot V \subseteq U$. Then $V \cap D_n$ is infinite for some $n \in \omega$, since $1_{\mathbb{G}}$ is a vD-point of $\mathcal{D}$. Also, $S_n \subseteq^* V$. Thus for some large enough $i \in \omega$, both $d_i \in V$ and $s_{m(i)} \in V$, showing that $e^n_i \in U$.

Let $C \subseteq B$ be an infinite sequence such that $C \to g$ for some $g \in \mathbb{G}$. Since each B_n is closed and discrete, we may assume that $C = \{ e^{n(k)}_{i(k)} : k \in \omega \}$, where $n(k)$ is strictly increasing. Since $e^n_i \in O_n$, $C \to 1_{\mathbb{G}}$. Thus, there exists a $j \in \omega$ such that $C \cdot p_j \subseteq^* P$. Pick a $k \in \omega$ large enough so that $n(k) > j$ and $e^{n(k)}_{i(k)} \cdot p_j \in P$. At the same time, $e^{n(k)}_{i(k)} \cdot p_j = s_{m(i(k))} \cdot d_{i(k)} \cdot p_j \notin P$ by the choice of $s_{m(i)}$, a contradiction. □

Lemma 23. *Let $\mathbb{G}$ be a countable nondiscrete sequential group. Suppose $\mathcal{P} \subseteq$ **nwd**$(\mathbb{G})$ is a countable family such that for every $S \to 1_{\mathbb{G}}$, there exists a $P \in \mathcal{P}$ such that $|S \cap P| = \omega$. Then $\mathbb{G}$ does not have a countable π-network at $1_{\mathbb{G}}$ that consists of dense in themselves sets.*

Proof. Note that $\mathbb{G}$ is not Fréchet by Lemma 6 and thus does not contain a closed subspace homeomorphic to $\mathbb{D}(\omega)$ by Proposition 2.

Let $\mathcal{D} = \{ D_n : n \in \omega \}$ be a π-network at $1_{\mathbb{G}}$ such that each $D \in \mathcal{D}$ is dense in itself. By translating each element of $\mathcal{D}$ if necessary, we may assume that $1_{\mathbb{G}} \in D$ for every $D \in \mathcal{D}$.

Fix open $O_n \ni 1_{\mathbb{G}}$ so that $\overline{O_{n+1}} \subseteq O_n$ and $\bigcap_{n\in\omega} O_n = \{1_{\mathbb{G}}\}$.

Let $\mathcal{P} = \{ P_n : n \in \omega \} \subseteq$ **nwd**$(\mathbb{G})$ be such that for every $S \to 1_{\mathbb{G}}$, there exists a $P \in \mathcal{P}$ such that $|S \cap P| = \omega$. Just as in the proof of Lemma 22, we may construct a $P \in$ **nwd**$(\mathbb{G})$ such that for every $S \to 1_{\mathbb{G}}$, $S \subseteq^* P$. By taking the closure of P if necessary, we may assume that P is closed.

Let $g \in \mathbb{G}$. Define $d(g) = \mathfrak{so}(g, \mathbb{G} \setminus P)$. Let $\alpha_P = d(1_{\mathbb{G}})$. Note that $\alpha_P > 1$ by the choice of P. We proceed to prove the following claim by induction on α.

(3) Let $p \in P$ and $d(p) = \alpha$ for some $\alpha < \omega_1$. There exists a $T \subseteq \mathbb{G} \setminus P$ and a neighbourhood assignment $W : T \to \tau(\mathbb{G})$ such that the following properties hold:
 (a) $p \in [T]_\alpha$, if $p' \in \overline{T}$, then $d(p') = \mathfrak{so}(p', T)$;
 (b) $g \in W(g) \setminus P$ for every $g \in T$, the $W(g)$ are disjoint; if $s_i \in W(g_i) \setminus P$ is such that $s_i \to g$ for some $g \in \mathbb{G}$ and all g_i are distinct, then $g_i \to g$;

Let $d(p) = \alpha$ for some $p \in P$. If $\alpha = 1$, there exists an infinite sequence of $g_i \in \mathbb{G} \setminus P$ such that $g_i \to p$. Thinning out the sequence and reindexing, if necessary, pick disjoint open $W(g_i) \ni g_i$ so that $W(g_i) \subseteq O_i \cdot p$. Put $T = \{ g_i : i \in \omega \}$. Properties (a) and (b) are easy to check.

Thus we can assume $\alpha > 1$. Let $p^n \to p$ and $\alpha_n < \alpha$ be such that $p^n \in O_n \cdot p$ and $p^n \in [\mathbb{G} \setminus P]_{\alpha_n}$ for every $n \in \omega$. Since $d(p^n) \le \alpha_n < \alpha$, by the induction hypothesis there exist $T_n \subseteq \mathbb{G} \setminus P$ and $W_n : T_n \to \tau(\mathbb{G})$ that satisfy (3). Pick a sequence of open disjoint $V_n \ni p^n$ such that $V_n \subseteq O_n \cdot p$ after thinning out and reindexing if necessary. By passing to subsets and reindexing again, if necessary, we may assume that the $\overline{T_n}$ are disjoint, $T_n \subseteq O_n \cdot p \cap V_n$, and $W_n(g) \subseteq O_n \cdot p \cap V_n$ for every $n \in \omega$ and $g \in T_n$. Let $T = \bigcup_{n\in\omega} T_n$, and define $W : T \to \tau(\mathbb{G})$ by $W(g) = W_n(g)$ whenever $g \in T_n$.

By the choice of T_n and O_n, $\overline{T} = \{p\} \cup \bigcup_{n\in\omega} \overline{T_n}$. If $p' \in \overline{T_n}$, then $d(p') = \mathfrak{so}(p', T_n) = \mathfrak{so}(p', T)$ by the inductive hypothesis and the choice of T_n. Since $d(p) = (\sup_n \alpha_n) + 1$ and $d(p^n) \le \alpha_n$, $d(p) = \mathfrak{so}(p, T)$.

Let $s_i \in W(g_i) \setminus P$ for some $g_i \in T$ be such that $s_i \to g$. By thinning out and reindexing, we may assume that either $g_i \in T_n$ for some fixed $n \in \omega$ or $g_i \in T_{n(i)} \subseteq O_{n(i)} \cdot p$ for some strictly increasing $n(i)$. In the first case, $g_i \to g \in P$ by the choice of T_n. Otherwise, $s_i \in W_{n(i)}(g_i) \subseteq O_{n(i)} \cdot p$, so $s_i \to p$ by the choice of O_n, contradicting $s_i \in \mathbb{G} \setminus P$ and $d(p) = \alpha > 1$.

Pick T and W that satisfy (3) for $p = 1_{\mathbb{G}}$, and let $T = \{ t_n : n \in \omega \}$ be a 1-1 enumeration of the points of T. Pick $U_n \subseteq (W(t_n) \setminus P) \cdot t_n^{-1}$ so that $U_{n+1} \subseteq U_n$, $\bigcap_{n\in\omega} U_n = \{1_{\mathbb{G}}\}$, and $1_{\mathbb{G}} \in \overline{\{ t_k : U_k \cdot t_k \subseteq O_n \}}$ for every $n \in \omega$. Note that each $U_n \cdot t_n \subseteq \mathbb{G} \setminus P$. Let $k \in \omega$, and show that

(4) there exists a closed discrete subset $E_k \subseteq D_k$ such that $E_k = \{\, e^k_n : n \in \omega \,\}$ and $e^k_n \in U_n$ for every $n \in \omega$.

If no such E_k exists, then $U_n \cap D_k$ form a countable base of neighbourhoods of $1_{\mathbb{G}}$ in D_k. Since $1_{\mathbb{G}} \in D_k$ and each D_k is dense in itself, this implies the existence of a closed copy of $\mathbb{D}(\omega)$ in $\mathbb{G}$, contradicting the sequentiality of $\mathbb{G}$ as noted at the beginning of this proof.

Consider the set $D^k = \{\, d^n : d^n = e^k_n \cdot t_n, U_n \cdot t_n \subseteq O_k, n \in \omega \,\}$. Then $D^k \subseteq O_k \setminus P$ for every $k \in \omega$ and $d^n \in W(t_n)$ for every $n \in \omega$ by $e^k_n \in U_n$ and the choice of U_n. Suppose $d^{n(i)} \to d$ for some $d \in \mathbb{G}$. By (3b) $t_{n(i)} \to d$ so $e^k_{n(i)} = d^{n(i)} \cdot t^{-1}_{n(i)} \to 1_{\mathbb{G}}$, contradicting the choice of e^k_n. Thus each $D^k \subseteq O_k$ is closed and discrete in $\mathbb{G}$.

Suppose $S \to g$ for some $g \in \mathbb{G}$ is an infinite sequence such that $S \subseteq \bigcup_{k\in\omega} D^k$. Since each D^k is closed discrete, we may assume that $S = \{\, s_n : n \in \omega \,\}$, where $s_n \in D^{k(n)}$ for some strictly increasing $k(n)$. Then $s_n \in O_{k(n)}$ and $S \to 1_{\mathbb{G}}$, contradicting $S \subseteq \mathbb{G} \setminus P$ and the choice of P.

Let $U \ni 1_{\mathbb{G}}$ be open, and find an open $V \ni 1_{\mathbb{G}}$ such that $V \cdot V \subseteq U$. Let $k \in \omega$ be such that $D_k \subseteq V$, and let t_n be such that $U_n \cdot t_n \subseteq O_k$ and $t_n \in V$. Then $d^n = e^k_n \cdot t_n \in D^k \cap V \cdot V$. Thus $1_{\mathbb{G}} \in \overline{\bigcup_{k\in\omega} D^k}$, contradicting the sequentiality of $\mathbb{G}$. □

Lemma 24. *Let X be a sequential space, $S \subseteq X$ and $x \in \overline{S}$ for some $x \in X$. Let $\mathcal{I} \subseteq 2^X$ be a cover of S. Then either there exists an $I \in \mathcal{I}$ such that $x \in \overline{S \cap I}$ or there is a countable $\mathcal{I}^* \subseteq [\mathcal{I}]^\omega$ such that whenever $\mathcal{I}' \subseteq \mathcal{I}$ is such that $\mathcal{I}' \cap \mathcal{I}''$ is infinite for every $\mathcal{I}'' \in \mathcal{I}^*$, $x \in \overline{S \cap \bigcup \mathcal{I}'}$.*

Proof. The proof proceeds by induction on $\mathfrak{so}(x,S)$. The case $\mathfrak{so}(x,S) = 0$ is trivial, so assume $\mathfrak{so}(x,S) = \alpha + 1$, and the lemma is proved for all successor $\beta \leq \alpha$. Pick a sequence $T \subseteq X$ such that $T \to x$ and $\mathfrak{so}(y,S) = \beta_y \leq \alpha$ for every $y \in T$, and consider the two alternatives that follow from the inductive hypothesis.

First, suppose the set $T' = \{\, y : y \in \overline{I_y \cap S}$ for some $I_y \in \mathcal{I} \,\}$ is infinite. If the family $\mathcal{I}' = \{\, I_y : y \in T' \,\}$ is finite, then there is an $I \in \mathcal{I}'$ such that $x \in \overline{S \cap I}$. Otherwise put $\mathcal{I}^* = \{\mathcal{I}'\}$.

Alternatively, assume for every $y \in T$, there is a countable $\mathcal{I}^*_y \subseteq [\mathcal{I}]^\omega$ such that $y \in \overline{S \cap \bigcup \mathcal{I}'}$ for any $\mathcal{I}'$ such that $\mathcal{I}' \cap \mathcal{I}''$ is infinite for every $\mathcal{I}'' \in \mathcal{I}^*_y$. Put $\mathcal{I}^* = \bigcup_{y\in T} \mathcal{I}^*_y$. □

Lemma 25. *Let X be a countable sequential space and $\mathcal{I} \subseteq 2^X$ be an ideal with the following properties: $\mathcal{I}$ contains all singletons, $\overline{I} \in \mathcal{I}$ for every $I \in \mathcal{I}$, and whenever $A \in \mathcal{I}^+$, there is a $Y \subseteq A$, $Y \in \mathcal{I}^+$ such that $\overline{Y \setminus I} = \overline{Y}$ for any $I \in \mathcal{I}$. Then $\mathcal{I}$ is tame.*

Proof. Suppose $\mathcal{I}$ is not tame, and let $A \in \mathcal{I}^+$, $f : A \to \omega$ witness this. Using the property of $\mathcal{I}$ from the statement of the lemma, find $Y \subseteq A$, $Y \in \mathcal{I}^+$ such that $\overline{Y \setminus I} = \overline{Y}$ for any $I \in \mathcal{I}$. Let $y \in Y$. Since $\mathcal{I}$ contains $\{y\}$ and X is sequential, there exists a sequence $T \subseteq \overline{Y} \setminus \{y\}$ such that $T \to y$. Let $T = \langle y_i : i \in \omega \rangle$, $I_i = f^{-1}(i) \in \mathcal{I}$, and for every $i \in \omega$, pick a subset $S_i \subseteq Y \setminus \bigcup_{j<i} I_j$ such that $y_i \in \overline{S_i} \not\ni y$.

Note that $\mathcal{I}_f = \{\, I_i : i \in \omega \,\}$ is a cover of $S = \bigcup_{i\in\omega} S_i$ and $y \in \overline{S}$. Since $I_i \cap S \subseteq \bigcup_{j\leq i} S_i$ and $y \notin \overline{\bigcup_{j\leq i} S_i}$, the first alternative of Lemma 24 fails, so there exists a countable $\mathcal{I}^*_y \subseteq [\mathcal{I}_f]^\omega$ such that $y \in \overline{S \cap \bigcup \mathcal{I}'}$ for any $\mathcal{I}' \subseteq \mathcal{I}$ with the property that $\mathcal{I}' \cap \mathcal{I}''$ is infinite for every $\mathcal{I}'' \in \mathcal{I}^*_y$.

Let $\mathcal{J} = \{\, f[\bigcup \mathcal{I}''] : \mathcal{I}'' \in \mathcal{I}^*_y, y \in Y \,\} \subseteq 2^\omega$. Since $f[\mathcal{I}|_A]$ is ω-hitting, there exists a $J \in f[\mathcal{I}|_A]$ such that $J \cap J'$ is infinite for every $J' \in \mathcal{J}$. Let $\mathcal{I}' = \{\, f^{-1}(n) : n \in J \,\}$. Then $\mathcal{I}' \cap \mathcal{I}''$ is infinite for every $\mathcal{I}'' \in \mathcal{I}^*_y$, so $y \in \overline{f^{-1}[J]}$ for every $y \in Y$. Thus $Y \subseteq \overline{f^{-1}[J]} \in \mathcal{I}$, contradicting $Y \in \mathcal{I}^+$. □

Lemma 26. *Let $\mathbb{G}$ be a countable nondiscrete sequential group. Then each of* **nwd**$(\mathbb{G})$, **cpt**$(\mathbb{G})$ *and* **csc**$(\mathbb{G})$ *is tame.*

Proof. For **cpt**$(\mathbb{G})$, the statement can be proved directly. For **nwd**$(\mathbb{G})$ and **csc**$(\mathbb{G})$, it is sufficient to establish the properties listed in Lemma 25. In the case of **nwd**$(\mathbb{G})$, one may pick $Y = \mathrm{Int}(\overline{A}) \cap A$, while for **csc**$(\mathbb{G})$, the choice of the full Cantor-Bendixson derivative of A as Y satisfies the conditions of Lemma 25. □

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Lemma 27. *Let $\mathbb{G}$ be a countable, sequential non-metrizable, non-k_ω group. Then one of the* $\mathbf{nwd}(\mathbb{G})$, $\mathbf{cpt}(\mathbb{G})$ *or* $\mathbf{csc}(\mathbb{G})$ *is a tame invariant ideal that satisfies neither 1 nor 2 of the* IIA.

Proof. That each of the ideals is tame follows from Lemma 26. The invariance is trivial.

Since $\mathbf{nwd}(\mathbb{G})$ never satisfies 2 for a nonmetrizable $\mathbb{G}$ (see [39], Proposition 5.2), we may assume that $\mathbf{nwd}(\mathbb{G})$ satisfies 1. Suppose $\mathbf{csc}(\mathbb{G})$ satisfies 2. Then there exists a countable $\mathcal{D} \subseteq \mathbf{csc}^+(\mathbb{G})$ such that for any nonempty open $U \subseteq \mathbb{G}$, there exists a $D \in \mathcal{D}$ with the property $D \setminus U \in \mathbf{csc}(\mathbb{G})$. By replacing each D with a full Cantor-Bendixson derivative of itself, we may assume that each D is dense in itself. Applying Lemma 23, we arrive at a contradiction. Thus $\mathbf{csc}(\mathbb{G})$ does not satisfy 2.

Suppose $\mathbf{csc}(\mathbb{G})$ satisfies 1. If $\mathbf{cpt}(\mathbb{G})$ satisfies 2, there exists a countable family $\mathcal{D}$ of closed, noncompact subsets of $\mathbb{G}$ such that for any open $U \subseteq \mathbb{G}$, there exists a $D \in \mathcal{D}$ such that $D \setminus U$ is compact. By picking an infinite closed discrete subset in each $D \in \mathcal{D}$ and applying Lemma 22, we arrive at a contradiction. Thus either $\mathbf{csc}(\mathbb{G})$ does not satisfy 1 or $\mathbf{cpt}(\mathbb{G})$ does not satisfy 2.

Since $\mathbb{G}$ is not k_ω, $\mathbf{cpt}(\mathbb{G})$ cannot satisfy 1. □

This concludes the proof of Theorem 19. The result has the following corollary, which illuminates the behavior of sequential groups under taking products (part (1) is an obvious corollary of Theorem 7 and has been included for completeness):

Corollary 28. *Assume IIA.*

1. *The product of at most countably many separable Fréchet groups is Fréchet, and*
2. *The product of finitely many countable sequential groups that are either discrete or not Fréchet is sequential.*

Proof. It suffices to note that

1. $\mathbb{Q}_{\omega^\alpha} \times \mathbb{Q}_{\omega^\beta} \simeq \mathbb{Q}_{\omega^\beta}$ if $\alpha < \beta < \omega_1$,
2. $\mathbb{Q}_{\omega^\alpha} \times \mathbb{Q}_{\omega_1}$ is not sequential if $0 < \alpha < \omega_1$,
3. $\mathbb{Q}_0 \times \mathbb{Q}_0 \simeq \mathbb{Q}_0$, and
4. $\mathbb{Q}_0 \times \mathbb{Q}_{\omega_1} \simeq \mathbb{Q}_{\omega_1} \times \mathbb{Q}_{\omega_1} \simeq \mathbb{Q}_{\omega_1}$. □

We do not know at the moment whether it is consistent that the product of two sequential groups that are not Fréchet is sequential (independently of their cardinality).

4. Examples, concluding remarks and open questions

The example below can probably be constructed using the techniques of [76], but we chose to provide a direct proof. An appeal to [76] would require a proof of the normality of finite powers of $\gamma\mathbb{N}$ spaces, as well as an adaptation of the free topological group arguments from [76] to the free boolean group construction used here.

The nontrivial case of the proof below assumes $\mathfrak{t} = \omega_1$; however, the statement of the example is meant to emphasize the fact that one of the two 'pathologies' exists in every model of ZFC: either there is a separable nonmetrizable Fréchet group or a (possibly uncountable) sequential group that is not k_ω. The authors do not know whether any separable locally compact first countable countably compact non-compact space may be used in place of $\gamma\mathbb{N}$.

Example 29. If there is no separable nonmetrizable Fréchet group, then there exists a separable sequential c_ω group $\mathbb{G}$ that is not k_ω.

Proof. Since $\mathfrak{t} > \omega_1$ implies the existence of a separable nonmetrizable Fréchet group (see, for example [38]), we may assume that $\mathfrak{t} = \omega_1$.

Let $X = D \cup \omega_1$ be a countably compact $\gamma\mathbb{N}$ space (see [48], Example 2.2) where D is the set of isolated points, disjoint from ω_1, which has the usual topology. Let $X \cup \{\infty\}$ be the one-point compactification of X, which is also a subspace of some boolean group H that is (algebraically) generated by $X \cup \{\infty\}$. We shall assume that $X \cup \{\infty\}$ is linearly independent over H (in particular, this means $0_H \notin X \cup \{\infty\}$). The free boolean group over $X \cup \{\infty\}$ (see [73]) would have all the desired properties (in fact, it is not difficult to show that any group satisfying the properties above is naturally isomorphic to the free boolean group over $X \cup \{\infty\}$). Below we use the convention that the elements of such a group are finite sets of elements of $X \cup \{\infty\}$ with the symmetric difference as the group operation.

Let $\mathbb{G}$ be the subgroup of H generated by X, and let $A \subseteq \mathbb{G}$ be such that $0_{\mathbb{G}} \in \overline{A} \setminus A$ (here the closure is taken in the topology induced by H). We must show that there exists a sequence $S \subseteq A$ such that $S \to x \in \mathbb{G} \setminus A$. Suppose no such sequence exists.

Let $n \in \omega$ be the smallest number such that $0_{\mathbb{G}} \in \overline{A \cap \sum^n X}$. Note that such an n exists by the definition of the topology on H. We will assume that $A \subseteq \sum^n X$. By the minimality of n (truncating A if necessary), we may assume that $|a| = n$ for every $a \in A$. Write an arbitrary $a \in A$ as $a = d + w$, where $d \in \langle D \rangle$ and $w \in \langle \omega_1 \rangle$, and put $\delta(a) = |d|$. Note that for every $k \leq n$ the set $A_k = \{ a \in A : \delta(a) \leq k \}$ is a sequentially closed subset of A.

Let $k \leq n$ be the smallest such that $0_{\mathbb{G}} \in \overline{A_k}$. Replacing A with A_k and using the minimality of k, we may assume (again truncating A if necessary) that $\delta(a) = k$ for every $a \in A$. Let $D_A = \bigcup \{t \in D : t \in a \in A\}$, and suppose D_A is infinite. Note that $\sum^n X$ is sequentially compact. Using this and the property of D_A, we may pick a convergent sequence $S \subseteq A$ such that $S \to x$ for some $x \in \sum^n X$ and for each $s \in S$, there is a $t_s \in s \cap D$ such that $\{ t_s : s \in S \} \to \alpha \in \omega_1$. Then $\delta(x) < k$, so $x \notin A$.

Thus we may assume that D_A is finite. Note that this implies that D_A is empty (otherwise $0_{\mathbb{G}} \notin \overline{A}$). Therefore $A \in \sum^n \omega_1$. Define $l \leq n$ to be the largest with the following property: for any $\alpha < \omega_1$, there exists an $a \in A$ such that $|a \setminus \alpha| \geq l$. Note that we may assume that $l \geq 1$; otherwise, A is countable with a metrizable closure.

Suppose $l \geq 2$. Recursively pick a sequence $a_i \in A$ such that for some distinct $\alpha_i, \beta_i \in a_i, \alpha_{i+1}, \beta_{i+1} > \max\{\alpha_i, \beta_i\}$. By passing to a subsequence if necessary, we may assume that $a_i \to a \in \sum^n \omega_1$. Since $\alpha_i, \beta_i \to \gamma$ for some $\gamma \in \omega_1$, $|a| < n$, showing that $a \notin A$. We may thus assume that $l = 1$.

This implies the existence of an $\alpha \in \omega_1$ such that every $a \in A$ can be written as $a = \{\beta_a\} + b_a$, where $\beta_a > \alpha + 1$ and $b_a \in [\alpha + 1]^{n-1}$. If $n > 1$, the set $A' = \{ b_a : a \in A \}$ is a sequentially closed (and therefore compact) subset of H such that $0_{\mathbb{G}} \notin A'$ (otherwise $A \cap \omega_1 \neq \emptyset$, contradicting the choice of A and $n > 1$). Now $U = H \setminus (A' + (\{ \beta : \beta > \alpha + 1 \} \cup \{\infty\}))$ is an open neighbourhood of $0_{\mathbb{G}}$ such that $U \cap A = \emptyset$, contradicting the choice of A. Hence $n = 1$, implying $0_{\mathbb{G}} \notin \overline{A}$, a contradiction. □

The statement of IIA given at the beginning of this paper may appear somewhat technical in that it lists several restrictions on both the space (groomed) as well as the ideal (tame, invariant, weakly closed). This complexity may be significantly reduced in most applications, however. Most natural ideals (including all used in this paper) in sequential spaces are generated by their (sequentially) closed members, while tameness can be replaced by the topological condition defined in Lemma 25, namely the existence of a 'kernel' in each positive set. One may prefer a weaker version of IIA that states that for every invariant ideal generated by sequentially closed sets for which the conditions in Lemma 25 are satisfied, one of the two alternatives in the statement of IIA holds.

Limiting the class of spaces may also make applications of IIA more transparent. Call the following statement the *Unrestricted Ideal Axiom* or UIA:

UIA: *For every space X in some class $\mathcal{P}$and every ideal $\mathcal{I} \subseteq 2^X$, one of the following holds for every $x \in X$:*

1. *there is a countable $\mathcal{S} \subseteq \mathcal{I}$ such that for every infinite sequence C convergent to x in X, there is an $I \in \mathcal{S}$ such that $C \cap I$ is infinite,*
2. *there is a countable $\mathcal{H} \subseteq \mathcal{I}^+$ such that for every non-empty open $U \subseteq X$, $x \in U$, there is an $H \in \mathcal{H}$ such that $H \setminus U \in \mathcal{I}$.*

It is not difficult to see that any countable space that is either k_ω or first countable satisfies UIA. Theorem 19 shows IIA implies UIA holds for the class of all countable sequential groups (note that there are no restrictions on the ideal whatsoever, not even invariance). The authors do not know at the moment

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if UIA for all *groomed* groups is implied by IIA or even whether it is consistent. There are countable Fréchet spaces for which UIA fails in ZFC with the ideal of the nowhere dense subsets as the witness (see [27]).

To shed some light on the topology of groomed spaces, the following more detailed treatment of the concept of a vD-point from Lemma 22 may be helpful.

Definition 30. Let X be a topological space. Let $\mathcal{D}$ be a countable family of infinite closed discrete subspaces of X. We call $\mathcal{D}$ a *(strict) vD-network* at $x \in X$ if for every open $U \ni x$, there is a $D \in \mathcal{D}$ such that $D \cap U$ is infinite ($D \subseteq^* U$).

If $\mathcal{D}$ is a (strict) vD-network at x, we will refer to the space $\bigcup \mathcal{D} \cup \{x\}$ as a *(strict) vD-subspace of* X and the point x as a *(strict) vD-point of* $\mathcal{D}$ *in* X.

Now the lemma below offers a topological description of countable groomed spaces. We omit an elementary proof.

Lemma 31. *A countable topological space X is groomed if and only if for every dense $D \subseteq X$, there exists a point $x \in X$ such that there exists either an infinite sequence $S \subseteq D$ such that $S \to x$ or a strict vD-network $\mathcal{D} \subseteq 2^D$ at x.*

As indicated by Corollary 28, the classes of (countable) Fréchet and sequential non-Fréchet groups are both finitely productive, assuming IIA holds. It appears the following question is open, including the intriguing possibility of the negative answer in ZFC.

Question 1. Does there exist a (separable, or even countable) sequential non-Fréchet group with a non-sequential square?

The answer to the next question is known to be independent of ZFC for separable groups, while the non-separable case in ZFC remains open.

Question 2. Does there exist a Fréchet group with a non-Fréchet square?

Example 29 shows that the dichotomy of Theorem 19 does not hold for general sequential groups even in the separable case. However, the more general question below appears to be open.

Question 3. Does there exist a separable sequential group that is neither c_ω nor metrizable?

Finally, while the class of k_ω spaces is finitely productive, it is not clear if the same is true about sequential c_ω spaces.

Question 4. Are sequential c_ω-spaces preserved by finite products?

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