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q-HYPERCYCLIC RINGS

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0. Introduction. A ring R is called q-hypercyclic (hypercyclic) if each cyclic ring R-module has a cyclic quasi-injective (injective) hull. A ring R is called a qc-ring if each cyclic right R-module is quasi-injective. Hypercyclic rings have been studied by Caldwell [**4**], and by Osofsky [**12**]. A characterization of qc-rings has been given by Koehler [**10**]. The object of this paper is to study q-hypercyclic rings. For a commutative ring R, R can be shown to be q-hypercyclic ($=$ qc-ring) if R is hypercyclic. (Theorems 4.2 and 4.3). Whether a hypercyclic ring (not necessarily commutative) is q-hypercyclic is considered in Theorem 3.11 by showing that a local hypercyclic ring R is q-hypercyclic if and only if the Jacobson radical of R is nil. However, we do not know if there exists a local hypercyclic ring with nonnil radical [**12**]. Example 3.10 shows that a q-hypercyclic ring need not be hypercyclic. A characterization of local q-hypercyclic rings is given in Theorem 3.9 by showing that local q-hypercyclic rings are precisely qc-rings. The structure of a semi-perfect q-hypercyclic ring is given in Theorem 5.7 whence it follows as a consequence that if R is a semi-perfect q-hypercyclic ring then each cyclic right R-module is a finite direct sum of indecomposable quasi-injective modules. Finally, a characterization of right or left perfect q-hypercyclic (hypercyclic) rings is given in Section 6. Our results depend upon a number of lemmas. Lemma 5.1 regarding the quasi-injective hull of $A \oplus B$, where B contains a copy of the injective hull $E(A)$ of A, though straightforward, is also perhaps of interest by itself, besides being a key lemma in the proof of our Theorem 5.5. We also make use of Koehler's characterization of qc-rings as those which are direct sum of rings each of which is semisimple artinian, or a rank 0 duo maximal valuation ring.

1. Notation and definitions. All rings considered have unity and unless otherwise stated all modules are unital right modules. If M is a module, then $E(M)$ (q.i.h. (M)) will denote the injective hull (quasi-injective hull) of M. An idempotent e of a ring R is called primitive if the module eR is indecomposable. J will denote the Jacobson radical of the ring R. $S(R_R)$ $(S(_RR))$ will denote the right (left) socle of R. Let $X \subseteq R$, then $r_R(X)$ $(l_R(X))$ will denote the right (left) annihilator of X in R.

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$N \subset' M$ will denote that N is a large submodule of M.

R is called right (left) duo if every right (left) ideal of R is a twosided ideal of R. R is a right (left) valuation ring if right (left) ideals of R are linearly ordered. R is called a right (left) bounded ring if every non-zero right (left) ideal of R contains a non-zero twosided ideal of R. R is called a duo (valuation, bounded) ring if R is both right and left duo (valuation, bounded).

A module M is called local if M has a unique submodule. A ring R is called semi-perfect if R/J is artinian and idempotents modulo J can be lifted, or equivalently every finitely generated module has a projective cover. R is called right (left) perfect if every right (left) R-module has a projective cover, or equivalently, R/J is artinian and every non-zero right (left) R-module has a maximal submodule. R is called uniserial if R is an artinian principal ideal ring. An R-module M is said to have finite Azumaya diagram (A.D) **[5]** if

$$M = \oplus \sum_{i=1}^{k} M_i,$$

where each R-submodule M_i has a local endomorphism ring.

2. Preliminary results.

LEMMA 2.1. *Let M be quasi-injective. If* $E(M) = \oplus \sum_{i=1}^{n} K_i$ *is a direct sum of submodules K_i, then*

$$M = \oplus \sum_{i=1}^{n} (M \cap K_i).$$

Proof. See (**[7]**, Theorem 1.1).

The following is a well known equivalence between mod-R, the category of right R-modules and mod-R_n, the category of right R_n-modules, where R_n is the $n \times n$ matrix ring over R.

LEMMA 2.2. *Let*

$$F = \sum_{i=1}^{n} x_i R$$

be a free R-module with free basis $\{x_i | 1 \leqq i \leqq n\}$. *Then* $M_R \to \mathrm{Hom}_R(F, M)$ *is a category isomorphism between* mod-R *and* mod-R_n *with inverse*

$$N_{R_n} \to N \otimes_{R_n} F.$$

LEMMA 2.3. *Let R/J be artinian, I a right ideal of R,*

$$R/I = \oplus \sum_{i=1}^{k} M_i.$$

Then $k \leqq$ composition length of R/J.

Proof. See ([**12**], Lemma 1.8).

LEMMA 2.4. *Let I be a two-sided ideal of R and let E be an injective R-module. Then*

$$0:_E I = \{x \in E | xI = 0\}$$

is injective as an R/I-module.

Proof. See ([**13**], Proposition 2.27).

LEMMA 2.5. *Let R be semiperfect and q-hypercyclic. Then R_R is self-injective.*

Proof. Let I be a right ideal of R such that R/I is the quasi-injective hull of R. Let $\phi:R \rightarrow R/I$ be the embedding. Since R/I contains a copy of R, R/I is injective. Let $\phi(R) = B/I$. Then $B/I \subset' R/I$. Hence $B \subset' R$. Since $R \cong B/I$, B/I is projective. Thus $B = I \oplus K$ for some $K_R \subseteq B_R$. Now

$$R \cong \frac{B}{I} = \frac{I \oplus K}{I} \cong K.$$

Therefore $E(R) \cong E(K)$. But then $I \oplus K \subseteq' R$ implies

$$E(R) = E(I) \oplus E(K) \cong E(I) \oplus E(R).$$

Since $E(R) \cong R/I$, $E(R)$ is a finite direct sum of indecomposable modules, by Lemma 2.3. Thus $E(R)$ has finite Azumaya-Diagram [**5**]. Therefore, $E(R) \cong E(R) \oplus E(I)$ implies $E(I) = 0$. Hence $I = 0$. Thus R is self-injective.

LEMMA 2.6. *Let R be q-hypercyclic. Then every homomorphic image of R is also q-hypercyclic.*

Proof. Let A be a twosided ideal of R. Let $\bar{R} = R/A$. Let $\bar{R}/\bar{I}$ be a cyclic $\bar{R}$-module, where $\bar{I} = I/A$. But $\bar{R}/\bar{I} \cong R/I$. Since $A \subset I$, A annihilates R/I. Let R/K be the quasi-injective hull of R/I as an R-module. Then

$$\frac{R}{K} \cong \operatorname{End}_R\left(E\left(\frac{R}{I}\right)\right)\frac{R}{I}.$$

Then it follows that A annihilates R/K. Thus R/K may be regarded as an $\bar{R}$-module. Since R/K is quasi-injective as an R-module, R/K is quasi-injective as an $\bar{R}$-module. Since A is a twosided ideal and annihilates

R/K, $A \subset K$. Hence

$$\frac{R}{K} \cong \frac{R}{A} \quad \frac{K}{A}.$$

Clearly $\bar{R}/\bar{K}$ is the quasi-injective hull of $\bar{R}/\bar{I}$ as an $\bar{R}$-module. Hence $\bar{R}$ is q-hypercyclic.

LEMMA 2.7. *Let R be a finite direct sum of rings, $\{R_i | 1 \leqq i \leqq n\}$. Then R is q-hypercyclic if and only if each R_i is q-hypercyclic for all i, $1 \leqq i \leqq n$.*

Proof. This is straightforward.

3. Local q-hypercyclic rings. In this section we study local q-hypercyclic rings and show that over such rings every cyclic module is quasi-injective. Throughout this section unlesss otherwise stated R will denote a local q-hypercyclic ring.

LEMMA 3.1. *If I is a right ideal of R, then $E(R/I)$ is indecomposable.*

Proof. Let q.i.h. $(R/I) = R/A$. Since R/A is indecomposable, $E(R/I)$ is indecomposable.

LEMMA 3.2. *Right ideals of R are linearly ordered.*

Proof. Let A and B be right ideals of R. Suppose

$$\frac{A}{A \cap B} \neq 0, \quad \frac{B}{A \cap B} \neq 0.$$

Then

$$\frac{A}{A \cap B} \oplus \frac{B}{A \cap B} \subseteq \frac{R}{A \cap B}.$$

Hence

$$E\left(\frac{R}{A \cap B}\right) = E\left(\frac{A}{A \cap B}\right) \oplus E\left(\frac{B}{A \cap B}\right) \oplus K.$$

By Lemma 3.1, $E\left(\frac{R}{A \cap B}\right)$ is indecomposable. Hence either

$$\frac{A}{A \cap B} = 0 \quad \text{or} \quad \frac{B}{A \cap B} = 0.$$

Thus either $A \subseteq B$ or $B \subseteq A$.

LEMMA 3.3. *Left ideals of R are linearly ordered.*

Proof. This follows by (**[8]**, Theorem 1) and Lemma 3.2.

LEMMA 3.4. *Let I be a non-zero right ideal of R. If* q.i.h. $(R/I) \cong R$, *then R/I is injective.*

Proof. Let $\phi:R/I \to R$ be the embedding. Let $\phi(1 + I) = x$. Then $R/I \cong xR$. Let $A = xR$. Then R is quasi-injective hull of A. Thus

$$R = \operatorname{End}_R(R)A = RA = RxR.$$

Therefore, $x \notin J$, and hence x is a unit. Thus $A = R$. Hence R/I is injective.

LEMMA 3.5. *Let I be a non-zero right ideal of R such that R/I is quasi-injective. Suppose $S({}_RR) = 0$. Then I contains a non-zero twosided ideal of R.*

Proof. Since R is local, $r_R(J) = S({}_RR) = 0$. We may assume that $I \neq J$. Let $x \in J$ and $x \notin I$. Then $I \subsetneq xR$. By linear ordering on right ideals either $x^{-1}I \subset I$ or $I \subset x^{-1}I$. Suppose $x^{-1}I \subset I$. Define

$$\phi:\frac{xR}{I} \to \frac{R}{I}$$

by $\phi(xa + I) = a + I$. Since $x^{-1}I \subset I$, ϕ is well defined. Then ϕ can be extended to $f:R/I \to R/I$. Let $f(1 + I) = t + I$. Then

$$1 + I = \phi(x + I) = f(x + I) = tx + I.$$

Therefore $1 - tx \in I$. Since $tx \in J$, $1 - tx$ is a unit. Thus $I = R$. Hence $I \subset x^{-1}I$. Let

$$y = xa \in xI, a \in I.$$

Since $a \in I \subset x^{-1}I$, $xa \in I$. Thus $xI \subset I$. Hence for all $x \in J$, $x \notin I$, $xI \subset I$. Thus $JI \subset I$. If $JI = 0$ then

$$I \subset r_R(J) = 0.$$

Since I is non-zero, $JI \neq 0$. Therefore JI is a non-zero twosided ideal of R contained in I.

LEMMA 3.6. *R is left bounded or R is right bounded.*

Proof. Case 1. If $\operatorname{Soc}({}_RR) \neq 0$, then by linear ordering on left ideals, $\operatorname{Soc}({}_RR)$ is a non-zero twosided ideal contained in each left ideal and hence R is left bounded.

Case 2. $\operatorname{Soc}({}_RR) = 0$.

Let I be a nonzero right ideal of R. If R is the quasi-injective hull of R/I, then R/I is quasi-injective by Lemma 3.4. Hence I contains a non-zero twosided ideal (Lemma 3.5).

Let R/A be the quasi-injective hull of R/I, for some non-zero right ideal A of R. Then by Lemma 3.5 A contains a non-zero twosided ideal,

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say B. Let $\phi:R/I \to R/A$ be the embedding and let $\phi(1 + I) = x + A$. Let $a \in B$. Then

$$\phi(a + I) = xa + A = A.$$

Therefore $a \in I$. Thus $B \subset I$. Therefore I contains a non-zero ideal B. Hence R is right bounded.

LEMMA 3.7. *J is nil.*

Proof. Let $a \in J$. Suppose $a^n \neq 0$ for any positive integer n. Let

$$S = \{a^n | n > 0\}.$$

By Zorn's lemma there exists an ideal P of R maximal with respect to the property that $P \cap S = \phi$. Then P is prime. Hence R/P is a prime local q-hypercyclic ring. Thus R/P is either left bounded or right bounded. Then it follows that R/P is a domain. Since R/P is also local and q-hypercyclic ring, R/P is self-injective and hence a division ring. Therefore P is a maximal ideal of R. Thus $P = J$, a contradiction. Hence J is nil.

LEMMA 3.8. *R is duo.*

Proof. It suffices to show that for $0 \neq y \in R$, $yR = Ry$. Let $0 \neq y \in R$. Suppose $yr \notin Ry$. By linear ordering on left ideals $Ry \subsetneq Ryr$. Therefore

$$y = xyr \quad \text{for some } x \in R.$$

If $x \in J$ then $x^n = 0$ for some n. Then

$$y = xyr = x^2yr^2 = \ldots = x^nyr^n = 0,$$

which is a contradiction. Thus x is a unit. Hence

$$xyr = y \Rightarrow yr = x^{-1}y \in Ry,$$

which is again a contradiction. Thus $yR \subseteq Ry$. By symmetry $Ry \subseteq yR$. Hence $Ry = yR$.

We now prove the main result of this section.

THEOREM 3.9. *Let R be a local ring. Then R is q-hypercyclic if and only if R is a qc-ring.*

Proof. Let R be q-hypercyclic and let A be a non-zero right ideal of R. Then by Lemma 3.8, A is a twosided ideal of R. But then by Lemma 2.6, R/A is a self-injective ring. Thus R/A is a quasi-injective R-module, proving that R is a qc-ring. The converse is obvious.

The following example shows that a q-hypercyclic ring need not be hypercyclic.

Example 3.10. *Let F be a field, x an indeterminant over F. Let*

$W = \{ \{\alpha_i\} \mid \{\alpha_i\}$ *is a well ordered sequence of nonnegative real numbers*$\}$.

Let

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^{\alpha_i} | a_i \in F, \{\alpha_i\} \in W \right\}.$$

Then T is a local, commutative ring and

$$J(T) = \left\{ \sum_{i=0}^{\infty} a_i x^{\alpha_i} \in T | \alpha_0 > 0 \right\}.$$

Let

$$R = \frac{T}{xJ(T)}.$$

Then as shown in [**4**], *R is a commutative local hypercyclic ring. Then R is q-hypercyclic* (*Theorem* 4.3). *But R/S, where S is the socle of R, is not hypercyclic. Since R/S is a homomorphic ring of R, R/S is q-hypercyclic by Lemma* 2.6. *Note that R/S is a commutative local ring with zero socle.*

A ring has rank 0 if every prime ideal is a maximal ideal. A valuation ring is called maximal if every family of pairwise solvable congruences of the form $x \equiv x_\alpha(K_\alpha)$ (each $x_\alpha \in R$, each K_α is an ideal of R) has a simultaneous solution [**9**].

We now give a necessary and sufficient condition for a local hypercyclic ring to be q-hypercyclic. In the next section we will show that a commutative hypercyclic ring is always q-hypercyclic.

THEOREM 3.11. *Let R be local and hypercyclic. Then the following conditions are equivalent.*

(i) *J is nil.*

(ii) *R is q-hypercyclic.*

Proof. (i) $\Rightarrow$ (ii). By [**12**], R is duo, valuation, and self-injective. But then R is maximal. Thus R is a qc-ring [**10**], and hence q-hypercyclic. (ii) $\Rightarrow$ (i) follows from Lemma 3.7.

Remark. 3.12. It is not known whether there exists a semi-perfect (or equivalently local) hypercyclic ring with a non-nil radical ([**12**], p. 339).

4. Commutative q-hypercyclic rings. We begin with

LEMMA 4.1. *Let R be commutative and q-hypercyclic. Then R is self-injective.*

Proof. This is obvious.

THEOREM 4.2. *Let R be a commutative ring. Then the following are equivalent.*

(i) *R is q-hypercyclic.*
(ii) *R is a qc-ring.*

Proof. This is similar to the proof of the Theorem 3.9.

THEOREM 4.3. *Let R be a commutative hypercyclic ring. Then R is q-hypercyclic.*

Proof. Let R be hypercyclic. Then by ([**4**], Theorem 2.5), R is a finite direct sum of commutative local hypercyclic rings. So it suffices to show that a commutative local hypercyclic ring is q-hypercyclic. Let R be commutative local and hypercyclic. Then by [**4**], R is valuation and self-injective, and J is nil. Then by ([**9**], Theorem 2.3), R is maximal. Since J is nil, R has rank 0. Then R is rank 0 maximal valuation ring. Thus R is a qc-ring [**10**], proving the theorem.

5. Semi-perfect q-hypercyclic rings.

LEMMA 5.1. *Let A and B be right R-modules. Let B be injective containing a copy of $E(A)$. Then*

$$\text{q.i.h. } (A \oplus B) = E(A) \oplus B.$$

Proof.

$$\text{q.i.h. } (A \oplus B) = \text{End}_R(E(A) \oplus B)(A \oplus B)$$
$$= \begin{pmatrix} \text{Hom}_R(E(A), E(A)) & \text{Hom}_R(B, E(A)) \\ \text{Hom}_R(E(A), B) & \text{Hom}_R(B, B) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$
$$= \begin{pmatrix} \text{Hom}_R(E(A), E(A)A & + \text{Hom}_R(B, E(A))B \\ \text{Hom}_R(E(A), B)A & + \text{Hom}_R(B, B)B \end{pmatrix}$$
$$= \begin{pmatrix} E(A) \\ B \end{pmatrix} = E(A) \oplus B.$$

The above lemma gives another proof of an interesting result of Koehler.

COROLLARY 5.2. ([**11**]). *If the direct sum of any two quasi-injective modules is quasi-injective, then every quasi-injective module is injective.*

Proof. Let M be a quasi-injective right R-module. By Lemma 5.1

$$\text{q.i.h. } (M \oplus E(M)) = E(M) \oplus E(M).$$

Then $M \oplus E(M) = E(M) \oplus E(M)$. Therefore $M \cong E(M)$, proving M is injective.

The proof of the next lemma is exactly similar to Osofsky's ([**12**], Corollary 1.9).

LEMMA 5.3. *Let R be semiperfect and q-hypercyclic and let e be an idempotent in R. Assume length of $eR/eJ = m$. Then any independent family of submodules of a quotient of eR has at most m elements.*

Proof. Let $\{M_i | 1 \leqq i \leqq k\}$ be an independent family of submodules of eR/eI. Then

$$\frac{R}{eI} \supseteq (1 - e)R \oplus \left(\oplus \sum_{i=1}^{k} M_i\right).$$

Therefore $E(R/eI)$ is a direct sum of length $R/J - m + s$ terms, where $s \geqq k$. Thus q.i.h. (R/eI) is a direct sum of length $R/J - m + s$ terms, by Lemma 2.1. Then Lemma 2.3 gives $s \leqq m$. Hence $k \leqq m$.

COROLLARY 5.4. *Let R be semi-perfect and q-hypercyclic, $e^2 = e \in R$, eR/eJ is simple. Then submodules of eR are linearly ordered.*

Proof. This follows from Lemma 5.3.

THEOREM 5.5. *Let R be a semi-perfect q-hypercyclic ring. Then R is a finite direct sum of q-hypercyclic matrix rings over local rings.*

Proof. $R = e_1R \oplus \ldots \oplus e_nR$, where e_i, $1 \leqq i \leqq n$ are primitive idempotents.

We will show that for $i \neq j$, either $e_iR \cong e_jR$, or

$$\mathrm{Hom}_R(e_iR, e_jR) = 0.$$

Suppose for some $i \neq j$, $\mathrm{Hom}_R(e_iR, e_jR) \neq 0$. By renumbering, if necessary, we may assume that $i = 1$, $j = 2$. Let $\alpha: e_1R \rightarrow e_2R$ be a non-zero R-homomorphism. Then $e_1R/\mathrm{Ker}\ \alpha$ embedds in e_2R. Since e_2R is indecomposable,

$$E(e_1R/\mathrm{Ker}\ \alpha) \cong e_2R.$$

Hence $B = e_2R \oplus \ldots \oplus e_nR$ contains a copy of $E(e_1R/\mathrm{Ker}\ \alpha)$. Now

$$R/\mathrm{Ker}\ \alpha \cong (e_1R)/\mathrm{Ker}\ \alpha \times e_2R \times \ldots \times e_nR.$$

Let $A = (e_1R)/\mathrm{Ker}\ \alpha$. Then B is injective and contains a copy of $E(A)$. Hence

$$\mathrm{Hom}_R(B, E(A))B = E(A).$$

Since R is q-hypercyclic, for some right ideal I,

$$R/I \cong \text{q.i.h.}\ (R/\mathrm{Ker}\ \alpha) \cong \text{q.i.h.}\ (A \times B) \cong E(A) \times B.$$

Thus $R/I \cong e_2R \times B$. Then R/I is projective. Hence $R = I \oplus K$ for some right ideal K. Then

$$K \cong R/I \cong e_2R \times e_2R \times \ldots \times e_nR.$$

Thus

$$R = I \oplus K \Rightarrow e_1R \times e_2R \times \ldots \times e_nR$$
$$\cong I \times e_2R \times e_2R \times \ldots \times e_nR.$$

Hence by Azumaya Diagram [**5**],

$$e_1R \cong I \times e_2R.$$

Since e_1R is indecomposable, $I = 0$. Consequently, $R = K$. Then

$$e_1R \times e_2R \times \ldots \times e_nR \cong e_2R \times e_2R \times \ldots \times e_nR.$$

Again by Azumaya Diagram, $e_1R \cong e_2R$. Thus for $i \neq j$, either

$$e_iR \cong e_jR \quad \text{or} \quad \mathrm{Hom}_R(e_iR, e_jR) = 0.$$

Set $[e_kR] = \sum e_iR$, $e_iR \cong e_kR$. Renumbering if necessary, we may write

$$R = [e_1R] \oplus \ldots \oplus [e_tR],\ t \leqq n.$$

Then for all $1 \leqq k \leqq t$, $[e_kR]$ is an ideal. Since for any k, $1 \leqq k \leqq n$, e_kR is indecomposable,

$$\mathrm{End}_R(e_kR) \cong e_kRe_k$$

is a local ring.

Thus $[e_kR] = \oplus \sum_i e_iR$ is the $n_k \times n_k$ matrix ring over the local ring e_kRe_k where n_k is the number of e_iR appearing in $\oplus \sum_i e_iR$. That the matrix ring is q-hypercyclic follows from Lemma 2.7.

We now proceed to study q-hypercyclic rings which are matrix rings over local rings.

THEOREM 5.6. *Let $S = T_n$ be the $n \times n$ q-hypercyclic matrix ring over a local ring T. Then T is q-hypercyclic.*

Proof. Let e be a primitive idempotent of S and let eS/eI be a quotient of eS. Since S is q-hypercyclic,

$$\text{q.i.h.}\left(\frac{eS}{eI}\right) \cong \frac{S}{A}$$

for some right ideal A of S. But since submodules of eS are linearly ordered, S/A is indecomposable. Thus $S/A \cong fS/fK$, ([**2**], Lemma 27.3), for some primitive idempotent f of S, which may be chosen to be e by itself. Thus $S/A \cong eS/eB$ for some right ideal B of S. Since category isomorphism (Lemma 2.2) takes T to eS, every quotient of T has quasi-injective hull a quotient of T, proving that T is a q-hypercyclic ring.

THEOREM 5.7. *Let R be a semi-perfect and q-hypercyclic ring. Then R is a finite direct sum of matrix rings over local qc-rings.*

Proof. Combine Theorems 5.5, 5.6 and 3.9.

We are unable to show if, in general, the $n \times n$ matrix ring S over a local q-hypercyclic ring is again q-hypercyclic. However we will show in the next section that the result is true if S is a perfect ring. In the following theorem we prove that each cyclic S-module is a finite direct sum of indecomposable quasi-injective modules and generalise this to the case when S is any semi-perfect q-hypercyclic ring in Theorem 5.9.

THEOREM 5.8. *Let $S = T_n$ be the $n \times n$ matrix ring over a local q-hypercyclic ring. Then every cyclic S-module is a direct sum of indecomposable quasi-injective S-modules.*

Proof. Let I be a right ideal of S. Let $e \in S$ be a primitive idempotent of S. Since the category isomorphism (Lemma 2.2) takes T to eS every quotient of eS is quasi-injective. Let

$$\frac{S}{I} = \oplus \sum_{i=1}^{k} M_i,$$

where the M_i are indecomposable S-modules. Since S is semi-perfect and M_i indecomposable,

$$M_i = (e_iS)/(e_iA),$$

where e_i is a primitive idempotent of S. Thus S/I is a direct sum of indecomposable quasi-injective S-modules.

THEOREM 5.9. *Let R be a semi-perfect and q-hypercyclic ring. Then every cyclic R module is a direct sum of indecomposable quasi-injective R-modules.*

Proof. Combine Theorems 5.7 and 5.8.

6. Perfect q-hypercyclic rings. A ring R is called right (left) perfect if every right (left) R-module has a projective cover. A theorem of Bass [**3**] states that the following conditions on a ring R are equivalent.

(i) R is right perfect.

(ii) R satisfies minimum conditions on principal left ideals.

(iii) R/J is artinian and every right R-module has a maximal submodule.

LEMMA 6.1. *Let R be a local right perfect and q-hypercyclic ring. Then R is hypercyclic.*

Proof. By Theorem 3.9, R is a qc-ring and hence R is duo. Let I be a nonzero right ideal of R. Then R/I is quasi-injective and indecomposable, and hence $E = E(R/I)$ is indecomposable.

First we show that the submodules of E are linearly ordered. Let aR and bR be submodules of E. Let $A = r_R(a)$. Since ideals of R are linearly ordered either

$$r_R(a) \subseteq r_R(b) \text{ or } r_R(b) \subseteq r_R(a).$$

To be specific let $A = r_R(a) \subseteq r_R(b)$. Let $E' = 0:_E A$. Then $aR, bR \subseteq E'$. By Lemma 2.4, E' is injective as an R/A module. Hence E' is quasi-injective as an R-module. Since $E' \subseteq E$ and E is injective and indecomposable, E' is indecomposable as an R-module and hence E' is indecomposable as an R/A-module. Let $\bar{R} = R/A$. Then $E' \cong E_{\bar{R}}(\bar{R})$, the injective hull of $\bar{R}$ as an $\bar{R}$-module. Hence $E' \cong R/A$. Since submodules of R/A are linearly ordered, submodules of E' are linearly ordered. Thus $aR \subseteq bR$ or $bR \subseteq aR$. Hence submodules of E are linearly ordered. But then E must be local, since R is right perfect. Hence E is cyclic. Therefore R is hypercyclic, proving the theorem.

THEOREM 6.2. *Let $S = T_n$ be the $n \times n$ matrix ring over a local ring T. Let S be right perfect. Then S is q-hypercyclic if and only if T is q-hypercyclic.*

Proof. Let T be q-hypercyclic. Then T is right perfect local and q-hypercyclic. Thus by Theorem 6.1, T is hypercyclic. Further, by Theorem 3.9, T is a qc-ring. Since T is hypercyclic, by ([**12**], Theorem 1.17), S is hypercyclic. Let $e \in S$ be a primitive idempotent. Then as before, the category isomorphism (Lemma 2.2) takes T to eS. Hence quotients of eS are quasi-injective and each quotient has injective hull a quotient of eS. Let I be a right ideal of S. By Theorem 5.8,

$$\frac{S}{I} = \oplus \sum_{i=1}^{k} M_i,$$

where for all $1 \leqq i \leqq k$, M_i are indecomposable and quasi-injective. Then

$$M_i \cong (e_iS)/(e_iA)$$

for some primitive idempotent $e_i \in S$. Hence

$$E(M_i) \cong E[\,(e_iS)/(e_iA)\,] \cong (e_iS)/(e_iB) \text{ for all } 1 \leqq i \leqq k.$$

Since S is right perfect and hypercyclic, submodules of $(e_iS)/(e_iB)$ are linearly ordered. Hence for all $1 \leqq i \leqq k$, submodules of $E(M_i)$ are linearly ordered. Let

$$H = \text{q.i.h. } (S/I).$$

Then

$$H = \oplus \sum_{i=1}^{k} (H \cap E(M_i)).$$

Let $K_i = H \cap E(M_i)$. Then submodules of K_i are linearly ordered for all $1 \leqq i \leqq k$. But then since S is right perfect, for all $1 \leqq i \leqq k$, K_i is cyclic. Therefore,

$$K_i = (f_iS)/(f_iD)$$

where $f_i \in S$ is a primitive idempotent, $1 \leqq i \leqq k$. Thus

$$H \cong \oplus \sum_{i=1}^{k} (f_iS)/(f_iD).$$

Then H is isomorphic to a quotient of S, proving that S is q-hypercyclic.

The converse follows from Theorem 5.6.

THEOREM 6.3. *Let R be right perfect. Then R is q-hypercyclic if and only if R is a finite direct sum of matrix rings over local qc-rings.*

Proof. Combine Theorems 5.5 and 6.2.

THEOREM 6.4. *Let R be right perfect and local. Then the following are equivalent.*

(i) *R is hypercyclic.*

(ii) *R is q-hypercyclic.*

Proof. (i) $\Rightarrow$ (ii). Then R is valuation. Let I be a non-zero right ideal of R. Then $E(R/I) \cong R/A$ for some right ideal A of R. Let

$$X = \text{q.i.h. } (R/I).$$

Since the submodules of R/A and hence those of X are linearly ordered, and R is right perfect, X is local. Thus X is a cyclic module, proving that R is a q-hypercyclic ring.

(ii) $\Rightarrow$ (i) is Theorem 6.1.

THEOREM 6.5. *Let R be right perfect. Then the following are equivalent.*

(i) *R is hypercyclic.*

(ii) *R is q-hypercyclic.*

Proof. (i) $\Rightarrow$ (ii). Let R by hypercyclic. Then by ([**12**], Theorem 1.18),

$$R = \oplus \sum_{i=1}^{t} M_{n_i}(T_i),$$

where $M_{n_i}(T_i)$ is the $n_i \times n_i$ matrix ring over a local hypercyclic ring T_i. Since R is right perfect, T_i is right perfect. Thus T_i is local right perfect and hypercyclic, and hence q-hypercyclic. Then by Theorem 6.2, $M_{n_i}(T_i)$ is q-hypercyclic, proving that R is q-hypercyclic.

(ii) $\Rightarrow$ (i). Proceed as in (i) $\Rightarrow$ (ii) and use Theorem 6.4.

LEMMA 6.6. *Let R be q-hypercyclic. Then R is left perfect if and only if R is right perfect.*

Proof. If R is right (or left) perfect ring then by Theorem 5.7,

$$R = \oplus \sum_{i=1}^{k} M_{n_i}(T_i),$$

where $M_{n_i}(T_i)$ are $n_i \times n_i$ matrix rings over local right (or left) perfect qc-rings T_i. Since T_i's are duo, R is left perfect if and only if R is right perfect.

THEOREM 6.7. *The following conditions on a ring R are equivalent:*

(i) *R is right perfect and hypercyclic.*

(ii) *R is left perfect and hypercyclic.*

(iii) *R is uniserial.*

(iv) *R is right perfect and q-hypercyclic.*

(v) *R is left perfect and q-hypercyclic.*

Proof. (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (i) is a theorem of Caldwell ([**4**], Theorem 1.5).

(i) $\Rightarrow$ (ii). By Theorem 6.4, R is q-hypercyclic. Then by Lemma 6.6, R is left perfect.

(i) $\Leftrightarrow$ (iv) is Theorem 6.5.

(iv) $\Leftrightarrow$ (v) is Lemma 6.6.

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