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ON THE PERMANENT OF A CERTAIN CLASS OF (0, 1)–MATRICES

BY
D. J. HARTFIEL AND J. W. CROSBY

Introduction. In [3, p. 77] Ryser notes the importance of the minimum of the permanent function on the class of (0, 1)-matrices having exactly k ones in each row and column. In [4] a lower bound was found for the minimum of the permanent on the class Λ_n of $n \times n$ (0, 1)-matrices with exactly three 1's in each row and column. The purpose of our work is to improve this result, in particular we show that $\min_{A \in \Lambda_n} (\text{per } A) \geq 3(n-1)$.

The following definitions and notation will be used in the paper.

An $n \times n$ (0, 1)-matrix A is said to be partly decomposable if there exist permutation matrices P and Q such that

$$PAQ = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

where A_1 and A_2 are square. If A is not partly decomposable then A is said to be fully indecomposable. If $a_{1\sigma(1)} = a_{2\sigma(2)} = \cdots = a_{n\sigma(n)} = 1$ where σ is a permutation of $1, 2, \ldots, n$ then A is said to have a positive diagonal. If $\sigma(i) = i$, $i \in \{1, \ldots, n\}$ then A is said to have a positive main diagonal.

$\Lambda_n^{(1)}$ denotes the class of $n \times n$ (0, 1)-matrices for which one row and one column have exactly two 1's and $n-1$ rows and $n-1$ columns have exactly three 1's. $\Lambda_n^{(2)}$ is the class of $n \times n$ (0, 1)-matrices for which two rows and two columns have exactly two 1's and $n-2$ rows and $n-2$ columns have exactly three 1's. Λ_n^* is the class of fully indecomposable matrices in $\Lambda_n^{(2)}$.

Let f denote any function from $\{1, 2, \ldots\}$ into $\{1, 2, \ldots\}$ with the following properties:

(1) $f(n) \leq \min_{A \in \Lambda_n^*} (\text{per } A)$ for $n \geq 2$;

(2) $f(n) \leq f(n-k) f(k-1)$ when $\min(n-k, k-1) \geq 2$;
$f(n) \leq f(n-2) f(2)$ when $n \geq 4$;

(3) f is monotone nondecreasing.

The following lemmas will be used in the paper.

LEMMA 1. *If A is an $n \times n$ (0, 1)-matrix with exactly three 1's in each row and column then each 1 is on a positive diagonal and per $A \geq n$* [4, p. 201].

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LEMMA 2. *If A is an $n \times n$ (0, 1)-matrix which is fully indecomposable then each* 1 *is on a positive diagonal* [4, p. 199].

LEMMA 3. *If $A \in \Lambda_n^{(2)}$ then A has a positive diagonal.*

Proof. Consider the $(n+1)\times(n+1)$ matrix B obtained by bordering A with 0's and 1's so that B has exactly three 1's in each row and column. The result now follows from Lemma 1.

LEMMA 4. *If $A \in \Lambda_n^{(2)}$ and is partly decomposable then there are permutation matrices P and Q such that*

$$PAQ = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

where A_1 and A_2 are square and

(a) *A_1 is fully indecomposable.*
(b) *PAQ has a positive main diagonal.*

Proof. Follows from Lemma 3.

Results and consequences.

LEMMA 5. *If $A \in \Lambda_n^{(2)}$, then* per $A \geq f(n)$.

Proof. By the previous remarks concerning $f(n)$ it suffices to show the inequality for partly decomposable matrices in $\Lambda_n^{(2)}$. We proceed by induction on n, the dimension of the matrices in the class $\Lambda_n^{(2)}$.

The class $\Lambda_1^{(2)}$ is undefined. $\Lambda_2^{(2)}$ contains only the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. All members of the class $\Lambda_3^{(2)}$ are fully indecomposable and so the claim holds for $n \in \{2, 3\}$.

Suppose the lemma holds for all matrices in $\Lambda_r^{(2)}$, $r \in \{2, 3, \ldots, n-1\}$. We show the lemma holds for matrices in $\Lambda_n^{(2)}$.

If $A \in \Lambda_n^{(2)}$ $(n \geq 4)$ is partly decomposable there are permutation matrices P and Q such that

$$PAQ = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix} \quad \text{(where } A_1 \text{ is } k \times k \ (k < n))$$

and fully indecomposable, and PAQ has a positive main diagonal. For convenience, the matrix PAQ will be referred to as A.

By summing the number of 1's in A_1, A_2 and B and comparing this result to the number of 1's in A we see that:

(1) B can have at most two 1's.

(2) B contains exactly one 1 if and only if A_1 has one deficient row and A_2 has two deficient columns *or* A_2 has one deficient column and A_1 has two deficient rows.

(3) B contains exactly two 1's if and only if A_1 has two deficient rows and A_2 has two deficient columns.

We now argue by cases.

Case I. B has no positive entries. This case is easily shown and hence will be neglected.

Case II. B has exactly one positive entry. We divide this case into two subcases.

(a) The 1 in B is on a deficient row. Suppose this $1 = a_{i_0 j_0}$. There is a 1 in A_1 in column j_0, say $a_{i_1 j_0}$ such that row i_1 is not a deficient row. Let $a_{i_1 j_1}$ denote some 1 in the i_1 row, $j_1 \neq j_0$. Now $a_{i_0 j_1} = 0$.

$$A = \begin{matrix} & \begin{matrix} j_0 & & j_1 & \cdots & i_0 \end{matrix} \\ \begin{matrix} i_1 \\ \vdots \\ i_0 \end{matrix} & \begin{pmatrix} 1 & 1 & \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & & \vdots & & \vdots \\ 1 & & \mathbf{0} & \cdots & \mathbf{1} \end{pmatrix} \end{matrix}.$$

Let $\hat{A}$ be the matrix formed from A by replacing $a_{i_1 j_1}$ by 0, $a_{i_1 i_0}$ by 1, $a_{i_0 j_1}$ by 1, and $a_{i_0 i_0}$ by 0.

$$\hat{A} = \begin{matrix} & \begin{matrix} j_0 & & j_1 & \cdots & i_0 \end{matrix} \\ \begin{matrix} i_1 \\ \vdots \\ i_0 \end{matrix} & \begin{pmatrix} 1 & 1 & \mathbf{0} & \cdots & \mathbf{1} \\ \vdots & & \vdots & & \vdots \\ 1 & \cdots & \mathbf{1} & \cdots & \mathbf{0} \end{pmatrix} \end{matrix}.$$

Now let d_1 denote the number of positive diagonals in A_1 through $a_{i_1 j_1}$; $\bar{d}_1$ denote the number of positive diagonals in A_1 not through $a_{i_1 j_1}$; d_2 denote the number of positive diagonals in A_2 through $a_{i_0 i_0}$; $\bar{d}_2$ denote the number of positive diagonals in A_2 not through $a_{i_0 i_0}$; and Q denote the number of positive diagonals in A_1 through $a_{i_1 j_0}$.

Now

$$\text{per } A = (d_1 + \bar{d}_1)(d_2 + \bar{d}_2) = d_1 d_2 + d_1 \bar{d}_2 + \bar{d}_1 d_2 + \bar{d}_1 \bar{d}_2;$$

$$\text{per } \hat{A} = d_1 d_2 + Q d_2 + \bar{d}_1 \bar{d}_2.$$

Since there are three 1's in row i_1 we see by Lemma 2 that $Q d_2 < \bar{d}_1 d_2$. Therefore per $\hat{A} <$ per A. Hence the minimum of the permanent function is not achieved on these matrices.

(b) The 1 in B is not on a deficient row. Suppose this $1 = a_{i_0 j_0}$. Pick $a_{i_0 j_1} = 1$ in A_2 so that $i_0 \neq j_1$. Now $a_{j_1 j_1} = 1$, $a_{j_1 j_0} = 0$, $a_{j_0 j_0} = 1$.

$$A = \begin{matrix} & \begin{matrix} j_0 & \cdots & j_1 \end{matrix} \\ \begin{matrix} j_0 \\ \vdots \\ j_1 \\ i_0 \end{matrix} & \begin{pmatrix} \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{1} \\ 1 & \cdots & 1 \end{pmatrix} \end{matrix}.$$

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Let $\hat{A}$ be the matrix formed from A by replacing $a_{j_1 j_0}$ by 1, $a_{j_0 j_0}$ by 0, $a_{j_0 j_1}$ by 1, $a_{j_1 j_1}$ by 0.

$$\hat{A} = \begin{array}{c} \\ j_0 \\ \vdots \\ j_1 \\ i_0 \end{array}\begin{array}{c} \begin{array}{ccc} j_0 & \cdots & j_1 \end{array} \\ \begin{pmatrix} \mathbf{0} & \cdots & \mathbf{1} \\ \vdots & & \vdots \\ \mathbf{1} & \cdots & \mathbf{0} \\ 1 & \cdots & \end{pmatrix} \end{array}.$$

Let d_1 denote the number of positive diagonals of A_1 through $a_{j_0 j_0}$; $\bar{d}_1$ denote the number of positive diagonals of A_1 not through $a_{j_0 j_0}$; d_2 denote the number of positive diagonals of A_2 through $a_{j_1 j_1}$; $\bar{d}_2$ denote the number of positive diagonals of A_2 not through $a_{j_1 j_1}$.

Now

$$\text{per } A = d_1 d_2 + d_1 \bar{d}_2 + \bar{d}_1 d_2 + \bar{d}_1 \bar{d}_2;$$
$$\text{per } \hat{A} \le d_1 d_2 + \bar{d}_1 \bar{d}_2 + d_1 \bar{d}_2$$

and since $\bar{d}_1 d_2 \neq 0$, per $\hat{A} <$ per A. Hence the minimum of the permanent function is not achieved on these matrices.

Case III. B has two positive entries. First suppose the 1's in B are in different rows and columns. Then $A_1 \in \Lambda_k^{(2)}$ and $A_2 \in \Lambda_{n-k}^{(2)}$. Therefore

$$\text{per } A = (\text{per } A_1)(\text{per } A_2) \ge f(k)f(n-k) \ge f(n).$$

Since A_1 is fully indecomposable, it is clear that the two 1's in B cannot lie in the same column. If the two 1's in B lie in the same row, then A_2 has a row with exactly one 1 in it, say $a_{ij}=1$. (It should be noted that in this situation A_2 must be larger than 2×2 otherwise A_2 would have a column with exactly one 1 in it.) Expanding per A_2 along this row it is clear that per $A_2=$ per $\hat{A}_2$ where $\hat{A}_2$ is the matrix formed by deleting the $(i-k)$th row and the $(j-k)$th column of A_2. Now it is possible that $\hat{A}_2$ is in either $\Lambda_{n-k-1}^{(1)}$ or $\Lambda_{n-k-1}^{(2)}$, but since per $\hat{A}_2 \ge \min_{C\in\Lambda_{n-k-1}^{(2)}}$ (per C) in either case, it follows that

$$\text{per } A = (\text{per } A_1)(\text{per } A_2) \ge \text{per } A_1 \min_{C\in\Lambda_{n-k-1}^{(2)}} (\text{per } C) \ge f(k)f(n-k-1) \ge f(n)$$

by the inductive hypothesis.

By expanding per A along a row we see that the following theorem now holds.

THEOREM. $\min_{A\in\Lambda_n}$ (per $A) \ge 3 \circ f(n-1)$.

We include the following example.

EXAMPLE. $n \le \min_{A\in\Lambda_n^*}$ (per A). See [2, p. 120].

Let

$$f(n) = \begin{cases} n, & n \neq 5 \\ 4, & n = 5 \end{cases}.$$

It can be shown that $f(n)$ satisfies the conditions of the theorem. Hence

$$\min_{A\in\Lambda_n} (\text{per } A) \geq \begin{cases} 3(n-1) & \text{if } n \neq 6 \\ 12 & n = 6 \end{cases}.$$

The exception $n=6$ is unnecessary, since it is fairly easy to check that $\min_{A\in\Lambda_6} (\text{per } A) \geq 15$. For this we see that modulo permutations, the only matrix $B \in \Lambda_5^{(2)}$ with per $B < 5$ is

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

If $A \in \Lambda_6$ and per $A < 15$, then A has to contain B as a submatrix, so

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

But then per $A = 20 \geq 15$.

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Texas A & M University,
College Station, Texas

University of Houston,
Houston, Texas