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# A CARDINAL STRUCTURE THEOREM FOR AN ULTRAPOWER

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ABSTRACT. In this note, we construct a model with a normal measure $U$ over a measurable cardinal $\kappa$ so that the cardinal structures of $V$ and $V^\kappa/U$ are the same $\leq 2^\kappa$. We then show that it is possible to construct a model where this is not true.

Ultrapowers have proven to be a very useful and powerful tool for set theorists in recent years. The entire litany of their applications is both extensive and well known.

Structurally-speaking, ultrapowers tend to be quite "thin" when compared with their underlying universe, i.e., they tend to omit many sets which the underlying universe possesses. For example, if $\kappa$ is a measurable cardinal, $U$ is a normal ultrafilter on $\kappa$, and $V$ is the universe, then $V^\kappa/U$ is "thin" in that it is $\kappa$ closed but not $\kappa^+$ closed. Indeed, for $M$ the transitive collapse of the above ultrapower, $j: V \to M$ the associated elementary embedding, $j''\kappa^+ \notin M$.

It can be of some interest to determine the nature of the cardinal structure of the ultrapower $V^\kappa/U$. In general, it is true that $(\kappa^+)^M = (\kappa^+)^V$ (this follows easily from the $\kappa$ closure of $M$), but if $2^\kappa > \kappa^+$, one might wonder as to what the cardinal structure above $\kappa^+$ of $V^\kappa/U$ looks like. However, as $V \models$ "$2^\kappa < j(\kappa) < (2^\kappa)^+$", the cardinal structure of $M$ can coincide with the cardinal structure of $V$ at most through $2^\kappa$.

The purpose of this note is to show that if $\kappa$ is supercompact, then it is possible to force and obtain a measure $U$ on $\kappa$ so that for any arbitrary cardinal $\delta > \kappa$ with $\text{cof}(\delta) > \kappa$, the cardinal structure of the transitive collapse of $V^\kappa/U$ and $V$ coincide exactly through $\delta$, and $V \models$ "$2^\kappa = \delta$". Specifically, we prove the following

THEOREM. *Suppose that $\bar{V} \models$ "$\kappa$ is a supercompact cardinal and $\delta > \kappa$ is a cardinal with* $\text{cof}(\delta) > \kappa$". *Then there is a generic extension $V \supseteq \bar{V}$ so that:*

1. $V \models$ "$2^\kappa = \delta$".

2. *There is a normal measure $U$ on $\kappa$ so that the cardinals $\leq \delta$ in $V$ are exactly the cardinals $\leq \delta$ in the transitive collapse of $V^\kappa/U$.*

To prove this theorem, let $\bar{V}$ be as above. By a theorem of Laver [1], we assume that $\bar{V} \models$ "$2^\kappa = \kappa^+$" and that $\kappa$ remains supercompact in any generic extension of $\bar{V}$ by a $\kappa$ directed closed partial ordering. In particular, as the standard Cohen partial ordering

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$P$ for making $2^\kappa = \delta$ is $\kappa$ directed closed, $V = \bar{V}^P$ will be so that $V \models$ "$2^\kappa = \delta$ and $\kappa$ is supercompact".

In $V$, let $\hat{U}$ be any normal ultrafilter on $P_\kappa(\delta)$, and let $U = \hat{U}|\kappa$ be the restriction ultrafilter to $\kappa$. For $M$ the transitive collapse of $V^\kappa/U$ and $M'$ the transitive collapse of $V^{P_\kappa(\delta)}/U$, a theorem of Menas [2] shows that there is an elementary embedding $k: M \to M'$ so that the least ordinal moved by $k$ is $((2^{[\kappa]^{<\kappa}})^+)^M = (2^\kappa)^{+M}$ (by the inaccessibility of $\kappa$ in $M$).

As $V \models$ "$2^\kappa = \delta$" and $M$ is $\delta$ closed, $M \models$ "$2^\kappa \geq \delta$". (A further argument will show that $M \models$ "$2^\kappa = \delta$".) Thus, if $\gamma \leq \delta$ is a cardinal in $M$, then $M' \models$ "$\kappa(\gamma)$ is a cardinal", i.e., $M' \models$ "$\gamma$ is a cardinal". Since $M'$ is $\delta$ closed, $V \models$ "$\gamma$ is a cardinal". This proves the theorem.

We remark that an observation of Woodin can be used to show that it is possible to construct a model $\tilde{V}$ with a normal measure $U$ on $\kappa$ so that the cardinal structure of $\tilde{V}$ and $\tilde{V}^\kappa/U$ do not correspond exactly through $2^\kappa$. Specifically, let us assume that we are forcing over the above model $V$ with the partial ordering $Q = \{f: \kappa^+ \to \delta : f$ is a function whose domain has cardinality $\kappa\}$, with the ordering given by $\subseteq$. As $Q$ is $\kappa^+$ directed closed, there are no new $\kappa$ sequences of ordinals in $V^Q = \tilde{V}$, and $U$ remains a normal measure on $\kappa$ in $\tilde{V}$. Further, from the fact that $\tilde{V} \models$ "$2^\kappa \geq \delta$", any ordinal $\alpha \leq \delta$ is represented in $\tilde{V}^\kappa/U$ by a function $f: \kappa \to \kappa$. This implies that if $\tilde{V}^\kappa/U \models$ "$[g] \subseteq [f]$", then $\{\beta : g(\beta) \subseteq f(\beta)\} \in U$, i.e., $g$ is a function with domain $\kappa$ whose values almost everywhere are subsets of ordinals $< \kappa$. By the facts that there are no new $\kappa$ sequences of ordinals in $\tilde{V}$ and any $\kappa$ sequence of subsets of ordinals $< \kappa$ can be coded by a $\kappa$ sequence of ordinals, $g$ can be chosen as an element of $V$. This means that the subsets of any ordinal $\alpha < \delta$ in $\tilde{V}^\kappa/U$ are the same as those in $V^\kappa/U$, i.e., any ordinal $\alpha \leq \delta$ which is a cardinal in $V$ or $V^\kappa/U$ is a cardinal in $\tilde{V}^\kappa/U$. However, it is clearly true that any ordinal in $\tilde{V}$ in the interval $((\kappa^+)^{\tilde{V}}, \delta]$ is no longer a cardinal.

In conclusion, we remark that Woodin can construct ultrapowers which contain a bit more information than the ultrapowers constructed above.

## References


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