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Shadowing and hyperbolicity for linear delay difference equations

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It is known that hyperbolic linear delay difference equations are shadowable on the half-line. In this article, we prove the converse and hence the equivalence between hyperbolicity and the positive shadowing property for the following two classes of linear delay difference equations: (a) for non-autonomous equations with finite delays and uniformly bounded compact coefficient operators in Banach spaces and (b) for Volterra difference equations with infinite delay in finite dimensional spaces.

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1. Introduction

Let $\mathbb{Z}$ and $\mathbb{C}$ denote the set of integers and the set of complex numbers, respectively. For $k \in \mathbb{Z}$, define $\mathbb{Z}_k^+ = \{n \in \mathbb{Z} : n \geq k\}$ and $\mathbb{Z}_k^- = \{n \in \mathbb{Z} : n \leq k\}$. Throughout the article, we shall assume as a standing assumption that $(X, |\cdot|)$ is a Banach space. The symbol $\mathcal{L}(X)$ will denote the space of all bounded linear operators $A\colon X \to X$ equipped with the operator norm, $|A| = \sup_{|x|=1} |Ax|$ for $A \in \mathcal{L}(X)$.

Consider the linear autonomous difference equation

$$x(n+1) = Ax(n), \tag{1.1}$$

where $A \in \mathcal{L}(X)$. Given $\delta \geq 0$, by a δ-*pseudosolution* of Eq. (1.1) on $\mathbb{Z}_0^+$, we mean a function $y\colon \mathbb{Z}_0^+ \to X$ such that

$$\sup_{n\geq 0} |y(n+1) - Ay(n)| \leq \delta.$$

Note that for $\delta = 0$, the pseudosolution becomes a true solution of Eq. (1.1) on $\mathbb{Z}_0^+$. We say that Eq. (1.1) is *shadowable* on $\mathbb{Z}_0^+$ or that it has the *positive shadowing property* if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every δ-pseudosolution y of (1.1) on $\mathbb{Z}_0^+$, there exists a true solution x of (1.1) on $\mathbb{Z}_0^+$ such that

$$\sup_{n\geq 0} |x(n) - y(n)| \leq \epsilon.$$

The shadowing of Eq. (1.1) is closely related to its hyperbolicity. Recall that Eq. (1.1) is *hyperbolic* if $\sigma(A)$ does not intersect the unit circle $|\lambda| = 1$ in $\mathbb{C}$, where $\sigma(A)$ denotes the spectrum of A.

In a recent article [5], Bernardes *et al.* have studied various shadowing properties of Eq. (1.1). Among others, they have shown that if Eq. (1.1) is hyperbolic, then it is shadowable on $\mathbb{Z}_0^+$ (see [5, Theorem 13]). Moreover, if the coefficient $A \in \mathcal{L}(X)$ in Eq. (1.1) is a compact operator, then the converse is also true. As usual, an operator $A\colon X \to X$ is *compact* if, for every bounded set $S \subset X$, the image $A(S)$ has compact closure in X. Thus, we have the following theorem:

Theorem 1.1 5, Theorem 15 *Let $A \in \mathcal{L}(X)$ be a compact operator. Then, the following statements are equivalent.*

(i) Eq. (1.1) is shadowable on $\mathbb{Z}_0^+$;

(ii) Eq. (1.1) is hyperbolic.

Remark 1.2. As noted previously, the implication (ii) $\Rightarrow$ (i) in Theorem 1.1 is true if we assume merely that $A \in \mathcal{L}(X)$. The compactness of $A \in \mathcal{L}(X)$ is important only for the validity of the converse implication (i) $\Rightarrow$ (ii) (see [5, Remark 14]).

Now let us consider the finite dimensional case $X = \mathbb{C}^d$, where d is a positive integer and $\mathbb{C}^d$ denotes the d-dimensional space complex column vectors. Then, the space $\mathcal{L}(\mathbb{C}^d)$ can be identified with $\mathbb{C}^{d\times d}$, the space of $d \times d$ matrices with complex entries. Since linear operators between finite dimensional spaces are compact and the spectrum of a square matrix $A \in \mathbb{C}^{d\times d}$ consists of the roots of its characteristic equation

$$\det(\lambda E - A) = 0, \tag{1.2}$$

where $E \in \mathbb{C}^{d\times d}$ is the unit matrix, in this case, Theorem 1.1 can be reformulated as follows:

Theorem 1.3 *Let $A \in \mathbb{C}^{d\times d}$. Then, the following statements are equivalent:*

(i) Eq. (1.1) is shadowable on $\mathbb{Z}_0^+$;

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(ii) The characteristic Eq. (1.2) has no root on the unit circle $|\lambda| = 1$.

Our aim in this article is to extend Theorems 1.1 and 1.3 to more general classes of linear difference equations with delay.

In §2, we will generalize Theorem 1.1 to the non-autonomous linear difference equation with finite delays

$$x(n+1) = \sum_{j=0}^{r} A_j(n)x(n-j), \tag{1.3}$$

where $r \in \mathbb{Z}_0^+$ is the maximum delay and the coefficients $A_j(n) \in \mathcal{L}(X)$, $0 \le j \le r$, $n \in \mathbb{Z}_0^+$, are compact linear operators that are uniformly bounded, i.e., there exists $K \ge 1$ such that

$$|A_j(n)| \le K, \qquad n \in \mathbb{Z}_0^+, \quad 0 \le j \le r. \tag{1.4}$$

The main result of §2 is formulated in Theorem 2.3, which may be viewed as a discrete analogue of our recent shadowing theorem for delay differential equations in $\mathbb{R}^d$ [2, Theorem 2.2]. It says that, under the above hypotheses, Eq. (1.3) is shadowable on $\mathbb{Z}_0^+$ if and only if it has an exponential dichotomy, which is a non-autonomous variant of hyperbolicity. To the best of our knowledge, this result is new even for ordinary difference equations ($r = 0$). We note that in the particular case when $r = 0$ and X is finite-dimensional, a version of this result was established in [1] (see [1, Corollary 2] and [1, Proposition 4]).

In §3, we will extend Theorem 1.3 to the linear Volterra difference equation with infinite delay

$$x(n+1) = \sum_{j=-\infty}^{n} A(n-j)x(j), \tag{1.5}$$

where $A\colon \mathbb{Z}_0^+ \to \mathbb{C}^{d\times d}$ satisfies

$$\sum_{j=0}^{\infty} |A(j)|e^{\gamma j} < \infty \qquad \text{for some } \gamma > 0. \tag{1.6}$$

The *characteristic equation* of Eq. (1.5) has the form

$$\det \mathit{\Delta}(\lambda) = 0, \qquad |\lambda| > e^{-\gamma}, \tag{1.7}$$

where

$$\mathit{\Delta}(\lambda) = \lambda E - \sum_{j=0}^{\infty} \lambda^{-j} A(j), \qquad |\lambda| > e^{-\gamma}. \tag{1.8}$$

The main result of §3, Theorem 3.2, says that in the natural (infinite dimensional) phase space $\mathcal{B}_\gamma$ defined below Eq. (1.5) is shadowable on $\mathbb{Z}_0^+$ if and only if its characteristic Eq. (1.7) has no root on the unit circle $|\lambda| = 1$.

The fact that non-autonomous linear delay difference equations, including (1.3) and (1.5), are shadowable whenever they are hyperbolic follows from [10, Theorem 1]. Therefore, in both cases (1.3) and (1.5), we need to prove only the converse result. In the case of the non-autonomous equation with finite delays (1.3), the proof follows similar lines as the proof our continuous time result [2, Theorem 2.2] with non-trivial modifications because, instead of $\mathbb{R}^d$, we consider Eq. (1.3) in a general infinite dimensional Banach space. It is based on the eventual compactness of the solution operator, combined with an input–output technique [3] and Schäffer's result about the existence of regular covariant subspaces of linear difference equation in a Banach space [27]. A similar argument for the Volterra equation with infinite delay (1.5) does not apply since its solution operator is not eventually compact. In this case, the proof will be based on the duality between Eq. (1.5) and its formal adjoint equation, which has been established by Matsunaga *et. al.* [18].

2. Shadowing of non-autonomous linear difference equation with finite delays

In this section, we consider the shadowing of the non-autonomous linear difference equation with finite delays (1.3), where $A_j\colon \mathbb{Z}_0^+ \to \mathcal{L}(X)$, $0 \le j \le r$, satisfy condition (1.4). The *phase space* for Eq. (1.3) is $(\mathcal{B}_r, \|\cdot\|)$, where $\mathcal{B}_r$ is the set of all functions $\phi\colon [-r,0]\cap\mathbb{Z} \to X$ and

$$\|\phi\| = \max_{-r\le\theta\le 0} |\phi(\theta)|, \qquad \phi \in \mathcal{B}_r.$$

Eq. (1.3) can be written equivalently in a form of a *functional difference equation*

$$x(n+1) = L_n(x_n), \tag{2.1}$$

where the *solution segment* $x_n \in \mathcal{B}_r$ is defined by

$$x_n(\theta) = x(n+\theta), \qquad \theta \in [-r,0]\cap\mathbb{Z},$$

and $L_n\colon \mathcal{B}_r \to X$ is a bounded linear functional defined by

$$L_n(\phi) = \sum_{j=0}^{r} A_j(n)\phi(-j), \qquad \phi\in\mathcal{B}_r, \quad n\in\mathbb{Z}_0^+, \quad 0\le j\le r.$$

In view of (1.4), we have that

$$\|L_n\| \le M := (r+1)K, \qquad n\in\mathbb{Z}_0^+. \tag{2.2}$$

Given $m \in \mathbb{Z}_0^+$ and $\phi \in \mathcal{B}_r$, there exists a unique function $x\colon \mathbb{Z}_{m-r}^+ \to X$ satisfying Eq. (1.3) and the *initial condition* $x(m+\theta) = \phi(\theta)$ for $\theta \in [-r,0]\cap\mathbb{Z}$. We shall call x the *solution of* Eq. (1.3) *with initial value* $x_m = \phi$. By a *solution of Eq. (1.3) on* $\mathbb{Z}_m^+$, we mean a solution x with initial value $x_m = \phi$ for some $\phi\in\mathcal{B}_r$.

For each $n, m \in \mathbb{Z}_0^+$ with $n \ge m$, the *solution operator* $T(n,m)\colon \mathcal{B}_r \to \mathcal{B}_r$ is defined by $T(n,m)\phi = x_n$ for $\phi\in\mathcal{B}_r$, where x is the unique solution of Eq. (1.3) with initial value $x_m = \phi$. It is easily seen that for all $n,k,m\in\mathbb{Z}_0^+$ with $n\ge k\ge m$,

$$T(m,m) = I, \tag{2.3}$$

$$T(n,m) = T(n,k)T(k,m), \tag{2.4}$$

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$$\|T(n,m)\| \le e^{\omega(n-m)}, \tag{2.5}$$

where I denotes the identity operator on $\mathcal{B}_r$ and $\omega = \log(M(1+r))$.

Now we can introduce the definitions of shadowing and exponential dichotomy for Eq. (1.3) (equivalently, (2.1)).

DEFINITION 2.1. *We say that Eq. (1.3) is shadowable on $\mathbb{Z}_0^+$ if, for each $\epsilon > 0$, there exists $\delta > 0$ such that for every function $y\colon \mathbb{Z}_{-r}^+ \to X$ satisfying*

$$\sup_{n\ge 0} |y(n+1) - L_n(y_n)| \le \delta,$$

there exists a solution x of (1.3) on $\mathbb{Z}_0^+$ such that

$$\sup_{n\ge 0} \|x_n - y_n\| \le \epsilon.$$

DEFINITION 2.2. *We say that Eq. (1.3) admits an exponential dichotomy (on $\mathbb{Z}_0^+$) if there exist a sequence of projections $(P_n)_{n\in\mathbb{Z}_0^+}$ on $\mathcal{B}_r$ and constants $D, \lambda > 0$ with the following properties:*

- *for $n, m \in \mathbb{Z}_0^+$ with $n \ge m$,*

$$P_n T(n,m) = T(n,m) P_m, \tag{2.6}$$

  and $T(n,m)|_{\ker P_m}\colon \ker P_m \to \ker P_n$ is onto and invertible;
- *for $n, m \in \mathbb{Z}_0^+$ with $n \ge m$,*

$$\|T(n,m)P_m\| \le De^{-\lambda(n-m)}; \tag{2.7}$$

- *for $n, m \in \mathbb{Z}_0^+$ with $n \le m$,*

$$\|T(n,m)Q_m\| \le De^{-\lambda(m-n)}, \tag{2.8}$$

  where $Q_m = I - P_m$ and $T(n,m) := \left(T(m,n)|_{\ker P_n}\right)^{-1}$.

The main result of this section is the following shadowing theorem for Eq. (1.3).

THEOREM 2.3 *Suppose that the coefficients $A_j(n) \in \mathcal{L}(X)$, $n \in \mathbb{Z}_0^+$, $0 \le j \le r$, of Eq. (1.3) are compact operators satisfying condition (1.4). Then, the following statements are equivalent.*

(i) Eq. (1.3) is shadowable on $\mathbb{Z}_0^+$;

(ii) Eq. (1.3) admits an exponential dichotomy.

REMARK 2.4. It is well known that the autonomous linear Eq. (1.1) admits an exponential dichotomy if and only if the spectrum $\sigma(A)$ does not intersect the unit circle $|\lambda| = 1$. Therefore, Theorem 2.3 is a generalization of Theorem 1.1 to the non-autonomous delay difference Eq. (1.3). Its conclusion is new even for ordinary difference equations ($r = 0$).

Remark 2.5. The implication (ii) $\Rightarrow$ (i) in Theorem 2.3 is a consequence of [10, Theorem 1] with $f = 0$, $c = 0$, and $\mu = 1$, which does not require the compactness of the coefficients. Thus, this implication is true even without the compactness assumption. However, for the validity of the converse implication (i) $\Rightarrow$ (ii), the compactness of the coefficient operators of Eq. (1.3) is essential (see Remark 1.2).

Remark 2.6. It follows from (2.7) and (2.8) that if Eq. (1.3) admits an exponential dichotomy, then the solution operator $T(m, n)$ of Eq. (1.3) exhibits the (one-sided) domination property in the sense of [22, p. 2]. In [22, Theorem 1.2], the authors have formulated sufficient conditions under which the solution operator associated with a non-autonomous difference equation (without delay)

$$x_{n+1} = A_n x_n, \qquad n \in \mathbb{Z},$$

on an arbitrary Banach space X exhibits the domination property. We stress that no compactness assumptions on the coefficients A_n, $n \in \mathbb{Z}$, are assumed. These sufficient conditions are expressed in terms of the so-called *uniform singular valued gap property* (see [22, (SVG)]). For related results in the case of linear cocycles over topological dynamical systems, we refer to the works of Bochi and Gourmelon [7] and Blumenthal and Morris [6], where the connection between this type of results and the Oseledets multiplicative ergodic theorem is discussed. Our Theorem 2.3 provides a characterization of the more restrictive notion of uniform exponential dichotomy, which is expressed in terms of the shadowing property instead of the singular values.

Proof of Theorem 2.3. As noted in Remark 2.5, we need to prove only the implication (i) $\Rightarrow$ (ii). Suppose that Eq. (1.3) is shadowable on $\mathbb{Z}_0^+$. We will show that it admits an exponential dichotomy. We split the proof into several auxiliary results, which we now briefly describe.

In Claim 1, we show that the shadowing property implies the so-called Perron property, which guarantees that for each bounded function $z \colon \mathbb{Z}_0^+ \to X$, the nonhomogeneous Eq. (2.9) has at least one bounded solution $x \colon \mathbb{Z}_0^+ \to X$.

The next four claims are preparatory results for the proof of the crucial Claim 6, which shows that the subspace $\mathcal{S}(0)$ of those initial functions in $\mathcal{B}_r$, which generate bounded solutions, is closed and complemented in $\mathcal{B}_r$. Claims 2 and 4 are rather simple observations, while Claim 3 is a straightforward consequence of Claim 1. A more involved argument is needed for the proof of Claim 5, which asserts that the solution operator $T(n, m)$ of Eq. (1.3) is a compact operator on $\mathcal{B}_r$ whenever $n \geq m + r + 1$. The proof of Claim 6 follows directly from Claims 2, 3, 4, and 5 by applying an abstract result from [27] formulated in Lemma 2.13.

As a consequence of Claim 6, we are able to construct the unstable subspace $\mathcal{U}$ at time $n = 0$ as a topological complement of $\mathcal{S}(0)$ (see (2.14)), and we can revisit the Perron property established in Claim 1. More precisely, in Claim 7, we show that for each bounded $z \colon \mathbb{Z}_0^+ \to X$, there exists a *unique* bounded solution $x \colon \mathbb{Z}_0^+ \to X$ of Eq. (2.9) with $x_0 \in \mathcal{U}$. Moreover, the supremum norm of x can be controlled by the supremum norm of z (see (2.15)).

In the next step, we construct the unstable subspace $\mathcal{U}(n)$ at each time $n \in \mathbb{Z}^+$. In Claims 8 and 9, we prove that $T(n, m)|_{\mathcal{U}(m)} \colon \mathcal{U}(m) \to \mathcal{U}(n)$ is an isomorphism

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whenever $n \geq m$ and the phase space $\mathcal{B}_r$ splits into stable and unstable subspaces at each moment $n \in \mathbb{Z}^+$. Note that both claims are consequences of Claim 7 we show that for each bound.

The desired exponential estimates along the stable and unstable directions are obtained in Claims 11 and 13, respectively. As a preparation for the proofs of these results, in Claims 10 and 12, we show that the dynamics along the stable and unstable directions is uniformly bounded forward and backward in time, respectively. These results also rely on Claim 7.

Finally, in Claim 14, we prove that there is a uniform bound for the norms of the projections onto the stable subspaces along the unstable ones.

We now proceed with the details. $\square$

Claim 1. Eq. (1.3) has the following *Perron-type property*: for each bounded function $z\colon \mathbb{Z}_0^+ \to X$, there exists a bounded function $x\colon \mathbb{Z}_{-r}^+ \to X$, which satisfies

$$x(n+1) = L_n(x_n) + z(n), \qquad n \in \mathbb{Z}_0^+. \tag{2.9}$$

Proof of claim 1. Let $z\colon \mathbb{Z}_0^+ \to X$ be an arbitrary bounded function. If $z(n) = 0$ for every $n \in \mathbb{Z}_0^+$, then (2.9) is trivially satisfied with $x(n) = 0$ for every $n \in \mathbb{Z}_{-r}^+$. Now suppose that $z(n) \neq 0$ for some $n \in \mathbb{Z}_0^+$ so that $\|z\|_\infty := \sup_{n\in\mathbb{Z}_0^+} |z(n)| > 0$. Choose a constant $\delta > 0$ corresponding to the choice of $\epsilon = 1$ in Definition 2.1. Take an arbitrary solution $y\colon \mathbb{Z}_{-r}^+ \to X$ of the non-homogeneous equation

$$y(n+1) = L_n(y_n) + \frac{\delta}{\|z\|_\infty} z(n), \qquad n \in \mathbb{Z}_0^+.$$

(The unique solution y with initial value $y_0 = 0$ is sufficient for our purposes.) Since

$$\sup_{n\geq 0} |y(n+1) - L_n(y_n)| \leq \delta,$$

according to Definition 2.1, there exists a solution $\tilde{x}$ of Eq. (2.1) on $\mathbb{Z}_0^+$ such that

$$\sup_{n\geq -r} |\tilde{x}(n) - y(n)| = \sup_{n\geq 0} \|\tilde{x}_n - y_n\| \leq 1.$$

Define a function $x\colon \mathbb{Z}_{-r}^+ \to X$ by

$$x(n) := \frac{\|z\|_\infty}{\delta} (\, y(n) - \tilde{x}(n)\,), \qquad n \in \mathbb{Z}_{-r}^+.$$

It can be easily verified that x satisfies (2.9) and

$$\sup_{n\in\mathbb{Z}_{-r}^+} |x(n)| \leq \frac{\|z\|_\infty}{\delta} < \infty.$$

The proof of the claim is complete. $\square$

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For each $m \in \mathbb{Z}_0^+$, define

$$\mathcal{S}(m) = \big\{\phi \in \mathcal{B}_r : \sup_{n\geq m} \|T(n,m)\phi\| < \infty\big\}.$$

Clearly, $\mathcal{S}(m)$ is a subspace of $\mathcal{B}_r$, which will be called the *stable subspace of Eq. (2.1) at* $m \in \mathbb{Z}_0^+$.

Claim 2. For each $n, m \in \mathbb{Z}_0^+$ with $n \geq m$, we have that

$$[\,T(n,m)\,]^{-1}(\mathcal{S}(n)) = \mathcal{S}(m).$$

Proof of claim 2. Let n and m be as in the statement. If $\phi \in \mathcal{S}(m)$, then (see (2.4))

$$\sup_{k\geq n} \|T(k,n)T(n,m)\phi\| = \sup_{k\geq n} \|T(k,m)\phi\| \leq \sup_{k\geq m} \|T(k,m)\phi\| < \infty.$$

This shows that $T(n,m)\phi \in \mathcal{S}(n)$, and hence, $\phi \in [\,T(n,m)\,]^{-1}(\mathcal{S}(n))$.

Now suppose that $\phi \in [\,T(n,m)\,]^{-1}(\mathcal{S}(n))$. Then, $T(n,m)\phi \in \mathcal{S}(n)$, which implies that

$$\sup_{k\geq n} \|T(k,m)\phi\| = \sup_{k\geq n} \|T(k,n)T(n,m)\phi\| < \infty.$$

Hence,

$$\sup_{k\geq m} \|T(k,m)\phi\| \leq \max_{m\leq k\leq n-1} \|T(k,m)\phi\| + \sup_{k\geq n} \|T(k,m)\phi\| < \infty.$$

Thus, $\phi \in \mathcal{S}(m)$. □

Claim 3. For $n, m \in \mathbb{Z}_0^+$ with $n \geq m$, we have the algebraic sum decomposition

$$\mathcal{B}_r = T(n,m)\mathcal{B}_r + \mathcal{S}(n). \tag{2.10}$$

Proof of claim 3. It is sufficient to prove the claim for $m = 0$. Indeed, assuming that the desired conclusion holds for $m = 0$, we now fix an arbitrary $m \geq 1$. Then, for every $n \geq m$ and $\phi \in \mathcal{B}_r$, there exist $\phi_1 \in \mathcal{B}_r$ and $\phi_2 \in \mathcal{S}(n)$ such that $\phi = T(n,0)\phi_1 + \phi_2$. Hence,

$$\phi = T(n,0)\phi_1 + \phi_2 = T(n,m)T(m,0)\phi_1 + \phi_2 \in T(n,m)\mathcal{B}_r + \mathcal{S}(n).$$

Thus, (2.10) holds. Therefore, from now on, we suppose that $m = 0$. Evidently, for every $\phi \in \mathcal{B}_r$,

$$\phi = T(0,0)\phi + 0 \in T(0,0)\mathcal{B}_r + \mathcal{S}(0).$$

Thus, (2.10) holds for $n = m = 0$. Now suppose that $n > 0$ and let $\phi \in \mathcal{B}_r$ be arbitrary. Define $v\colon \mathbb{Z}_{n-r}^+ \to X$ by

$$v(k) = \begin{cases} \phi(k-n) & \text{for } n-r \leq k \leq n, \\ 0 & \text{for } k \geq n+1 \end{cases}$$

so that $v_n = \phi$ and $z\colon \mathbb{Z}_0^+ \to X$ by

$$z(k) = \begin{cases} 0 & \text{for } 0 \le k \le n-1, \\ v(k+1) - L_k(v_k) & \text{for } k \ge n. \end{cases}$$

Clearly, v is bounded on $\mathbb{Z}_{n-r}^+$. From this and (2.2), we find that $\sup_{k\ge 0} |z(k)| < \infty$. By Claim 1, there exists a function $x\colon \mathbb{Z}_{-r}^+ \to X$ such that $\sup_{k\ge -r} |x(k)| < \infty$ and (2.9) holds. Moreover, it follows from the definition of z that

$$v(k+1) = L_k(v_k) + z(k), \qquad k \ge n.$$

From this and (2.9), we conclude that $x - v$ is a solution of Eq. (2.1) on $\mathbb{Z}_n^+$ and thus

$$x_k - v_k = T(k,n)(x_n - v_n) = T(k,n)(x_n - \phi), \qquad k \ge n. \tag{2}$$

Since both x and v are bounded on $\mathbb{Z}_{n-r}^+$, this implies that $x_n - \phi \in \mathcal{S}(n)$. On the other hand, since $z(k) = 0$ for $0 \le k < n$, we have that $x_n = T(n,0)x_0$. This implies that

$$\phi = x_n + (\phi - x_n) = T(n,0)x_0 + (\phi - x_n) \in T(n,0)\mathcal{B}_r + \mathcal{S}(n).$$

Since $\phi \in \mathcal{B}_r$ was arbitrary, we conclude that (2.10) holds for $m = 0$. □

Claim 4. For each $m \in \mathbb{Z}_0^+$, $\mathcal{S}(m)$ is the image of a Banach space under the action of a bounded linear operator.

Proof of claim 4. Fix $m \in \mathbb{Z}_0^+$ and let $\mathcal{X}$ denote the Banach space of all bounded functions $x\colon \mathbb{Z}_{m-r}^+ \to X$ equipped with the supremum norm,

$$\|x\|_{\mathcal{X}} := \sup_{k\ge m-r} |x(k)| < \infty, \qquad x \in \mathcal{X}.$$

Let $\mathcal{X}'$ denote the set of all $x \in \mathcal{X}$, which are solutions of (2.1) on $\mathbb{Z}_0^+$. We will show that $\mathcal{X}'$ is a closed subspace of $\mathcal{X}$. To this end, let $(x^j)_{j\in\mathbb{Z}_0^+}$ be a sequence in $\mathcal{X}'$ such that $x^j \to y$ in $\mathcal{X}$ as $j \to \infty$ for some $y \in \mathcal{X}$. Then, $x^j(k) \to y(k)$ as $j \to \infty$ for every $k \ge m - r$. Moreover, for each $n \ge m$, we have that $x_n^j \to y_n$ in $\mathcal{B}_r$ as $j \to \infty$. From this, by letting $j \to \infty$ in the equation

$$x^j(n+1) = L_n(x_n^j), \qquad n \ge m,$$

and using the continuity of the coefficients L_n, we conclude that

$$y(n+1) = L_n(y_n), \qquad n \ge m.$$

Thus, y is a solution of Eq. (2.1) on $\mathbb{Z}_m^+$ and hence $y \in \mathcal{X}'$. This shows that $\mathcal{X}'$ is a closed subspace of $\mathcal{X}$, and hence, it is a Banach space. Now define $\Phi\colon \mathcal{X}' \to \mathcal{B}_r$ by

$\Phi(x) = x_m$ for $x \in \mathcal{X}'$. Clearly, Φ is a bounded linear operator with $\|\Phi\| \leq 1$ and $\Phi(\mathcal{X}') = \mathcal{S}(m)$. □

Claim 5. For $n, m \in \mathbb{Z}_0^+$ with $n \geq m + r + 1$, the solution operator $T(n, m) \colon \mathcal{B}_r \to \mathcal{B}_r$ is compact.

Proof of claim 5. Suppose that $m \in \mathbb{Z}_0^+$ and $\phi \in \mathcal{B}_r$. Let x denote the unique solution of (2.1) with initial value $x_m = \phi$. From (2.1) and (2.2), we obtain for $n \geq m$,

$$|x(n+1)| \leq \|L_n\| \|x_n\| \leq M \|x_n\|,$$

and hence,

$$\|x_{n+1}\| \leq |x(n+1)| + \|x_n\| \leq (1 + M)\|x_n\|.$$

Since $\|x_m\| = \|\phi\|$, this implies by induction on n that

$$\|x_n\| \leq (1+M)^{n-m} \|\phi\| \qquad \text{for } n \geq m. \tag{2.11}$$

Clearly, $\iota(\phi) = (\phi(-r), \phi(-r+1), \ldots, \phi(0))$ is an isometric isomorphism between the phase space $\mathcal{B}_r$ and the $(r+1)$-fold product space X^{r+1} endowed with the maximum norm, $\|x\| = \max_{1 \leq j \leq r+1} |x_j|$ for $x = (x_1, x_2, \ldots, x_{r+1}) \in X^{r+1}$. With this identification, we have that

$$T(m+r+1, m)\phi = x_{m+r+1} = (x(m+1), x(m+2), \ldots, x(m+r+1)),$$

which, together with Eq. (2.1), implies that

$$T(m+r+1, m)\phi = \big(L_m(x_m), L_{m+1}(x_{m+1}), \ldots, L_{m+r}(x_{m+r})\big). \tag{2.12}$$

Let S be an arbitrary bounded subset of $\mathcal{B}_r$. Then, there exists $\rho > 0$ such that $\|\phi\| \leq \rho$ for all $\phi \in S$. From this and (2.11), we obtain that

$$\|x_{m+j}\| \leq (1+M)^r \rho \qquad \text{whenever } \phi \in S \text{ and } \ 0 \leq j \leq r.$$

Therefore, if $\phi \in S$, then the segments $x_m, x_{m+1}, \ldots, x_{m+r}$ of the corresponding solution x of (2.1) with initial value $x_m = \phi$ belong to the closed ball of radius $(1+M)^r \rho$ around zero in $\mathcal{B}_r$, which will be denoted by D. From this and (2.12), we conclude that

$$T(m+r+1, m)(S) \subset C := \overline{L_m(D)} \times \overline{L_{m+1}(D)} \times \cdots \times \overline{L_{m+r}(D)}. \tag{2.13}$$

Since $L_m, L_{m+1}, \ldots, L_{m+r}$ are compact operators and D is a bounded set, the closures of the image sets $L_m(D), L_{m+1}(D), \ldots, L_{m+r}(D)$ are compact subsets of X. Therefore, C is a product of $r+1$ compact subsets of X, which is a compact subset of the product space X^{r+1}. In view of (2.13), $T(m+r+1, m)(S)$ is a subset of the compact set $C \subset X^{r+1}$, and hence, it is relatively compact in X^{r+1}. Since S was an arbitrary bounded subset of $\mathcal{B}_r$, this proves that $T(m+r+1,m)\colon \mathcal{B}_r \to \mathcal{B}_r$ is a compact operator. Finally, if $n > m+r+1$, then

$$T(n,m) = T(n,m+r+1)T(m+r+1,m)$$

is a product of the bounded linear operator $T(n,m+r+1)$ and the compact operator $T(m+r+1,m)$, and hence, it is compact (see, e.g., [28]). $\square$

CLAIM 6. The stable subspace $\mathcal{S}(0)$ of Eq. (2.1) is closed and has finite codimension in $\mathcal{B}_r$.

Before giving a proof of Claim 6, let us recall the following notions [27]. Let B be a Banach space. A subspace S of B is called *subcomplete in* B if there exist a Banach space Z and a bounded linear operator $\Phi\colon Z \to B$ such that $\Phi(Z) = S$.

Let $\mathcal{A}\colon \mathbb{Z}_0^+ \to \mathcal{L}(B)$ be an operator-valued map. For $n,m \in \mathbb{Z}_0^+$ with $n \geq m$, define the corresponding *transition operator* $U(n,m)\colon B \to B$ by

$$U(n,m) = \mathcal{A}(n-1)\mathcal{A}(n-2)\cdots\mathcal{A}(m) \qquad \text{for } n \geq m$$

and $U(n,n) = I$ for $n \in \mathbb{Z}_0^+$, where I denotes the identity operator on B. A sequence $Y = (Y(n))_{n\in\mathbb{Z}_0^+}$ of subspaces in B is called a *covariant sequence for* $\mathcal{A}$ if

$$[\,\mathcal{A}(n)\,]^{-1}(Y(n+1)) = Y(n) \qquad \text{for all } n \in \mathbb{Z}_0^+.$$

A covariant sequence $Y = (Y(n))_{n\in\mathbb{Z}_0^+}$ for $\mathcal{A}$ is called *algebraically regular* if

$$U(n,0)B + Y(n) = B \qquad \text{for each } n \in \mathbb{Z}_0^+.$$

Finally, a covariant sequence $Y = (Y(n))_{n\in\mathbb{Z}_0^+}$ for $\mathcal{A}$ is called *subcomplete* if the subspace $Y(n)$ is subcomplete in B for all $n \in \mathbb{Z}_0^+$.

The proof of Claim 6 will be based on the following result due to Schäffer [27].

LEMMA 2.13. *([27, Lemma 3.4]) Let B be a Banach space and $\mathcal{A}\colon \mathbb{Z}_0^+ \to \mathcal{L}(B)$. Suppose that $Y = (Y(n))_{n\in\mathbb{Z}^+}$ is a subcomplete algebraically regular covariant sequence for $\mathcal{A}$. If the transition operator $U(n,m)\colon B \to B$ is compact for some $n,m \in \mathbb{Z}_0^+$ with $n > m$, then the subspaces $Y(n)$,$n \in \mathbb{Z}_0^+$, are closed and have constant finite codimension in B.*

Now we can give a proof of Claim 6.

Proof of claim 6. Claims 2, 3, and 4 guarantee that the stable subspaces $Y(n) := \mathcal{S}(n) \subset \mathcal{B}_r$ of Eq. (2.1) form a subcomplete algebraically regular covariant sequence for $\mathcal{A}\colon \mathbb{Z}^+ \to \mathcal{L}(\mathcal{B}_r)$ defined by

$$\mathcal{A}(n) := T(n+r+1,n), \qquad n \in \mathbb{Z}_0^+.$$

According to Claim 5, the associated transition operator $U(n+1,n) = \mathcal{A}(n)$ is compact. By the application of Lemma 2.13, we conclude that $Y(0) = \mathcal{S}(0)$ is closed and has finite codimension in $\mathcal{B}_r$. $\square$

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By Claim 6, the stable subspace $\mathcal{S}(0)$ is closed and has finite codimension in $\mathcal{B}_r$. This implies that $\mathcal{S}(0)$ is complemented in $\mathcal{B}_r$ (see, e.g., [23, Lemma 4.21, p. 106]). More precisely, there exists a subspace $\mathcal{U}$ of $\mathcal{B}_r$ such that $\dim \mathcal{U} = \operatorname{codim} \mathcal{S}(0) < \infty$ and

$$\mathcal{B}_r = \mathcal{S}(0) \oplus \mathcal{U}. \tag{2.14}$$

Claim 7. For each bounded function $z\colon \mathbb{Z}_0^+ \to X$, there exists a unique bounded function $x\colon \mathbb{Z}_{-r}^+ \to X$ with $x_0 \in \mathcal{U}$ which satisfies (2.9). Moreover, there exists a constant $C>0$, independent of z, such that

$$\sup_{n\geq -r} |x(n)| \leq C \sup_{n\geq 0} |z(n)|. \tag{2.15}$$

Proof of claim 7. By Claim 1, there exists a bounded function $\tilde{x}\colon \mathbb{Z}_{-r}^+ \to X$ that satisfies

$$\tilde{x}(n+1) = L_n(\tilde{x}_n) + z(n), \qquad n \in \mathbb{Z}_0^+.$$

On the other hand, (2.14) implies the existence of $\phi_1 \in \mathcal{S}(0)$ and $\phi_2 \in \mathcal{U}$ such that

$$\tilde{x}_0 = \phi_1 + \phi_2.$$

Define $x\colon \mathbb{Z}_{-r}^+ \to X$ by

$$x(n) = \tilde{x}(n) - y(n), \qquad n \geq -r,$$

where y is a solution of Eq. (2.1) with initial value $y_0 = \phi_1$. Since $y_0 = \phi_1 \in \mathcal{S}(0)$, we have that $\sup_{n\geq -r} |y(n)| < \infty$. Then, x satisfies (2.9), $x_0 = \tilde{x}_0 - \phi_1 = \phi_2 \in \mathcal{U}$ and $\sup_{n\geq -r} |x(n)| < \infty$. We claim that x with the desired properties is unique. Indeed, if $\bar{x}$ is an arbitrary function with the desired properties, then $x_0 - \bar{x}_0 \in \mathcal{U} \cap \mathcal{S}(0) = \{0\}$. Thus, $x_0 = \bar{x}_0$ and hence $x = \bar{x}$ identically on $\mathbb{Z}_{-r}^+$.

Finally, we show the existence of a constant $C>0$ such that (2.15) holds. Let $\mathcal{X}_0$ and $\mathcal{X}_{-r}$ denote the Banach space of all bounded X-valued functions defined on $\mathbb{Z}_0^+$ and $\mathbb{Z}_{-r}^+$, respectively, equipped with the supremum norm. For $z \in \mathcal{X}_0$, define $\mathcal{F}(z) = x$, where x is the unique bounded solution of the non-homogeneous Eq. (2.9) with $x_0 \in \mathcal{U}$. (The existence and uniqueness of x is guaranteed by the first part of the proof.) Evidently, $\mathcal{F}(z) = x \in \mathcal{X}_{-r}$ for $z \in \mathcal{X}_0$ and $\mathcal{F}\colon \mathcal{X}_0 \to \mathcal{X}_{-r}$ is a linear operator. We will now observe that $\mathcal{F}$ is a closed operator. Indeed, let $(z^k)_{k\in\mathbb{Z}^+}$ be a sequence in $\mathcal{X}_0$ such that $z^k \to z$ for some $z \in \mathcal{X}_0$ and $x^k := \mathcal{F}(z^k) \to x$ for some $x \in \mathcal{X}_{-r}$. Then, letting $k \to +\infty$ in

$$x^k(n+1) = L_n(x_n^k) + z^k(n)$$

for each fixed n, we get that

$$x(n+1) = L_n(x_n) + z(n), \qquad n \in \mathbb{Z}_0^+.$$

That is, x satisfies (2.9). Now, since $x_0^k \in \mathcal{U}$ for $k \in \mathbb{Z}_0^+$ and $\mathcal{U}$ is a finite-dimensional and hence a closed subset of $\mathcal{B}_r$, we have that $x_0 = \lim_{k\to\infty} x_0^k \in \mathcal{U}$. Therefore,

$x \in \mathcal{X}_{-r}$ is a bounded function satisfying (2.9) with $x_0 \in \mathcal{U}$. Hence $\mathcal{F}(z) = x$, which shows that $\mathcal{F}\colon \mathcal{X}_0 \to \mathcal{X}_{-r}$ is a closed operator. According to the Closed Graph Theorem (see, e.g., [28, Theorem 4.2-I, p. 181]), $\mathcal{F}$ is bounded, which implies that (2.15) holds with $C = \|\mathcal{F}\|$, the operator norm of $\mathcal{F}$. □

For $n \in \mathbb{Z}^+$, define

$$\mathcal{U}(n) = T(n,0)\mathcal{U}$$

so that $\mathcal{U}(0) = \mathcal{U}$. It is easily seen that

$$T(n,m)\mathcal{S}(m) \subset \mathcal{S}(n) \quad \text{and} \quad T(n,m)\mathcal{U}(m) = \mathcal{U}(n) \tag{2.16}$$

whenever $n, m \in \mathbb{Z}_0^+$ with $n \geq m$.

Claim 8. For $n, m \in \mathbb{Z}_0^+$ with $n \geq m$, $T(n,m)|_{\mathcal{U}(m)} : \mathcal{U}(m) \to \mathcal{U}(n)$ is invertible.

Proof of claim 8. In view of (2.16), we only need to show that the operator above is injective. Let $n, m \in \mathbb{Z}_0^+$ with $n \geq m$ and $\phi \in \mathcal{U}(m)$ be such that $T(n,m)\phi = 0$. Since $\phi \in \mathcal{U}(m)$, there exists $\bar{\phi} \in \mathcal{U}(0) = \mathcal{U}$ such that $\phi = T(m,0)\bar{\phi}$. Let $x\colon \mathbb{Z}_{-r}^+ \to X$ be the solution of (2.1) with initial value $x_0 = \bar{\phi}$. Since $x(k) = 0$ for all sufficiently large $k \in \mathbb{Z}^+$, we have that $\sup_{k \geq -r} |x(k)| < \infty$. It follows from the uniqueness in Claim 7, applied for $z \equiv 0$, that $x \equiv 0$. This implies that $\bar{\phi} = \phi = 0$. □

Claim 9. For each $n \in \mathbb{Z}_0^+$, $\mathcal{B}_r$ can be decomposed into the direct sum

$$\mathcal{B}_r = \mathcal{S}(n) \oplus \mathcal{U}(n). \tag{2.17}$$

Proof of claim 9. Since $\mathcal{U}(0) = \mathcal{U}$, for $n = 0$, the decomposition (2.17) follows immediately from (2.14). Now suppose that $n \geq 1$ and let $\phi \in \mathcal{B}_r$ be arbitrary. Let $v\colon \mathbb{Z}_{n-r}^+ \to X$ and $z\colon \mathbb{Z}_0^+ \to X$ be as in the proof of Claim 3. Since $\sup_{k \geq 0} |z(k)| < \infty$, by Claim 7, there exists a unique function $x\colon \mathbb{Z}_{-r}^+ \to X$ such that $x_0 \in \mathcal{U}$, $\sup_{k \geq -r} |x(k)| < \infty$, and (2.9) holds. By the same reasoning as in the proof of Claim 3, we have that $x_n - \phi \in \mathcal{S}(n)$. Moreover, $x_n = T(n,0)x_0 \in \mathcal{U}(n)$. Consequently,

$$\phi = (\phi - x_n) + x_n \in \mathcal{S}(n) + \mathcal{U}(n).$$

Suppose now that $\phi \in \mathcal{S}(n) \cap \mathcal{U}(n)$. Then, there exists $\bar{\phi} \in \mathcal{U}$ such that $\phi = T(n,0)\bar{\phi}$. Consider the unique solution $x\colon \mathbb{Z}_{-r}^+ \to X$ of Eq. (2.1) with $x_0 = \bar{\phi}$. Then, x satisfies (2.9) with $z \equiv 0$, $x_0 = \bar{\phi} \in \mathcal{U}$ and $\sup_{k \geq -r} |x(k)| < \infty$. By the uniqueness in Claim 7, we conclude that $x \equiv 0$. Therefore, $\bar{\phi} = 0$, which implies that $\phi = 0$. The proof of the Claim is completed. □

Claim 10. There exists $Q > 0$ such that

$$\|T(n,m)\phi\| \leq Q\|\phi\|,$$

for every $n, m \in \mathbb{Z}_0^+$ with $n \geq m$ and $\phi \in \mathcal{S}(m)$.

Proof of claim 10. Fix $m \in \mathbb{Z}_0^+$ and $\phi \in \mathcal{S}(m)$. Let $u \colon \mathbb{Z}_{m-r}^+ \to X$ be the solution of Eq. (2.1) with initial value $u_m = \phi$. Define $x \colon \mathbb{Z}_{-r}^+ \to X$ and $z \colon \mathbb{Z}_0^+ \to X$ by

$$x(k) = \begin{cases} u(k) & \text{for } k \geq m+1; \\ 0 & \text{for } -r \leq k \leq m \end{cases}$$

and

$$z(k) = x(k+1) - L_k(x_k), \qquad k \geq 0,$$

respectively. Evidently, (2.9) is satisfied, and since $\phi \in \mathcal{S}(m)$, we have that $\sup_{k \geq -r} |x(k)| < \infty$. Moreover, $x_0 = 0 \in \mathcal{U}$. Furthermore, $z(k) = 0$ for $0 \leq k \leq m-1$ and $k \geq m+r+1$. Thus, using (2.2), (2.5), and the fact that $\|x_k\| \leq \|u_k\|$, we find that

$$\begin{aligned}
\sup_{k \geq 0} |z(k)| &= \sup_{m \leq k \leq m+r} |z(k)| \\
&\leq \sup_{m \leq k \leq m+r} |x(k+1)| + \sup_{m \leq k \leq m+r} |L_k x_k| \\
&\leq \sup_{m \leq k \leq m+r} |u(k+1)| + M \sup_{m \leq k \leq m+r} \|x_k\| \\
&\leq \sup_{m \leq k \leq m+r} |L_k(u_k)| + M \sup_{m \leq k \leq m+r} \|u_k\| \\
&\leq 2M \sup_{m \leq k \leq m+r} \|u_k\| \\
&= 2M \sup_{m \leq k \leq m+r} \|T(k,m)\phi\| \\
&\leq 2M e^{\omega r} \|\phi\|.
\end{aligned}$$

From the last inequality and conclusion (2.15) of Claim 7, we conclude that

$$\sup_{k \geq m+1} |u(k)| \leq \sup_{k \geq -r} |x(k)| \leq C \sup_{k \geq 0} |z(k)| \leq 2CMe^{\omega r} \|\phi\|.$$

Hence,

$$\|T(n,m)\phi\| = \|u_n\| \leq 2CMe^{\omega r} \|\phi\|, \qquad n \geq m+r+1.$$

Since (2.2) implies that

$$\|T(n,m)\phi\| \leq e^{\omega r} \|\phi\|, \qquad m \leq n \leq m+r,$$

the conclusion of the claim holds with

$$Q := \max\{\, e^{\omega r}, 2CMe^{\omega r} \,\} > 0.$$

$\square$

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Claim 11. There exist $D, \lambda > 0$ such that

$$\|T(n,m)\phi\| \leq De^{-\lambda(n-m)}\|\phi\|$$

for every $n, m \in \mathbb{Z}_0^+$ with $n \geq m$ and $\phi \in \mathcal{S}(m)$.

Proof of claim 11. We claim that if

$$N > eCMQ^2(r+1) + r + 1 \tag{2.18}$$

with Q as in Claim 10, then for every $m \in \mathbb{Z}_0^+$ and $\phi \in \mathcal{S}(m)$,

$$\|T(n,m)\phi\| \leq \frac{1}{e}\|\phi\| \qquad \text{for } n \geq m+N. \tag{2.19}$$

Suppose, for the sake of contradiction, that (2.18) holds and there exist $m \in \mathbb{Z}_0^+$ and $\phi \in \mathcal{S}(m)$ such that

$$\|T(n,m)\phi\| > \frac{1}{e}\|\phi\| \qquad \text{for some } n \geq m+N. \tag{2.20}$$

Fix $n \geq m+N$ such that the first inequality in (2.20) holds and let u denote the solution of the homogeneous equation Eq. (2.1) with initial value $u_m = \phi$ so that $u_n = T(n,m)\phi$. Therefore, the first inequality in (2.20) can be written as

$$\|u_n\| > \frac{1}{e}\|\phi\|. \tag{2.21}$$

In view of (2.16), $\phi \in \mathcal{S}(m)$ implies that $u_j = T(j,m)\phi \in \mathcal{S}(j)$ for $j \geq m$. Therefore, by Claim 10, we have that

$$\|u_n\| = \|T(n,j)u_j\| \leq Q\|u_j\| \qquad \text{whenever } m \leq j \leq n.$$

From the last inequality and (2.21), we find that $\|u_j\| > 0$ whenever $m \leq j \leq n$. This, together with the fact that $n \geq m+N > m+r+1$ and hence, $n-r-1 > m$, implies that we can define a function $x\colon \mathbb{Z}_{-r}^+ \to X$ by

$$x(k) = \begin{cases} \chi(k)u(k) & \text{for } k \geq m, \\ 0 & \text{for } -r \leq k \leq m-1, \end{cases}$$

where

$$\chi(k) = \begin{cases} 0 & \text{for } -r \leq k \leq m, \\ \displaystyle\sum_{j=m}^{k-1} \|u_j\|^{-1} & \text{for } m+1 \leq k \leq n-r-1, \\ \displaystyle\sum_{j=m}^{n-r-1} \|u_j\|^{-1} & \text{for } k \geq n-r. \end{cases}$$

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Evidently, x satisfies the non-homogeneous equation

$$x(k+1) = L_k(x_k) + z(k), \qquad k \in \mathbb{Z}_0^+, \tag{2.22}$$

where $z\colon \mathbb{Z}_0^+ \to X$ is defined by

$$z(k) = x(k+1) - L_k(x_k), \qquad k \in \mathbb{Z}_0^+.$$

Since $u_m = \phi \in \mathcal{S}(m)$ implies that u is bounded on $\mathbb{Z}_{m-r}^+$ and $0 \le \chi(k) \le \chi(n-r)$ for $k \ge -r$, it follows that x is bounded on $\mathbb{Z}_{-r}^+$. From this and (2.2), we obtain that z is also bounded on $\mathbb{Z}_0^+$. Since $x_0 = 0 \in \mathcal{U}$, by Claim 7, we have that

$$\sup_{k \ge -r} |x(k)| \le C \sup_{k \ge 0} |z(k)|. \tag{2.23}$$

Our objective now is to estimate the norm of $z(k)$. Since $x(k) = 0$ for $-r \le k \le m$, we have that $z(k) = 0$ for $0 \le k \le m-1$. The function χ is constant on $\mathbb{Z}_{n-r}^+$, therefore x is a constant multiple of the solution u of the homogeneous Eq. (2.1) on $\mathbb{Z}_{n-r}^+$. Thus, x also satisfies the homogeneous Eq. (2.1) for $k \ge n$, and hence, $z(k) = 0$ for $k \ge n$. It remains to consider the case when $m \le k \le n-1$. Let such a k be fixed. By definition, we have

$$\begin{aligned} z(k) &= x(k+1) - L_k(x_k) \\ &= \chi(k+1)u(k+1) - L_k(\chi_k u_k) \\ &= \chi(k+1)L_k(u_k) - L_k(\chi_k u_k) \\ &= L_k(\chi(k+1)u_k - \chi_k u_k). \end{aligned}$$

Therefore, using (2.2), it follows that

$$|z(k)| \le M \|\chi(k+1)u_k - \chi_k u_k\|.$$

Let $\theta \in [-r, 0] \cap \mathbb{Z}$. Then,

$$|\left(\chi(k+1)u_k - \chi_k u_k\right)(\theta)| = |\left(\chi(k+1) - \chi(k+\theta)\right) u(k+\theta)|,$$

and hence,

$$|\left(\chi(k+1)u_k - \chi_k u_k\right)(\theta)| \le \sum_{j=m}^{k} \|u_j\|^{-1} |u(k+\theta)| \qquad \text{whenever } k+\theta \le m$$

and

$$|\left(\chi(k+1)u_k - \chi_k u_k\right)(\theta)| \le \sum_{j=k+\theta}^{k} \|u_j\|^{-1} |u(k+\theta)| \qquad \text{whenever } k+\theta > m.$$

In view of (2.16), $u_m = \phi \in \mathcal{S}(m)$ implies that $u_j = T(j,m)\phi \in \mathcal{S}(j)$ for $j \ge m$. Therefore, by Claim 10, for $m \le j \le k$, we have

$$|u(k+\theta)| \le \|u_k\| = \|T(k,j)u_j\| \le Q\|u_j\|$$

so that $\|u_j\|^{-1}|u(k+\theta)| \leq Q$. From this, we conclude that if $k+\theta \leq m$ so that $k-m \leq -\theta \leq r$, then

$$\sum_{j=m}^{k} \|u_j\|^{-1}|u(k+\theta)| \leq Q(k-m+1) \leq Q(r+1),$$

while in case $k+\theta > m$, we have

$$\sum_{j=k+\theta}^{k} \|u_j\|^{-1}|u(k+\theta)| \leq Q(-\theta+1) \leq Q(r+1).$$

Therefore, in both cases $k+\theta \leq m$ and $k+\theta > m$, we have that

$$|\left(\chi(k+1)u_k - \chi_k u_k\right)(\theta)| \leq Q(r+1).$$

Since $\theta \in [-r, 0] \cap \mathbb{Z}$ was arbitrary, this implies that

$$\|\chi(k+1)u_k - \chi_k u_k\| \leq Q(r+1),$$

which, combined with (2.2), yields

$$|z(k)| \leq MQ(r+1).$$

We have shown that the last inequality is valid whenever $m \leq k \leq n-1$ and $z(k) = 0$ otherwise. Hence,

$$\sup_{k \geq 0} |z(k)| \leq MQ(r+1).$$

This, together with (2.23), implies that

$$\sup_{k \geq -r} |x(k)| \leq CMQ(r+1). \tag{2.24}$$

Since $n-r \geq n-N \geq m$ and χ is non-decreasing on $\mathbb{Z}^+_{-r}$, we have for $\theta \in [-r, 0] \cap \mathbb{Z}$,

$$|x(n+\theta)| = \chi(n+\theta)|u(n+\theta)| \geq \chi(n-r)|u(n+\theta)| = |u(n+\theta)| \sum_{j=m}^{n-r-1} \|u_j\|^{-1}.$$

Hence,

$$\|x_n\| \geq \|u_n\| \sum_{j=m}^{n-r-1} \|u_j\|^{-1}.$$

According to Claim 10, $u_m = \phi \in \mathcal{S}(m)$ implies that $\|u_j\| = \|T(j,m)\phi\| \leq Q\|\phi\|$ for $j \geq m$. This, together with the previous inequality, yields

$$\|x_n\| \geq \|u_n\| \sum_{j=m}^{n-r-1} \|u_j\|^{-1} \geq \|u_n\| \frac{n-m-r}{Q\|\phi\|}.$$

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The last inequality, combined with (2.21) and (2.24), implies that

$$CMQ(r+1) \geq \|x_n\| \geq \|u_n\| \frac{n-m-r}{Q\|\varphi\|} > \frac{n-m-r}{eQ}.$$

However, this contradicts the fact that $n \geq m+N$ with N satisfying (2.18). Thus, (2.19) holds whenever $m \in \mathbb{Z}_0^+$ and $\phi \in \mathcal{S}(m)$.

Using (2.19), we can easily complete the proof. Choose an integer N satisfying (2.18). Let $n \geq m$. Then, $n-m = kN+h$ for some $k \in \mathbb{Z}_0^+$ and $0 \leq h \leq N-1$. From (2.19) and Claim 10, we obtain for $\phi \in \mathcal{S}(m)$,

$$\begin{aligned}
\|T(n,m)\phi\| &= \|T(m+kN+h,m)\phi\| \\
&= \|T(m+kN+h,m+kN)T(m+kN,m)\phi\| \\
&\leq Q\|T(m+kN,m)\phi\| \\
&\leq Qe^{-k}\|\phi\| \\
&\leq eQe^{-\frac{n-m}{N}}\|\phi\|.
\end{aligned}$$

Hence, the conclusion of the claim holds with $D = eQ$ and $\lambda = 1/N$. □

Claim 12. There exists $Q' > 0$ such that

$$\|T(n,m)\phi\| \leq Q'\|\phi\|$$

for every $n, m \in \mathbb{Z}_0^+$ with $n \leq m$ and $\phi \in \mathcal{U}(m)$.

Proof of claim 12. Given $m \in \mathbb{Z}_0^+$ and $\phi \in \mathcal{U}(m)$, there exists $\bar{\phi} \in \mathcal{U}$ such that $\phi = T(m,0)\bar{\phi}$. Let $u\colon \mathbb{Z}_{-r}^+ \to X$ be the solution of Eq. (2.1) such that $u_0 = \bar{\phi}$. Consider $x\colon \mathbb{Z}_{-r}^+ \to X$ and $z\colon \mathbb{Z}_0^+ \to X$ given by

$$x(k) = \begin{cases} u(k) & \text{for } -r \leq k \leq m \\ 0 & \text{for } k \geq m+1, \end{cases}$$

and

$$z(k) = x(k+1) - L_k(x_k), \qquad k \in \mathbb{Z}_0^+,$$

so that (2.9) is satisfied. Moreover, since $x(k) = 0$ for $k \geq m+1$, it follows that $\sup_{k \geq -r} |x(k)| < \infty$. Furthermore, $x_0 = u_0 \in \mathcal{U}$ and $z(k) = 0$ for $0 \leq k \leq m-1$ and $k \geq m+r+1$. Proceeding as in the proof of Claim 10, it can be shown that

$$\sup_{k \geq 0} |z(k)| \leq 2Me^{\omega r}\|\phi\|.$$

From conclusion (2.15) of Claim 7, we conclude that

$$\sup_{-r \leq k \leq m} |u(k)| \leq \sup_{k \geq -r} |x(k)| \leq C \sup_{k \geq 0} |z(k)| \leq 2CMe^{\omega r}\|\phi\|.$$

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This implies that the conclusion of the claim holds with

$$Q' := 2CMe^{\omega r} > 0.$$

□

CLAIM 13. There exist $D', \lambda' > 0$ such that

$$\|T(n,m)\phi\| \leq D' e^{-\lambda'(m-n)} \|\phi\|$$

for every $n, m \in \mathbb{Z}_0^+$ with $n \leq m$ and $\phi \in \mathcal{U}(m)$.

Proof of Claim 13. We claim that if

$$N' > eCM(Q')^2(r+1) \tag{2.25}$$

with Q' as in Claim 12, then for every $m \geq N'$ and $\phi \in \mathcal{U}(m)$,

$$\|T(n,m)\phi\| \leq \frac{1}{e}\|\phi\| \qquad \text{whenever } 0 \leq n \leq m - N'. \tag{2.26}$$

Suppose, for the sake of contradiction, that (2.25) holds and there exist $m \geq N'$ and $\phi \in \mathcal{U}(m)$ such that

$$\|T(n,m)\phi\| > \frac{1}{e}\|\phi\| \qquad \text{for some } n \text{ with } 0 \leq n \leq m - N'. \tag{2.27}$$

Fix n with $0 \leq n \leq m - N'$ such that the first inequality in (2.27) holds. Evidently, (2.27) implies that $\phi \in \mathcal{U}(m)$ is non-zero. Therefore, there exists a non-zero $\bar{\phi} \in \mathcal{U}$ such that $\phi = T(m,0)\bar{\phi}$. Let u denote the unique solution of Eq. (2.1) with $u_0 = \bar{\phi}$ so that $u_m = T(m,0)u_0 = T(m,0)\bar{\phi} = \phi$. Since $u_n = T(n,m)u_m = T(n,m)\phi$, the first inequality in (2.27) can be written as

$$\|u_n\| > \frac{1}{e}\|\phi\|. \tag{2.28}$$

Choose a sequence $\psi \colon \mathbb{Z}_0^+ \to [0,1]$ such that

$$\psi(j) = 1 \quad \text{for } 0 \leq j \leq m \qquad \text{and} \qquad \psi(j) = 0 \quad \text{for } j \geq m+1. \tag{2.29}$$

By Claim 8, $0 \neq \bar{\phi} \in \mathcal{U}$ implies that $u_j = T(j,0)\bar{\phi} \neq 0$ for $j \geq 0$. Therefore, we can define a function $x \colon \mathbb{Z}_{-r}^+ \to X$ by

$$x(k) = \chi(k)u(k) \qquad \text{for } k \geq -r,$$

where

$$\chi(k) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \psi(j)\|u_j\|^{-1} & \text{for } -r \leq k \leq 0, \\ \displaystyle\sum_{j=k}^{\infty} \psi(j)\|u_j\|^{-1} & \text{for } k \geq 1. \end{cases}$$

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Note that

$$x_0 = cu_0 = c\bar{\phi} \in \mathcal{U}, \qquad \text{where } c = \chi(0) = \sum_{j=0}^{m} \psi(j)\|u_j\|^{-1}.$$

Since $\psi(k) = 0$ and hence $x(k) = 0$ for $k \geq m+1$, we have that $\sup_{k\geq -r} |x(k)| < \infty$. Moreover, x satisfies (2.9) with $z\colon \mathbb{Z}_0^+ \to X$ defined by

$$z(k) = x(k+1) - L_k(x_k), \qquad k \geq 0. \tag{2.30}$$

Since $x_k = 0$ for $k \geq m + r + 1$, it follows that $z(k) = 0$ for $k \geq m + r + 1$. In particular, z is bounded on $\mathbb{Z}_0^+$. From (2.30) and Claim 7, we conclude that

$$\sup_{k\geq -r} |x(k)| \leq C \sup_{k\geq 0} |z(k)|. \tag{2.31}$$

Let $k \geq 0$ be arbitrary. By the same calculations as in the proof of Claim 10, we have

$$z(k) = L_k(\chi(k+1)u_k - \chi_k u_k).$$

From this and (2.2), we find that

$$|z(k)| \leq M\|\chi(k+1)u_k - \chi_k u_k\|. \tag{2.32}$$

Let $\theta \in [-r, 0] \cap \mathbb{Z}$. Then,

$$|\,(\chi(k+1)u_k - \chi_k u_k)\,(\theta)| = |\,(\chi(k+1) - \chi(k+\theta))\,u(k+\theta)|.$$

From this and the definition of χ, taking into account that $0 \leq \psi \leq 1$ on $\mathbb{Z}_0^+$, we conclude that

$$|\,(\chi(k+1)u_k - \chi_k u_k)\,(\theta)| \leq \sum_{j=0}^{k} \|u_j\|^{-1}|u(k+\theta)| \qquad \text{whenever } k+\theta \leq 0. \tag{2.33}$$

and

$$|\,(\chi(k+1)u_k - \chi_k u_k)\,(\theta)| \leq \sum_{j=k+\theta}^{k} \|u_j\|^{-1}|u(k+\theta)| \qquad \text{whenever } k+\theta > 0. \tag{2.34}$$

By Claim 8, $u_0 = \bar{\phi} \in \mathcal{U}$ implies that $u_j = T(j,0)u_0 \in \mathcal{U}(j)$ for $j \geq 0$. Therefore, according to Claim 12, if $k + \theta < 0$, then

$$|u(k+\theta)| \leq \|u_0\| = \|T(0,j)u_j\| \leq Q'\|u_j\|,$$

and hence, $\|u_j\|^{-1}|u(k+\theta)| \leq Q'$ for $j \geq 0$. If $k + \theta \leq 0$ so that $k \leq -\theta \leq r$, then from the last inequality and (2.33), we obtain that

$$|\,(\chi(k+1)u_k - \chi_k u_k)\,(\theta)| \leq Q'(r+1) \qquad \text{whenever } k+\theta \leq 0. \tag{2.35}$$

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It follows by similar arguments that if $k+\theta>0$, then

$$|u(k+\theta)| \le \|u_{k+\theta}\| = \|T(k+\theta,j)u_j\| \le Q'\|u_j\|,$$

and hence, $\|u_j\|^{-1}|u(k+\theta)| \le Q'$ for $j \ge k+\theta$. This, together with (2.34), yields

$$|\,(\chi(k+1)u_k - \chi_k u_k)\,(\theta)| \le Q'(r+1) \qquad \text{whenever } k+\theta>0. \tag{2.36}$$

Since $\theta \in [-r,0]\cap\mathbb{Z}$ was arbitrary, (2.35) and (2.36) imply that

$$\|\chi(k+1)u_k - \chi_k u_k\| \le Q'(r+1).$$

Since $k \ge 0$ was arbitrary, the last inequality, combined with (2.32), implies that

$$\sup_{k\ge 0}|z(k)| \le MQ'(r+1).$$

This, together with (2.31), yields

$$\sup_{k\ge -r}|x(k)| \le CMQ'(r+1). \tag{2.37}$$

Since χ is non-increasing on $\mathbb{Z}^+_{-r}$, we have for $\theta \in [-r,0]\cap\mathbb{Z}$,

$$|x(n+\theta)| = \chi(n+\theta)|u(n+\theta)| \ge \chi(n)|u(n+\theta)| = |u(n+\theta)|\sum_{j=n}^{m}\|u_j\|^{-1},$$

the last equality being a consequence of (2.29). Hence,

$$\|x_n\| \ge \|u_n\|\sum_{j=n}^{m}\|u_j\|^{-1}.$$

Since $u_m = \phi \in \mathcal{U}(m)$, by Claim 12, we have that $\|u_j\| = \|T(j,m)\phi\| \le Q'\|\phi\|$ for $0 \le j \le m$. This, together with the previous inequality, gives

$$\|x_n\| \ge \|u_n\|\frac{m-n+1}{Q'\|\phi\|}.$$

This, combined with (2.28) and (2.37), yields

$$CMQ'(r+1) \ge \|x_n\| \ge \|u_n\|\frac{m-n+1}{Q'\|\phi\|} > \frac{m-n+1}{eQ'}.$$

The last inequality contradicts the fact that $n-m \ge N'$ with $N^{'}$ satisfying (2.25). Thus, (2.26) holds whenever $m \ge N'$ and $\phi \in \mathcal{U}(m)$.

Now, using (2.26), we can easily complete the proof. Let $0 \le n \le m$ and $\phi \in \mathcal{U}(m)$. Choose an integer $N^{'}$ satisfying (2.25). Then, $m - n = kN' + h$ for some $k \in \mathbb{Z}_0^+$ and $0 \le h \le N' - 1$. From (2.26) and Claim 12, we obtain

$$\begin{aligned}\|T(n,m)\phi\| &= \|T(n, n+kN'+h)\phi\| \\ &= \|T(n,n+kN')T(n+kN', n+kN'+h)\phi\| \\ &\le e^{-k}\|T(n+kN', n+kN'+h)\phi\| \\ &\le Q'e^{-k}\|\phi\| \\ &\le eQ'e^{-\frac{m-n}{N'}}\|\phi\|.\end{aligned}$$

Thus, the conclusion of the claim holds with $D' = eQ'$ and $\lambda' = 1/N'$. □

For each $n \in \mathbb{Z}_0^+$, let P_n denote the projection of $\mathcal{B}_r$ onto $\mathcal{S}(n)$ along $\mathcal{U}(n)$ associated with the decomposition (2.17).

CLAIM 14. The projections P_n, $n \in \mathbb{Z}_0^+$, are uniformly bounded, i.e.,

$$\sup_{n\ge 0} \|P_n\| < \infty.$$

Proof of claim 14. Since $\ker P_n = \mathcal{U}(n)$ and $\operatorname{im} P_n = \mathcal{S}(n)$ for $n \in \mathbb{Z}_0^+$, Claims 11 and 13 show that the hypotheses of Lemma 3.1 of Huy and Van Minh [13] are satisfied with $X = \mathcal{B}_r$ and $A_n = T(n+1, n)$. Therefore, the desired conclusion follows from [13, Lemma 3.1]. □

Now we can complete the proof of Theorem 2.3. Let $\phi \in \mathcal{B}_r$ and $n, m \in \mathbb{Z}_0^+$ with $n \ge m$ be fixed. From $P_m\phi \in \mathcal{S}(m)$ and (2.16), we have that $T(n,m)P_m\phi \in \mathcal{S}(n)$. Hence,

$$P_nT(n,m)P_m\phi = T(n,m)P_m\phi.$$

Similarly, considering $Q_m = I - P_m$, $Q_m\phi \in \mathcal{U}(m)$ and (2.16) imply that $T(n,m)Q_m\phi \in \mathcal{U}(n)$. Hence,

$$P_nT(n,m)Q_m\phi = 0.$$

From the above relations, taking into account that $\phi = P_m\phi + Q_m\phi$, we conclude that

$$P_nT(n,m)\phi = P_nT(n,m)P_m\phi + P_nT(n,m)Q_m\phi = T(n,m)P_m\phi.$$

Since $\phi \in \mathcal{B}_r$ was arbitrary, this proves (2.6).

Evidently, $\ker P_m = \mathcal{U}(m)$ for $m \in \mathbb{Z}^+$. Therefore, from Claim 8 and (2.16), it follows that the restriction $T(n,m)|_{\ker P(m)} \colon \ker P(m) \to \ker P(n)$ is invertible and onto. Furthermore, by Claim 14, the projections P_n, $n \in \mathbb{Z}_0^+$, are uniformly bounded. Combining this fact with Claims 11 and 13, we conclude that the exponential estimates (2.7) and (2.8) are also satisfied. Thus, (2.1) admits an exponential dichotomy.

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Remark 2.22. In the proof of Theorem 2.3, we have shown that the Perron property (see Claim 1) implies the existence of an exponential dichotomy for Eq. (1.3). Results of this type have a long history that goes back to the pioneering works of Perron [21] for ordinary differential equations and Li [15] for difference equations. Subsequent important contributions are due to Massera and Schäffer [16, 17], Daleckiı and Kreın [9], Coppel [8] and Henry [11], who was the first to consider the case of non-invertible dynamics. For more recent contributions, we refer to [12–14, 24–26, 29] and the references therein. A comprehensive overview of the relationship between hyperbolicity and the Perron property is given in [3].

3. Shadowing of linear Volterra difference equations with infinite delay

In this section, we are interested in the shadowing of the Volterra difference equation with infinite delay (1.5), where the kernel A satisfies condition (1.6). The phase space for Eq. (1.5) is the Banach space $(\mathcal{B}_\gamma, \|\cdot\|)$ given by

$$\mathcal{B}_\gamma = \left\{ \phi\colon \mathbb{Z}_0^- \to \mathbb{C}^d : \sup_{\theta\in\mathbb{Z}_0^-} |\phi(\theta)|e^{\gamma\theta} < \infty \right\}, \qquad \|\phi\| = \sup_{\theta\in\mathbb{Z}_0^-} |\phi(\theta)|e^{\gamma\theta}, \qquad \phi \in \mathcal{B}_\gamma.$$

Under condition (1.6), Eq. (1.5) can be written equivalently in the form

$$x(n+1) = L(x_n), \tag{3.1}$$

where $x_n \in \mathcal{B}_\gamma$ is defined by $x_n(\theta) = x(n+\theta)$ for $\theta \in \mathbb{Z}_0^-$ and $L\colon \mathcal{B}_\gamma \to \mathbb{C}^d$ is a bounded linear functional defined by

$$L(\phi) = \sum_{j=0}^{\infty} A(j)\phi(-j), \qquad \phi \in \mathcal{B}_\gamma.$$

It is known (see, e.g., [18], [20]) that if (1.6) holds, then for every $\phi \in \mathcal{B}_\gamma$, there exists a unique function $x\colon \mathbb{Z} \to \mathbb{C}^d$ satisfying Eq. (1.5) (equivalently, Eq. (3.1)) such that $x(\theta) = \phi(\theta)$ for $\theta \in \mathbb{Z}_0^-$. We shall call x the *solution of Eq. (1.5) (or (3.1)) on $\mathbb{Z}_0^+$ with initial value $x_0 = \phi$*. By a *solution of Eq. (1.5) on $\mathbb{Z}_0^+$*, we mean a solution x with initial value $x_0 = \phi$ for some $\phi \in \mathcal{B}_\gamma$.

For Eq. (1.5), the definition of shadowing can be modified as follows.

Definition 3.1. *We say that Eq. (1.5) is shadowable on $\mathbb{Z}_0^+$ if, for each $\epsilon > 0$, there exists $\delta > 0$ such that for every function $y\colon \mathbb{Z} \to \mathbb{C}^d$ satisfying*

$$\sup_{n\geq 0} |y(n+1) - L(y_n)| \leq \delta,$$

there exists a solution x of (1.5) on $\mathbb{Z}_0^+$ such that

$$\sup_{n\geq 0} \|x_n - y_n\| \leq \epsilon.$$

The main result of this section is the following theorem, which shows that, under condition (1.6), Eq. (1.5) is shadowable on $\mathbb{Z}_0^+$ if and only if it is hyperbolic.

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THEOREM 3.2 *Suppose that (1.6) holds. Then, the following statements are equivalent.*

(i) Eq. (1.5) is shadowable on $\mathbb{Z}_0^+$*;*

(ii) The characteristic equation (1.7) has no root on the unit circle $|\lambda| = 1$*.*

Before we give a proof of Theorem 3.2, we summarize some facts from the spectral theory of linear Volterra difference equations with infinite delay ([18], [19], [20]).

For each $n \in \mathbb{Z}_0^+$, define $T(n)\colon \mathcal{B}_\gamma \to \mathcal{B}_\gamma$ by $T(n)\phi = x_n(\phi)$ for $\phi \in \mathcal{B}_\gamma$, where $x(\phi)$ is the unique solution of Eq. (1.5) on $\mathbb{Z}_0^+$ with initial value $x_0(\phi) = \phi$. It is well known (see [18, 19]) that $T(n)$ is a bounded linear operator in $\mathcal{B}_\gamma$ which has the semigroup property $T(0) = I$, the identity on $\mathcal{B}_\gamma$, and $T(n+m) = T(n)T(m)$ for $n, m \in \mathbb{Z}_0^+$. As a consequence, we have that

$$T(n) = T^n, \qquad n \in \mathbb{Z}_0^+, \qquad \text{where} \quad T := T(1).$$

From the definition of the *solution operator* $T = T(1)$ and Eq. (1.5), we have that

$$[T(\phi)](\theta) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} A(j)\phi(-j) & \text{for } \theta = 0, \\ \phi(\theta+1) & \text{for } \theta \le -1. \end{cases} \tag{3.2}$$

If (1.6) holds, then the characteristic function $\varDelta$ defined by (1.8) is an analytic function of the complex variable λ in the region $|\lambda| > e^{-\gamma}$. Denote by $\varSigma$ the set of characteristic roots of Eq. (1.5),

$$\varSigma = \{\, \lambda \in \mathbb{C} : |\lambda| > e^{-\gamma},\, \det \varDelta(\lambda) = 0 \,\},$$

and define

$$\varSigma^{cu} = \{\, \lambda \in \varSigma : |\lambda| \ge 1 \}.$$

It follows from the analyticity of $\varDelta$ that $\varSigma^{cu}$ is a finite spectral set for T. The corresponding spectral projection $\varPi^{cu}$ on $\mathcal{B}_\gamma$ is defined by

$$\varPi^{cu} = \frac{1}{2\pi i} \int_C (\lambda I - T)^{-1}\, d\lambda,$$

where C is any rectifiable Jordan curve, which is disjoint with $\varSigma$ and contains $\varSigma^{cu}$ in its interior, but no point of $\varSigma^s := \varSigma \setminus \varSigma^{cu}$. The phase space $\mathcal{B}_\gamma$ can be decomposed into the direct sum

$$\mathcal{B}_\gamma = \mathcal{B}_\gamma^{cu} \oplus \mathcal{B}_\gamma^s$$

with $\mathcal{B}_\gamma^{cu} = \Pi^{cu}(\mathcal{B}_\gamma)$ and $\mathcal{B}_\gamma^{s} = \Pi^{s}(\mathcal{B}_\gamma)$, where $\Pi^s = I - \Pi^{cu}$. The subspaces $\mathcal{B}_\gamma^{cu}$ and $\mathcal{B}_\gamma^{s}$ are called the *centre-unstable subspace* and the *stable subspace* of $\mathcal{B}_\gamma$, respectively. The spectra of the restrictions $T^{cu} := T|_{\mathcal{B}_\gamma^{cu}}$ and $T^s := T|_{\mathcal{B}_\gamma^{s}}$ satisfy

$$\sigma(T^{cu}) = \Sigma^{cu} \quad \text{and} \quad \sigma(T^s) = \sigma(T) \setminus \Sigma^{cu} = \{\, \lambda \in \sigma(T) : |\lambda| < 1 \,\}.$$

If Σ^{cu} is non-empty, then it consists of finitely many eigenvalues of T,

$$\Sigma^{cu} = \{\, \lambda_1, \dots, \lambda_r \,\},$$

and $\mathcal{B}_\gamma^{cu}$ can be written as a direct sum of the nullspaces

$$\mathcal{B}_\gamma^{cu} = \ker((T - \lambda_1 I)^{p_1}) \oplus \cdots \oplus \ker((T - \lambda_r I)^{p_r}),$$

where p_j is the index (ascent) of λ_j, $j = 1, \dots, r$ (see [18, Remark 2.1, p. 62]).

An explicit representation of the spectral projection Π^{cu} can be given using the duality between Eq. (1.5) and its *formal adjoint equation*

$$y(n-1) = \sum_{j=0}^{\infty} y(n+j)A(j), \qquad n \in \mathbb{Z}_0^+, \tag{3.3}$$

where $y(n) \in \mathbb{C}^{d*}$. Here $\mathbb{C}^{d*}$ denotes the d-dimensional space of complex row vectors with a norm $|\cdot|$, which is compatible with the given norm on $\mathbb{C}^d$, i.e., $|x^*x| \le |x^*||x|$ for all $x \in \mathbb{C}^d$. The superscript * indicates the conjugate transpose. The phase space for the formal adjoint Eq. (3.3) is the Banach space $(\mathcal{B}_{\tilde\gamma}^{\sharp}, \|\cdot\|)$ defined by

$$\mathcal{B}_{\tilde\gamma}^{\sharp} = \left\{ \psi \colon \mathbb{Z}_0^+ \to \mathbb{C}^{d*} : \sup_{\zeta \in \mathbb{Z}_0^+} |\psi(\zeta)| e^{-\tilde\gamma\zeta} < \infty \right\}, \quad \|\psi\| = \sup_{\zeta \in \mathbb{Z}_0^+} |\psi(\zeta)| e^{-\tilde\gamma\zeta}, \quad \psi \in \mathcal{B}_{\tilde\gamma}^{\sharp},$$

where $\tilde\gamma$ is a fixed number such that $0 < \tilde\gamma < \gamma$. The solution operator $T^\sharp \colon \mathcal{B}_{\tilde\gamma}^{\sharp} \to \mathcal{B}_{\tilde\gamma}^{\sharp}$ of Eq. (3.3) is given by

$$[T^\sharp(\psi)](\zeta) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \psi(j)A(j) & \text{if } \zeta = 0, \\ \psi(\zeta - 1) & \text{if } \zeta \ge 1. \end{cases} \tag{3.4}$$

Define a *bilinear form* $\langle \cdot, \cdot \rangle \colon \mathcal{B}_{\tilde\gamma}^{\sharp} \times \mathcal{B}_\gamma \to \mathbb{C}$ by

$$\langle \psi, \phi \rangle = \psi(0)\phi(0) + \sum_{j=1}^{\infty} \sum_{\zeta=0}^{j-1} \psi(\zeta+1)A(j)\phi(\zeta - j), \qquad \phi \in \mathcal{B}_\gamma, \quad \psi \in \mathcal{B}_{\tilde\gamma}^{\sharp}.$$

As shown in [18, In eq. (3.3), p. 64], this bilinear form is bounded, i.e., there exists $K > 0$ such that

$$|\langle \psi, \phi \rangle| \le K \|\psi\| \|\phi\|, \qquad \phi \in \mathcal{B}_\gamma, \quad \psi \in \mathcal{B}_{\tilde\gamma}^{\sharp}. \tag{3.5}$$

Moreover, between Eqs. (1.5) and (3.3), we have the following *duality relation* (see [18, Lemma 3.1])

$$\langle \psi, T\phi \rangle = \langle T^\sharp \psi, \phi \rangle, \qquad \phi \in \mathcal{B}_\gamma, \quad \psi \in \mathcal{B}^\sharp_{\tilde{\gamma}}. \tag{3.6}$$

It is known that T and $T^\sharp$ have the same spectrum and the dimension of the subspace

$$\mathcal{N}^\sharp := \ker((T^\sharp - \lambda_1 I)^{p_1}) \oplus \cdots \oplus \ker((T^\sharp - \lambda_r I)^{p_r})$$

of $\mathcal{B}^\sharp_{\tilde{\gamma}}$ is the same as the (finite) dimension of the centre-unstable subspace $\mathcal{B}^{cu}_\gamma$, which will be denoted by s. Let $\{ \phi_1, \dots, \phi_s \}$ and $\{ \psi_1, \dots, \psi_s \}$ be bases for $\mathcal{B}^{cu}_\gamma$ and $\mathcal{N}^\sharp$, respectively. Define $\Phi = (\phi_1, \dots, \phi_s)$ and $\Psi = \mathrm{col}(\psi_1, \dots, \psi_s)$. Then, the $s \times s$ matrix $\langle \Psi, \Phi \rangle$ given by $\langle \Psi, \Phi \rangle = (\langle \psi_i, \phi_j \rangle)_{i,j=1,\dots,s}$ is non-singular, therefore, by replacing Ψ with $\langle \Psi, \Phi \rangle^{-1}\Psi$, we may (and do) assume that $\langle \Psi, \Phi \rangle = E$, the $s \times s$ unit matrix. The projection $\Pi^{cu} \colon \mathcal{B}_\gamma \to \mathcal{B}^{cu}_\gamma$ can be given explicitly by (see [18, Theorem 3.1])

$$\Pi^{cu}\phi = \Phi\langle \Psi, \phi \rangle, \qquad \phi \in \mathcal{B}_\gamma, \tag{3.7}$$

where $\langle \Psi, \phi \rangle$ denotes the column vector $\mathrm{col}(\langle \psi_1, \phi \rangle, \dots, \langle \psi_s, \phi \rangle)$.

The subspace $\mathcal{B}^{cu}_\gamma$ is invariant under the solution operator T. If B denotes the representation matrix of the linear transformation $T|_{\mathcal{B}^{cu}_\gamma}$ with respect to the basis Φ of $\mathcal{B}^{cu}_\gamma$, then

$$T\Phi = \Phi B \qquad \text{and} \qquad \sigma(B) = \Sigma^{cu}. \tag{3.8}$$

A similar argument yields the existence of a square matrix C such that

$$T^\sharp \Psi = C\Psi \qquad \text{and} \qquad \sigma(C) = \Sigma^{cu}. \tag{3.9}$$

Now suppose that x is a solution of the non-homogeneous equation

$$x(n+1) = L(x_n) + p(n), \qquad n \in \mathbb{Z}^+_0. \tag{3.10}$$

with initial value $x_0 = \phi$ for some $\phi \in \mathcal{B}_\gamma$. Then, x satisfies the following representation formula in $\mathcal{B}_\gamma$ (see [19, Theorem 2.1]), which is called the *variation of constants formula* for Eq. (3.10) in the phase space,

$$x_n = T(n)\phi + \sum_{j=0}^{n-1} T(n-1-j)\Gamma p(j), \qquad n \in \mathbb{Z}^+_0, \tag{3.11}$$

where the operator $\Gamma \colon \mathbb{C}^d \to \mathcal{B}_\gamma$ is defined by

$$[\Gamma x](\theta) = \begin{cases} x & \text{if } \theta = 0, \\ 0 & \text{if } \theta \le -1. \end{cases} \tag{3.12}$$

Evidently,

$$\|\Gamma x\| = |x|, \qquad x \in \mathbb{C}^d. \tag{3.13}$$

Finally, let $z(n)$ be the coordinate of the projection $\Pi^{cu} x_n$ with respect to the basis Φ, i.e., $\Pi^{cu} x_n = \Phi z(n)$ for $n \in \mathbb{Z}_0^+$. In view of (3.7), $z(n)$ is given explicitly by

$$z(n) = \langle \Psi, x_n \rangle, \qquad n \in \mathbb{Z}_0^+. \tag{3.14}$$

Moreover, it is known (see [20, Theorem 3]) that z satisfies the following first order difference equation in $\mathbb{C}^s$,

$$z(n+1) = Bz(n) + \langle \Psi, \Gamma p(n) \rangle, \qquad n \in \mathbb{Z}_0^+, \tag{3.15}$$

with B as in (3.8).

Now we are in a position to give a proof of Theorem 2.3.

Proof of Theorem 2.3. (*i*) $\Rightarrow$ (*ii*). Suppose, for the sake of contradiction, that Eq. (1.5) is shadowable, but (ii) does not hold. The shadowing property of Eq. (1.5) implies the following Perron-type property.

CLAIM 15. For every bounded function $p \colon \mathbb{Z}_0^+ \to \mathbb{C}^d$, there exists a function $x \colon \mathbb{Z} \to \mathbb{C}^d$ satisfying the non-homogeneous Eq. (3.10) with

$$\sup_{n \geq 0} |x(n)| < \infty. \tag{3.16}$$

The proof of Claim 15 is almost identical with that of Claim 1 in the proof of Theorem 2.3, therefore we omit it. Since (ii) does not hold, there exists a characteristic root $\lambda \in \Sigma$ with $|\lambda| = 1$. Evidently, $\lambda \in \Sigma^{cu}$, therefore the second relation in (3.8) implies the existence of a non-zero vector $v \in \mathbb{C}^{s*}$ such that

$$vB = \lambda v. \tag{3.17}$$

Define $p \colon \mathbb{Z}_0^+ \to \mathbb{C}^d$ by

$$p(n) = \lambda^{n+1} (v\Psi(0))^*, \qquad n \in \mathbb{Z}_0^+. \tag{3.18}$$

Since $\sup_{n \geq 0} |p(n)| = |(v\Psi(0))^*| < \infty$, by Claim 15, the non-homogeneous Eq. (3.10) has at least one solution x on $\mathbb{Z}_0^+$ such that (3.16) holds. The corresponding coordinate function z defined by (3.14) satisfies the difference Eq. (3.15) From (3.5), (3.13), (3.14), and (3.16), we obtain that z and hence the function $u \colon \mathbb{Z}_0^+ \to \mathbb{C}$ defined by

$$u(n) = vz(n), \qquad n \in \mathbb{Z}_0^+, \tag{3.19}$$

is bounded on $\mathbb{Z}_0^+$. Multiplying Eq. (3.15) from the left by v and using (3.17), we obtain that

$$u(n+1) = \lambda u(n) + v\langle \Psi, \Gamma p(n) \rangle, \qquad n \in \mathbb{Z}_0^+. \tag{3.20}$$

It follows from (3.12), (3.18), and the definition of the bilinear form $\langle \cdot, \cdot \rangle$ that

$$\langle \Psi, \Gamma p(n) \rangle = \Psi(0)p(n) = \lambda^{n+1}\Psi(0)(v\Psi(0))^*.$$

This, together with (3.20), implies that

$$u(n+1) = \lambda u(n) + c\lambda^{n+1}, \qquad n \in \mathbb{Z}_0^+, \tag{3.21}$$

with

$$c = (v\Psi(0))(v\Psi(0))^* = |(v\Psi(0))^*|_2^2, \tag{3.22}$$

where $|\cdot|_2$ denotes the l_2-norm on $\mathbb{C}^d$. From Eq. (3.21), it follows by the variation of constants formula that

$$u(n) = \lambda^n (u(0) + cn), \qquad n \in \mathbb{Z}_0^+. \tag{3.23}$$

It follows from the duality (3.6) that $B = C$, where B and C have the meaning from (3.8) and (3.9), respectively. Indeed, (3.6) implies that

$$B = \langle \Psi, \Phi \rangle B = \langle \Psi, \Phi B \rangle = \langle \Psi, T\Phi \rangle = \langle T^\sharp \Psi, \Phi \rangle = \langle C\Psi, \Phi \rangle = C\langle \Psi, \Phi \rangle = C.$$

From (3.2) and the relation $T\Phi = \Phi B$ (see (3.8)), we find that

$$\Phi(\theta) = \Phi(0)B^\theta, \qquad \theta \in \mathbb{Z}_0^-. \tag{3.24}$$

Similarly, from (3.4) and the relation $T^\sharp \Psi = B\Psi$ (see (3.9)), we have that

$$\Psi(\zeta) = B^{-\zeta}\Psi(0), \qquad \zeta \in \mathbb{Z}_0^+. \tag{3.25}$$

Next we will show that

$$v\Psi(0) \neq 0. \tag{3.26}$$

Suppose, for the sake of contradiction, that $v\Psi(0) = 0$. Then, by (3.17) and (3.25), we have

$$v\Psi(\zeta) = vB^{-\zeta}\Psi(0) = \lambda^{-\zeta}v\Psi(0) = 0 \qquad \text{for all } \zeta \in \mathbb{Z}_0^+.$$

Thus, $v\Psi$ is identically zero on $\mathbb{Z}_0^+$. On the other hand, $v \neq 0$ implies that $v\Psi$ is a non-trivial linear combination of the basis elements $\psi_1, \ldots, \psi_s$ of $\mathcal{N}^\sharp$, and hence, it cannot be identically zero on $\mathbb{Z}_0^+$. This contradiction proves that (3.26) holds.

From (3.22) and (3.26), we find that $c > 0$. From this and (3.23), taking into account that $|\lambda| = 1$, we obtain that

$$|u(n)| = |u(0) + cn| \to \infty, \qquad n \to \infty,$$

which contradicts the boundedness of u.

$(ii) \Rightarrow (i)$. Suppose that the characteristic Eq. (1.7) has no root with $|\lambda| = 1$. Then, the exponential estimates of the solution operator on the stable and unstable subspaces of $\mathcal{B}_\gamma$ (see, e.g., [20, Theorem 1]) imply that Eq. (1.5) admits an exponential dichotomy on $\mathbb{Z}_0^+$ as defined in [10] with projections $P_n = \Pi^s$ for $n \in \mathbb{Z}_0^+$. By the application of [10, Theorem 1] with $f_n = 0$, $c = 0$ and $\mu = 1$, we conclude that Eq. (1.5) is shadowable on $\mathbb{Z}_0^+$. $\square$

Remark 3.4. It is known that certain solutions of Eq. (1.5) can be continued backward in the sense that they satisfy Eq. (1.5) for all $n \in \mathbb{Z}$. Such solutions are sometimes called *global solutions* or *entire solutions*. In a recent article [4], Barreira and Valls have considered the Ulam–Hyers stability, a special case of shadowing, of the global solutions for difference equations with finite delays. In [4, Theorem 5], they have proved the analogue of a recent shadowing theorem [10, Theorem 1] for global solutions. Moreover, in the special case of linear autonomous equations in finite dimensional spaces, they have shown that the Ulam–Hyers stability of the global solutions is equivalent to the existence of an exponential dichotomy whenever the coefficients are scalar (see [4, Theorem 8]) or the Jordan blocks associated with the central directions are diagonal (see [4, Theorem 9]). Using a similar argument as in the proof of Theorem 3.2, it can be shown that the assumption about the Jordan blocks in [4, Theorem 9] is superfluous. In this sense, Theorem 3.2 may be viewed as an improvement of [4, Theorem 9].

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