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BOOK REVIEWS

BORCEUX, F. AND JANELIDZE, G. *Galois theories* (Cambridge Studies in Advanced Mathematics, no. 72, Cambridge University Press, 2001), xiv+341 pp., 0 521 80309 8 (hardback), £50 (US$74.95).

The developments that led to this book are due to deeply based analogies between two initially very different looking mathematical theories, one in algebra, the other in topology.

Galois originally developed the elements of what was to become group theory in an attempt to understand polynomial equations. In modern language, working over a base field K, a field extension $K \subset L$ is a Galois extension when every element of L is the root of a polynomial in $K[X]$ which factors in $L[X]$ into linear factors and all of whose roots are simple. The group $\mathrm{Gal}[L : K]$ of this extension is the group of field automorphisms of L which fix all elements of K, and the classical Galois theorem asserts that, when L, considered as a K-vector space, is finite dimensional, the subgroups $G \subseteq \mathrm{Gal}[L : K]$ of the Galois group classify the intermediate field extensions $K \subseteq M \subseteq L$. The themes to note in this classical theory are (i) splitting into simpler structures, (ii) groups of automorphisms, and (iii) intermediate structures classified by subgroups.

The second theory is that of covering spaces in topology. A covering map $\alpha : X \to B$ is one that has the property that every point of B has an open neighbourhood whose inverse image by α is a disjoint union of open subsets each of which is mapped homeomorphically onto it by α. Given any reasonably 'locally nice' space, there is a universal connected covering space $p : \tilde{B} \to B$ such that all connected covering spaces of B are quotients of $\tilde{B}$. Classification of the connected coverings is by subgroups of the automorphism group of p. Of course, this automorphism group is isomorphic to the (Poincaré) fundamental group of B (under suitable local conditions).

This topological theory of covering spaces has some similarities to Galois theory. Again one has automorphism groups and a correspondence between intermediate structures (this time quotients not subobjects) and perhaps some notion of splitting—there is an open cover of B over each part of which the covering splits up as a family of isomorphic pieces.

These two theories, therefore, do look vaguely similar, but automorphism groups are very common in mathematics and even that sort of 'Galois' correspondence is not that uncommon, so surely any similarities must not be due to anything really deep! The story of how the deep connections between them became apparent is quite long and here is not the place to explore it in any detail. It involves function spaces and Riemann surfaces as well as much else that is central to modern pure mathematics, as you probably know (if you want a good source for the theory see the beautiful book by Douady and Douady [**1**]). The present book traces a greatly extended mathematical path beyond that semi-classical link between Galois theory and coverings, but starts at an elementary beginning. It describes classical Galois theory, then turns to its extension to infinitary field extensions, to étale algebras that became the foundation for the work of Grothendieck on the fundamental group of schemes (SGA1, see [**2**]). This was at the same time Galois theory and covering space theory, but for spaces for which there was no question of being able to define a fundamental group using paths.

This beautifully written book explores the connections between these theories, searching for further cases of the general 'scenario' and stripping back the superficial structure to reveal aspects of what are the essential features of all these theories.

The book is structured in a very well thought out way. It assumes a certain knowledge of algebra and general topology, together with some familiarity with the elementary language of category theory (categories, functors, natural transformations, limits and adjoint functors). It starts, as mentioned above, with a short trip through the theory of field extensions, then goes on to look at Grothendieck's extension of this to algebras. This is clearly introduced and discussed, although it does assume some knowledge of the tensor product. Chapter 3 handles infinitary Galois theory. Here, profinite spaces and profinite groups are introduced. They are also useful in the following chapter, where the extension of Galois theory to commutative rings (due to Magid [**3**]) is treated. The Pierce spectrum and Stone duality are handled clearly and simply, laying the base for Magid's profinite Galois groupoid. This material is not as well known as it deserves to be and its clear exposition here is very much to be welcomed.

The gradual introduction of categorical machinery to describe and manipulate the objects is masterly. The simplifications that arise in the descriptions as a result of the greater abstraction not only fully justify it but provide a gentle well-motivated introduction to some very deep and important categorical ideas that will be used time and time again in the remainder of the book.

With that first layer of categorical abstraction (Janelidze's abstract categorical Galois theory) in place, other applications can be explored. Given the introduction to the area in this review, it may seem strange that covering maps are only introduced in Chapter 6, but here they are very neatly described and handled, beautifully illustrating and enriching the earlier abstraction.

The final chapter describes one of the most important recent advances in topos theory, giving an introduction to the Joyal–Tierney classification of Grothendieck toposes as sheaves on localic groupoids. The book ends with a look at other directions the theory has taken beyond those handled in detail. This section is particularly valuable as it should set the scene for future research.

The book is remarkably self-contained. It starts with prerequisites that, in the UK, might correspond to MSc level, but it introduces many deep important concepts of algebra and category theory—introduces, motivates and uses in an interesting way. As one would expect, this means that the later chapters are harder going than the earlier ones, but the writing and structure of the book are exemplary, so the transition is gradual. It will be a very useful addition to the algebraic and categorical literature. It is not a category-theory monograph: it handles much more than that as it links the algebra and topology firmly into a categorical setting.

This book is highly recommended for anyone wishing to learn the mathematical side of category theory (rather than its computer-science aspect) or who needs to gain a deeper insight into what makes Galois theory work. I enjoyed reading it very much.

References

1. R. Douady and A. Douady, *Algèbre et théories galoisiennes* (Fernand Nathan, 1977).
2. A. Grothendieck, *Revêtements étales et groupe fondamental*, SGA1, exposé V, Springer Lecture Notes in Mathematics, vol. 224 (Springer, 1971).
3. A. R. Magid, *The separable Galois theory of commutative rings* (Marcel Dekker, 1974).

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