Can. J. Math., Vol. XXXIX, No. 6, 1987, pp. 1434-1445

FUCHSIAN EMBEDDINGS IN THE BIANCHI GROUPS

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1. Introduction. If d is a positive square free integer we let O_d be the ring of integers in $Q(\sqrt{-d})$ and we let $\Gamma_d = PSL_2(O_d)$, the group of linear fractional transformations

$$z' = az + b/cz + d, \quad ad - bc = 1$$

and entries from O_d $\{if\ d = 1,\ ad - bc = \pm 1\}$. The Γ_d are called collectively the *Bianchi groups* and have been studied extensively both as abstract groups and in automorphic function theory {see references}. Of particular interest has been Γ_1 – the *Picard group*. Group theoretically Γ_1 is very similar to the classical modular group $M = PSL_2(Z)$ both in its total structure [**4**, **6**], and in the structure of its congruence subgroups [**8**]. Where Γ_1 and M differ greatly is in their action on the complex place **C**. M is Fuchsian and therefore acts discontinuously in the upper half-plane and every subgroup has the same property. In distinction Γ_1 (and Γ_d in general) is nowhere discontinuous in C [**14**], and therefore contains no Fuchsian subgroups of finite index. A question then is how Fuchsian subgroups are embedded in Γ_1 and in Γ_d. What we prove is that to obtain a torsion-free, non-free Fuchsian subgroup F of Γ_1, (in particular a faithful Fuchsian representation of a Riemann surface group), F must be embedded in Γ_1 in such a way that F has cyclic intersection with every conjugate in Γ_1 of the modular group M. Maskit [**17**], Mennicke [**19**], and Fine [**4**] have used the Picard group to generate faithful representations of surface groups. An identical embedding property is shown to hold for Γ_d, $d = 2, 7, 11$ while the remaining cases require some modification. If F is Fuschian and has torsion and is embedded in Γ_d, $d = 1, 2, 7, 11$, then we show that F is finite or a free product of cyclics unless it has a certain intersection property with the modular group. Finally we show that for all non-real subrings R of **C**, $PSL_2(R)$ will contain no normal Fuchsian subgroups.

2. Preliminaries. In this section we recall some definitions and facts that will be crucial to the subsequent development. A *Fuchsian group F* is a discrete subgroup of $PSL_2(\mathbf{R})$ or a conjugate in $PSL_2(\mathbf{C})$ of a discrete subgroup of $PSL_2(\mathbf{R})$. Equivalently a Fuchsian group can be defined as a discontinuous (and therefore discrete) subgroup of $PSL_2(\mathbf{C})$ which maps

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Received February 1, 1985 and in revised form January 7, 1987.

a circle C and the interior of C on itself. C is called the *fixed circle* of the group F.

Since conjugation preserves traces, the transformations

$$z' = az + b/cz + d$$

in a Fuchsian group F all have real trace. Thus the transformations in a Fuchsian group are either elliptic (trace with absolute value less than 2), hyperbolic (trace with absolute value greater than 2) or parabolic (trace of ± 2). The fix points of hyperbolics and parabolics in F must lie on the fixed circle C, while the fixed points of elliptics in F are inverse with respect to C. Finally elements of finite order must be elliptic. If F consists solely of hyperbolic transformations we call F a *hyperbolic Fuchsian group*. Similarly for *parabolic Fuchsian groups* and *elliptic Fuchsian groups*.

If a Fuchsian group is finitely generated it possesses a standard presentation [**14**],

$$F = \left\langle e_1, \dots e_k, p_1, \dots p_t, a_1, b_1, \dots a_g, b_g; \right.$$
$$\left. e_i^{r_i} = 1, i = 1, \dots k, e_1 \dots e_k p_1 \dots p_t \prod_1^g [a_i, b_i] = 1 \right\rangle.$$

The $e_1, \dots e_k$ are the elliptic generators, $p_1 \dots p_t$ the parabolic generators and $a_1, b_1, \dots a_g, b_g$ the hyperbolic generators. If $t > 0$ then F is a free product of cyclics [**14**]. If $k = 0$, F is torsion-free and is then either a free group or has a presentation

$$F = \left\langle a_1, b_1, \dots a_g, b_g; \prod_1^g [a_i, b_i] = 1 \right\rangle.$$

This is isomorphic to the fundamental group of a Riemann surface of genus g. We will denote this by ϕ_g and call any group which is isomorphic to ϕ_g a *surface group*. Thus a torsion-free Fuchsian group is either a free group or a surface group. By a *Fuchsian surface group* we mean a Fuchsian group in the sense defined earlier which is also a surface group. Many of our embedding results apply to general surface groups as well as Fuchsian groups.

3. The Picard group. If Γ_1 is the Picard group, crucial to our discussion is the following result:

THEOREM [**6**]. *Γ_1 is a free product with amalgamation*

$$\Gamma_1 = G_1 \underset{M}{*} G_2$$

with

$$G_1 \simeq S_3 \underset{H}{*} A_4, \quad H \simeq Z_3$$

$$G_2 \simeq S_3 \underset{H_1}{*} D_2, \quad H_1 \simeq Z_2$$

and the amalgamated subgroup M is isomorphic to the modular group.

Different results are obtained if the Fuchsian groups are torsion-free or not. Therefore in all cases we separate these two situations. Our first result is on torsion-free Fuchsian subgroups of Γ_1.

THEOREM 1. *Let F be a torsion-free Fuchsian subgroup of* Γ_1. *Then F is a free group unless it satisfies*:

a) *F has at most cyclic intersection with all conjugates in* Γ_1 *of the modular group M and*

b) *F has non-trivial intersection with at least one conjugate of M.*

Proof. Suppose F is torsion-free and Fuchsian in Γ_1. Since

$$\Gamma_1 = G_1 \underset{M}{*} G_2,$$

if F has trivial intersection with all conjugates of M in Γ_1 it follows from [**12**] or [**21**] that F must be a free product of free groups and subgroups of conjugates of G_1 and G_2.

Now G_1 and G_2 are themselves free products with amalgamations of finite groups. These satisfy the "torsion-free subgroup property"; that is torsion-free subgroups must be free [**9**]. Thus since F is torsion-free, the subgroups of G_1 and G_2 contained in F must be free. F is then a free product of free groups and thus free.

We must now show that if F has non-cyclic intersection with some conjugate of M then F is also free.

Since F is Fuschian and torsion-free it contains only hyperbolic and parabolic maps. If it contains a parabolic, F is a free product of cyclics [**14**] and since it is torsion-free it must be free. Thus we can assume that F is totally hyperbolic. Our result is then a consequence of the following lemma.

LEMMA A. *Let R be a subring of* **C** *such that* $R \cap \mathbf{R} = Z$. *If F is a hyperbolic Fuchsian subgroup of* $PSL_2(R)$ *and* $F \cap g^{-1}Mg$ *is non-cyclic for some* $g \in PSL_2(R)$ (*M is the Modular group*) *then* $F \subset g^{-1}Mg$.

Proof. (lemma) Since $R \cap \mathbf{R} = Z$ we have

$$PSL_2(R) \cap PSL_2(\mathbf{R}) = PSL_2(Z) = M.$$

Now suppose that $F \cap g^{-1}Mg$ is non-cyclic for some g in $PSL_2(R)$.

By conjugation we can suppose that we have a Fuchsian subgroup F^* with $F^* \cap M$ non-cyclic. We show that $F^* \subset M$ and thus $F \subset g^{-1}Mg$.

Since F is totally hyperbolic so is its conjugate F^*. Since $F^* \cap M$ is non-cyclic and Fuchsian there exists two non-commuting hyperbolic elements T, U in $F^* \cap M$. (Since abelian subgroups of Fuchsian groups are cyclic.)

From $[T, U] \neq 1$, it follows that T, U have at least 3 distinct fix points Z_1, Z_2, Z_3 for T and U would commute if their fix points coincided.

Let C be the fix circle of F^*. Since T, U are hyperbolic, Z_1, Z_2, Z_3 are on C. But T, U are in M so Z_1, Z_2, Z_3 are real; fix points of hyperbolic maps in M are real. Therefore C has 3 points in common with $\mathbf{R}$ so C must be the real line.

Therefore the fix circle of F^* is the real line $\mathbf{R}$ so $F^* \subset PSL_2(\mathbf{R})$ and

$$F^* \subset PSL_2(\mathbf{R}) \cap PSL_2(R) = M$$

and then

$$F \subset g^{-1}Mg.$$

Now we can complete the proof of Theorem 1.

If F is hyperbolic and has non-cyclic intersection with some conjugate of M and since

$$\Gamma_1 \cap PSL_2(\mathbf{R}) = M$$

we have from Lemma A $F \subset g^{-1}Mg$ for some g in Γ_1 or $F^* \subset M$ by conjugation.

But F^* is a torsion-free subgroup of M and therefore must be free [**20**]. So F must be free.

Theorem 1 has an interesting application to representations of surface groups in Γ_1. Maskit [**17**], Mennicke [**19**], and Fine [**4**], have all generated faithful representations of surface groups in Γ_1. A result of [**18**] guarantees that there will be Fuchsian representations in Γ_1, although he does not explicitly construct any. (We call a representation of a group in $PSL_2(\mathbf{C})$ Fuchsian if the image is a Fuchsian subgroup of $PSL_2(\mathbf{C})$). Recall that if $g > 0$, then the surface group of genus g, ϕ_g, is the fundamental group of a Riemann surface of genus g. As mentioned in Section 2, finitely generated Fuchsian groups are either free groups or surface groups if they are torsion-free. Thus we get.

COROLLARY 1. *Suppose $F \subset \Gamma_1$ with F Fuchsian. If $F = \phi_g$ (F provides a faithful Fuchsian representation of ϕ_g in Γ_1) then F has non-trivial intersection with some conjugate of the modular group and cyclic (possibly trivial) intersection with all conjugates of M.*

Since surface groups do not decompose as free products the first part of the corollary applies to any representation of ϕ_g in Γ_1. That is

COROLLARY 2. *If $A \subset \Gamma_1$ with $A \simeq \phi_g$ then A has non-trivial intersection with some conjugate of the modular group M. (A need not be Fuchsian).*

If F has elliptic elements, that is F has torsion, the situation becomes somewhat more complicated. First we prove a general embedding result which is an extension of Lemma A. This will be used in analyzing the cases with torsion. This result was suggested by the referee.

LEMMA B. *Let G be a subgroup of $PSL_2(\mathbf{C})$ and suppose that H and F are Fuchsian subgroups of G. Suppose that H is the Fuchsian stabilizer in G of the circle C; that is H is the subgroup of G which fixes C and maps the interior of C on itself. Then $F \subset H$ or $F \cap H$ is cyclic.*

Proof. Since both F and H are Fuchsian, $F \cap H$ is also Fuchsian. If $F \cap H$ is non-elementary (that is has infinitely many limit points) then $F \cap H$ must contain at least two non-commuting hyperbolic elements. This follows from the fact that the limit points are the closure of the hyperbolic fixed points [**14**]. Suppose T, U are hyperbolic with $[T, U] \neq 1$ and $T, U \in F \cap H$. Then as in Lemma A, T, U must have at least three distinct fixed points on the fixed circle C of H. But then C must also be the fixed circle of F. Since H is the Fuchsian stabilizer in G of C it follows that $F \subset H$.

Now suppose that $F \cap H$ is elementary; $F \cap H$ has 0, 1 or 2 limit points. Since $F \cap H$ is Fuchsian there are 4 possibilities; parabolic cyclic, hyperbolic cyclic, elliptic cyclic or a group generated by two elliptic transformations of order 2 whose product is hyperbolic. Abstractly this is $Z_2 * Z_2$. The group generated by two elliptics of order 2 whose product is parabolic is elementary but does not appear as a Fuchsian group. We show that in the last case $F \subset H$. Suppose T_1, T_2 generate $F \cap H$ with $T_1^2 = T_2^2 = 1$ and T_1T_2 hyperbolic. Then the fixed points z_1, z_2 are the only limit points of $F \cap H$. These then must be interchanged by T_1. However given two points which are interchanged by an elliptic transformation there is a unique invariant circle passing through them. This must then be the fixed circle of F which is also the fixed circle of H. As above then $F \subset H$.

COROLLARY. *Let R be a subring of $\mathbf{C}$ such that $R \cap \mathbf{R} = Z$ and suppose that F is a Fuchsian subgroup of $PSL_2(R)$. Then if $F \cap g^{-1}Mg$ is non-cyclic for some g in $PSL_2(R)$ then $F \subset g^{-1}Mg$.*

Proof. Since $R \cap \mathbf{R} = Z$ it follows that the Fuchsian stabilizer of the real axis in $PSL_2(R)$ is the modular group M. The corollary then follows directly.

We now handle the case of a Fuchsian subgroup with torsion.

THEOREM 2. *Let F be a finitely generated Fuchsian subgroup of Γ_1.*

1) *If F has trivial intersection with all conjugates in Γ_1 of the modular group M then F is either finite or a free product of cyclics.*

2) *If F has non-cyclic intersection with some conjugate of M then F is a non-trivial free product of cyclics.*

3) *The possible finite Fuchsian groups which can appear are the cyclic groups* Z_2 *and* Z_3.

Proof. Suppose F has trivial intersection with all conjugates of M.

As in the proof of Theorem 1, F must then be a free product of free groups and conjugates of subgroups of G_1 and G_2.

Since finitely generated Fuchsian groups are not non-trivial free products unless they are free products of cyclics we conclude that F is a free product of cyclics or F is a conjugate of a subgroup of G_1 or G_2.

Suppose by conjugation that $F \subset G_1$ or $F \subset G_2$. If F is not finite or a free product of cyclics then by the Fenchel-Fox Theorem [**15**] F contains a torsion-free F-group (surface group) of finite index. But as remarked in the proof of Theorem 1, G_1 and G_2 satisfy the torsion-free subgroup property; that is torsion-free subgroups must be free. Thus if $F \subset G_1$ or $F \subset G_2$ or a conjugate of these it must be finite or a free product of cyclics. This proves part 1) of the theorem.

The proof of part 3) follows using the same analysis. A finite subgroup must be conjugate to a subgroup of the factors; thus it must be conjugate to a subgroup of S_3, A_4 or D_2. However finite Fuchsian groups must be cyclic giving only the possibilities Z_2 or Z_3.

Now suppose that F has non-cyclic intersection with some conjugate of M. By conjugation we can assume a Fuchsian group F^* with non-cyclic intersection with M.

Since $\Gamma_1 \cap PSL_2(R) = M$ it follows from the corollary to Lemma B that either $F^* \subset M$ or $F^* \cap M$ is cyclic. Since the intersection is non-cyclic F^* is contained in M. Therefore F^* is either finite or a free product of cyclics. Since F^* is non-cyclic in M it cannot be finite so F^* must be a non-trivial free product of cyclics. By conjugation F is then also a non-trivial free product of cyclics.

Lemma B and its corollary can also be used to consider the Fuchsian triangle groups in $PSL_2(R)$ where R is a subring of $\mathbf{C}$ with $R \cap \mathbf{R} = Z$.

Recall that a *triangle group* $T = T(m, n, p)$ is a group with a presentation of the form

$$\langle u, v: u^m = v^n = (uv)^p = 1\rangle.$$

A *Fuchsian triangle group* is a Fuchsian group which provides a faithful representation of a triangle group. These are precisely the Fuchsian groups with signatures $(m, n, p; 3; 0)$; that is those Fuchsian groups generated by three elliptic maps. We will use *F-triangle group* to denote a Fuchsian triangle group. We then have from the corollary to Lemma B and from Theorem 2:

COROLLARY 3. (1) *If R is a subring of* $\mathbf{C}$ *with* $R \cap \mathbf{R} = Z$ *and T is an F-triangle subgroup of* $PSL_2(R)$, *then T must have at most cyclic intersection with all conjugates of M in* $PSL_2(R)$.

(2) *In* Γ_1 *an F-triangle group must also have non-trivial intersection with at least* 1 *conjugate of M.*

Proof. The triangle groups do not decompose as free products of cyclics. Thus if T is an F-triangle group and T had non-cyclic intersection with M it would decompose. This handles part (1). Further if in Γ_1 it had trivial intersection with all conjugates of M it follows from the proof of Theorem 2 that it either decomposes as a non-trivial free product of cyclics or is finite; both impossible for F-triangle groups.

3. The Euclidean Bianchi groups. If $d = 1, 2, 3, 7, 11$ the ring O_d has a Euclidean algorithm. For $d = 2, 7, 11$ the groups are HNN groups while Γ_3 does not decompose as either an HNN group or a free product with amalgamation [**6**]. In this section we show that the results on embedding Fuchsian groups in Γ_1 can be extended to these groups, with again the modular group playing a prominent role. A similar result does not follow directly for Γ_3. First:

THEOREM 3. *Let F be a torsion-free Fuchsian subgroup of* Γ_d, $d = 2, 7, 11$. *Then F is free unless it has at most cyclic intersection with all conjugates of M in Γ_d and non-trivial intersection with at least one conjugate of M.*

Proof. Let F be torsion-free and Fuchsian in Γ_d, $d = 2, 7, 11$. If F contains parabolics it must be a free group so suppose that F is totally hyperbolic.

Since $\Gamma_d \cap PSL_2(\mathbf{R}) = M$ Lemma A applies so if $F \cap g^{-1}Mg$ is non-cyclic for some g in Γ_d then $F \subset g^{-1}Mg$. Since F is torsion-free it then must be free as in the proof of Theorem 1.

Next suppose that $F \subset \Gamma_d$ intersects all conjugates of the modular group M trivially.

From [**6**] we have that for $d = 2, 7, 11$, Γ_d is an HNN group with respective base groups K_2, K_7, K_{11}. The free part in each case has rank 1 and each has a single associated subgroup isomorphic to the modular group M. Further generators can be chosen so that in terms of these generators the associated subgroup is precisely M.

Now since F intersects all conjugates of M trivially and since M is the only associated subgroup we have from the Karrass-Solitar subgroup theorems on HNN groups [**14**] that F is a free product of a free group and conjugates of subgroups of the base.

If $d = 2$, the base K_2 is itself a free product with amalgamation [**6**].

$$K_2 = H_1 \underset{H}{*} H_2$$

with

$$H_1 = D_2,\ H_2 = A_4 \text{ and } H = Z_2.$$

Since F is torsion-free its intersections with conjugates of K_2 are also torsion-free. However K_2 is a free product with amalgamation of finite groups and thus K_2 has the torsion-free subgroup property.

Therefore F is a free product of free groups and thus must be free.

An identical argument works for Γ_7 and Γ_{11}. From **[6]** we have the structure of the base groups K_7 and K_{11}.

$$K_7 = H_1 \underset{H}{*} H_2$$

with

$$H_1 = S_3,\ H_2 = S_3 \text{ and } H = Z_2$$

and

$$K_{11} = H_1 \underset{H}{*} H_2$$

with

$$H_1 = A_4,\ H_2 = A_4 \text{ and } H = Z_3.$$

Thus both of these bases are free products with amalgamation of finite groups and thus have the torsion-free subgroup property.

As in Γ_1 the non-torsion-free cases must be modified slightly.

THEOREM 4. *Let F be a finitely generated Fuchsian subgroup of Γ_d with* $d = 2, 7$ *or* 11.

1) *If F has trivial intersection with all conjugates in Γ_d of the modular group M the F is either finite or a free product of cyclics.*

2) *If F has non-cyclic intersection with some conjugate of M then F is a non-trivial free product of cyclics.*

3) *The possible finite Fuchsian subgroups which can appear are the cyclic groups Z_2 and Z_3 in all three cases.*

Proof. Let F be finitely generated and contained in Γ_2, Γ_7 or Γ_{11}.

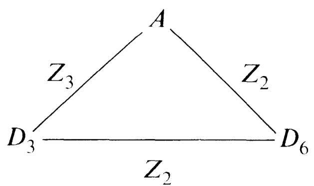
If F is Fuchsian and has non-cyclic intersection with some conjugate of M then the corollary to Lemma B holds and $F \subset g^{-1}Mg$. Thus F must be finite or a free product of cyclics. Since it has non-cyclic intersection with a conjugate of M it cannot be finite so it must be a non-trivial free product of cyclics. This handles part 2).

Further a finite subgroup of an HNN group must be conjugate to a finite subgroup of the base. The finite cyclic subgroups of the base groups in all three cases are Z_2 or Z_3. Since a finite Fuchsian group must be cyclic if F is finite it must be one of these.

Next suppose F is Fuchsian, $F \subset \Gamma_d$ and F has trivial intersection with all conjugates of M. Since Γ_d is an HNN group with single associated subgroup M, from the Karrass-Solitar subgroup theorems it follows that F is a free product of free groups and subgroups of conjugates of the base. As before since finitely generated Fuchsian groups are not free products unless free products of cyclics, then F is either a free product of cyclics or $F \subset g^{-1}K_dg$ where K_d is the base.

By conjugation we can assume a conjugate $F^* \subset K_d$. If F^* is not finite or a free product of cyclics it is an F-group. By the Fenchel-Fox Theorem it contains a surface group of finite index. But each of the base groups K_2, K_7 and K_{11} have the torsion-free subgroup property. Therefore in this case F must be finite or a free product of cyclics.

The remaining Euclidean case $d = 3$ must be handled differently since Karrass-Solitar [**6**] have shown that Γ_3 is not a free product with amalgamation or an HNN group. However Brunner, Lee and Wielenberg [**3**] have shown that $PGL_2(O_3)$ is a *triangular product* (see [**3**] for terminology) and Γ_3 is of course a subgroup of $PGL_2(O_3)$. In particular $PGL_2(O_3)$ has the structure



Although $PGL_2(O_3)$ has finite factors, triangular products do not necessarily satisfy the torsion-free subgroup property and there are non-free torsion-free subgroups of Γ_3 [**2**]. Therefore the same type of analysis as in the other Euclidean cases does not go through. However Lemmas A and B are still valid so:

THEOREM 5. *Suppose $F \subset \Gamma_3$ is Fuchsian. Then if F has non-cyclic intersection with some conjugate of M then F is a non-trivial free product of cyclics.*

4. The non-Euclidean cases. The remaining O_d, $d \neq 1, 2, 3, 7, 11$ are non-Euclidean. A method to compute presentations for $\Gamma_d = PSL_2(O_d)$ in all cases was given by Swan [**22**]. These methods depend on developing a geometric *Bianchi diagram* for Γ_d. (See [**22**] for terminology.) Several cases were explicitly worked out by Swan and more recently by Floge [**10**]. In these cases ($d = 5, 6, 10, 19$), Γ_d can again be described as an HNN group or free product with amalgamation. However the modular group, while appearing either as one of several associated subgroups or as part of the amalgamated subgroup does not have the pivotal role in the structure of Γ_d

that it has in the Euclidean cases. Therefore we cannot conclude that a non-free, torsion-free Fuchsian subgroup must have cyclic intersection with all conjugates of M. Whether it is possible to have a faithful Fuchsian representation of ϕ_g in Γ_d with trivial intersection with all conjugates of M is an open question. However Lemmas A and B are valid for all the Γ_d so we can extend Theorem 5.

THEOREM 5′. *For all $d > 0$, d square-free, if F is Fuchsian and $F \subset \Gamma_d$ then*

(1) *If F has non-cyclic intersection with some conjugate of M in Γ_d and F is torsion-free then F is free.*

(2) *If F has non-cyclic intersection with some conjugate of M in Γ_d then F is a non-trivial free product of cyclics.*

The non-Euclidean cases are precisely those where

$$PE_2(O_d) \neq PSL_2(O_d)$$

where $PE_2(O_d)$ is the projective elementary group. This is the subgroup of $PSL_2(O_d)$ generated by the images of the elementary matrices. If $d \neq 1, 2, 3, 7, 11$ it was shown in [**4**] that all the $PE_2(O_d)$ are isomorphic. In [**4**] it was shown that for all $d \neq 1, 2, 3, 7, 11$ $PE_2(O_d)$ has the presentation

$$PE_2(O_d) = \langle a, t, u; a^2 = (at)^3 = 1, u^{-1}tu = t\rangle$$

where a is the transformation $z' = -1/z$, t is $z' = z + 1$ and u is $z' = z + w$ where $\{1, w\}$ constitute an integral basis for O_d. Therefore $\{a, t\}$ generate the modular group M for all d and thus $PE_2(O_d)$ is an HNN group whose base is the modular group. Further it has free part of rank 1 and its only associated subgroup is a free group of rank 1. If F is Fuchsian and contained $PE_2(O_d)$ and if F has trivial intersection with all conjugates of M in $PE_2(O_d)$ it must be a free group. From [**14**] if a subgroup of an HNN group has trivial intersection with all conjugates of the base it must be a free group. Therefore:

THEOREM 6. *If F is a Fuchsian subgroup of $PE_2(O_d)$ with $d \neq 1, 2, 3, 7, 11$ then*

(1) *If F has trivial intersection with all conjugates of M is must be a free group.*

(2) *If F has non-cyclic intersection with some conjugate of M then F is a non-trivial free product of cyclics.*

Since $PE_2(O_d) \cap PSL_2(\mathbf{R}) = M$ part (2) follows from Lemmas A and B.

5. Normal Fuchsian subgroups. Our final result shows that no Fuchsian subgroup of a Bianchi group can be normal. In fact $PSL_2(R)$ contains no normal Fuchsian subgroups for any non-real subring R of C.

THEOREM 7. *If R is a non-real subring of* **C** *then $PSL_2(R)$ contains no normal Fuchsian subgroup.*

Proof. Let $G = PSL_2(R)$ and suppose $F \subset G$ is Fuchsian with fixed circle C.

Since R is a subring of **C**, $1 \in R$ and thus G contains the translation

$$T:z' = z + 1.$$

If F were normal in G then $T^{-1}FT = F$ and thus $T^{-1}FT$ is also Fuchsian with the same fixed circle as F. However $T^{-1}FT$ fixes $T(C)$ and the interior of $T(C)$ since F fixes C and its interior. The same argument applies to show that $T^n(C)$ is also a fixed circle for F.

Since all of the fix points of the elements of F are either on C or inverse to C and $T^n(C)$ translates C parallel to the real axis and $T^n(C)$ is also a fixed circle of F it follows that C must be a line parallel to the real axis.

However R is assumed to be non-real so there is a non-real element w in R. Therefore the non-real translation $U:z' = z + w$ is in G. Conjugating F by the powers $\{U^n\}$ and using the same argument as above we get that if F were normal in G its fixed circle C would be a line parallel to the direction of the translation $z' = z + w$. Since w is non-real this is not parallel to the real axis and therefore F cannot be normal.

6. Closing questions. For the non-Euclidean Bianchi groups our methods left open the question as to whether there can be Fuchsian subgroups which have trivial intersection with all conjugates of M and are not free products of cyclics. We close by stating this formally.

(1) Is it possible to find a Fuchsian subgroup $F \subset \Gamma_d$, $d \neq 1, 2, 3, 7, 11$ which has trivial intersection with all conjugates of M and is not a free product of cyclics?

or

(2) Is it possible to find a faithful Fuchsian representation of ϕ_g (for some $g > 1$) in Γ_d which has trivial intersection with all conjugates of M.

Acknowledgement. I would like to thank the referee for many helpful comments and especially for suggesting Lemma B and its proof.

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