

AN INVERSION THEOREM FOR SET-VALUED MAPS

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The aim of this paper is to give an inversion theorem for set-valued maps involving both some known results for functions and set-valued maps. To do this we introduce a notion of strict differentiability for set-valued maps and we use a Newton like method assuming the derivative to be surjective. Moreover we prove the pseudo-Lipschitz regularity of the inverse.

1. INTRODUCTION

In this note, we prove an open mapping and inversion theorem for set-valued maps which involves the results of [15] and [23]. To do this we introduce a notion of differentiability for set-valued maps similar to the one given in [8] (see also [6], [13], [19], [20] and [22]). Some inversion theorems for multifunctions have already been proved in particular in [3], [5], [11] and also [20], [23]. The favourable situation occurs when the multifunction takes on its values in a finite dimensional space. When this is not the case the problem is slightly different. In the particular case of functions, it is proved (see [7], [12]) that the assumption of Halkin's Theorem [16] (surjectivity of the derivative) is not sufficient to obtain the open mapping and local inversion theorem. In this case some uniformity is needed near the point where the inversion is studied. When working with set-valued maps, this uniformity is obtained by introducing a notion of strict differentiability and the inversion result is proved with the help of an iteration procedure of Newton type quite similar to the one used in the proof of the Ljusternik - Graves Theorem ([15], [18] and also [1]).

2. STATEMENT AND PROOF OF THE RESULT

In the sequel X, Y denote Banach spaces with closed unit balls B_X, B_Y . For any non-void subsets C, D of a metric space (E, d) , the Hausdorff excess of C into D is defined by $e(C, D) = \sup_{x \in C} d(x, D)$ where $d(x, D) = \inf_{y \in D} d(x, y)$. We also set $e(\emptyset, D) = 0$ and $e(C, \emptyset) = +\infty$ if $C \neq \emptyset$. A multifunction or set-valued map $F: X \rightrightarrows Y$ is a subset of $X \times Y$. For $x \in X$, $F(x) := \{y \in Y; (x, y) \in F\}$, the domain of F is the set $\text{dom } F = \{x \in X, F(x) \neq \emptyset\}$. A convex process is a multifunction $L: X \rightrightarrows Y$ such

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that L is a convex cone (see [24] for instance). The closed convex processes play the role of continuous linear maps in the theory of differentiability of multifunctions. A multifunction $F \rightrightarrows Y$ is said to be pseudo-Lipschitzian at $(x_0, y_0) \in F$ if there exists $\ell \geq 0$ and neighbourhoods U of x_0 and V of y_0 such that $U \subseteq \text{dom } F$ and, for each $x_1, x_2 \in U$, $e(F(x_1) \cap V, F(x_2)) \leq \ell \|x_1 - x_2\|$.

DEFINITION. We say that the multifunction $F: X \rightrightarrows Y$ is strictly differentiable at $x_0 \in \text{dom } F$ if there exists a closed convex process $L: X \rightrightarrows Y$ such that, for each $\varepsilon > 0$, there exist $r > 0$, $\alpha > 0$ such that

$$(1) \quad \forall x \in x_0 + rB_X, \forall u \in \alpha B_X, e(L(u) \cap B_Y + F(x), F(x + u)) < \varepsilon \|u\|.$$

REMARK: The process L is not unique since every subprocess $L' \subseteq L$ (in particular $L' = \{(0, 0)\}$) also satisfies (1).

EXAMPLES: 1. A map $f: U \rightarrow Y$, $U \subseteq X$ open, which is strictly differentiable at $x_0 \in U$ is also strictly differentiable as a multifunction at x_0 . More generally,

2. Let $f: U \rightarrow Y$ be strictly differentiable at x_0 . If we define $F(x) = f(x) + K$ where K is a convex closed cone, then F is strictly differentiable at x_0 with $L(u) = f'(x_0)(u) + K$. Indeed $e(f'(x_0)(u) + K + f(x) + K, f(x + u) + K) \leq \|f(x + u) - f(x) - f'(x_0)(u)\|$.

3. A closed convex process L is strictly differentiable at every $x_0 \in \text{dom } L$. The following inversion theorem extends both Theorem 1 of [23] and Theorem 1 of [15].

THEOREM. Assume that the multifunction F is closed and strictly differentiable at $x_0 \in \text{dom } F$ and that L is onto, then there exist $r > 0$, $\eta > 0$ such that

$$(2) \quad F(x_0) + \eta B_Y \subseteq F(x_0 + rB_X).$$

Moreover, F^{-1} is pseudo-Lipschitz near (y_0, x_0) for each $y_0 \in F(x_0)$ that is there exist $\eta_0, r_0, \ell_0 > 0$ such that $\forall y_1, y_2 \in y_0 + \eta_0 B_Y$

$$(3) \quad e(F^{-1}(y_1) \cap (x_0 + r_0 B_X), F^{-1}(y_2) \cap (x_0 + 2r_0 B_X)) \leq \ell_0 \|y_1 - y_2\|.$$

PROOF: We follow the lines of [1, Theorem 2.1]. As L is onto, there exists (see [4, Corollary 3 p. 132]) $\gamma > 0$ such that $B_Y \subseteq L(\gamma B_X)$. Thus, for each $v \in Y$, there exists $u \in X$ with $v \in L(u)$ and $\|u\| \leq \gamma \|v\|$, we write $u = L^{-1}(v)$. Consider now $0 < \varepsilon < 1$ and r, α associated with $\frac{\varepsilon}{\gamma}$ by (1). Let us choose $\eta > 0$ such that $\eta < \inf\left(1, \frac{\alpha}{\gamma}, \frac{r(1-\varepsilon)}{\gamma}\right)$. Using (1) we derive that

$$(4) \quad \forall v \in \eta B_Y, \forall x \in x_0 + rB_X, e(v + F(x), F(x + L^{-1}(v))) < \varepsilon \|v\|.$$

Let us consider $y_0 \in F(x_0)$ and $\bar{y} \in y_0 + \eta B_Y$. We set $x_{-1} = x_0$ and we assume that, for $n \in \mathbb{N}$ and $i \in [0, n]$

$$(5) \quad x_i \in x_0 + rB_X, y_i \in F(x_i), \|x_i - x_{i-1}\| \leq \gamma \|\bar{y} - y_0\| \varepsilon^{i-1}, \|y_i - \bar{y}\| \leq \varepsilon^i \|y_0 - \bar{y}\|.$$

Replacing in (4) v by $\bar{y} - y_n$, x by x_n and using the fact that $y_n \in F(x_n)$ we obtain $d(\bar{y}, F(x_n + L^{-1}(\bar{y} - y_n))) < \varepsilon \|\bar{y} - y_n\|$. Thus there exists $y_{n+1} \in F(x_n + L^{-1}(\bar{y} - y_n))$ such that $\|y_{n+1} - \bar{y}\| \leq \varepsilon \|y_n - \bar{y}\|$. Let us set $x_{n+1} = x_n + L^{-1}(\bar{y} - y_n)$, then $\|x_{n+1} - x_n\| \leq \gamma \|\bar{y} - y_n\| \leq \gamma \|\bar{y} - y_0\| \varepsilon^n$. Hence (5) holds for $n+1$. The sequences (x_n) and (y_n) converge respectively to $\bar{x} \in x_0 + \frac{\gamma \|\bar{y} - y_0\|}{1-\varepsilon} B_X \subseteq x_0 + rB_X$ and to \bar{y} . As $y_n \in F(x_n)$ and F is closed, one has $\bar{y} \in F(\bar{x})$ and (2) holds. \blacksquare

Lipschitz dependence.

Let us assume now that $\eta < \inf\left(1, \frac{\alpha}{\gamma}, \frac{r(1-\varepsilon)}{2\gamma}\right)$. Thanks to the first step of the proof, there exists, for $y_1 \in y_0 + \frac{\eta}{2} B_Y$ an element $x_1 \in x_0 + \frac{r}{2} B_X$ such that $y_1 \in F(x_1)$. Consider $y_2 \in y_0 + \frac{\eta}{2} B_Y$, one has $y_2 \in y_1 + \eta B_Y$, $y_1 \in F(x_1)$ and, by (4) $\forall v \in \eta B_Y$, $\forall x \in x_1 + \frac{r}{2} B_X$, $e(v + F(x), F(x + L^{-1}(v))) < \varepsilon \|v\|$. Hence, by using the same device as in the first step, there exists $x_2 \in X$ such that $y_2 \in F(x_2)$ and $\|x_2 - x_1\| \leq \frac{\gamma}{1-\varepsilon} \|y_2 - y_1\|$, then (3) holds with $\eta_0 = \frac{\eta}{2}$, $r_0 = \frac{r}{2}$, $\ell_0 = \frac{\gamma}{1-\varepsilon}$.

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