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On the continuity of Arthur's trace formula: the semisimple terms

Tobias Finis and Erez Lapid

Abstract

We show that the semisimple part of the trace formula converges for a wide class of test functions.

1. Introduction

Let Γ be a lattice in a reductive Lie group G. The right regular representation of G on $L^2(\Gamma\backslash G)$ gives rise to integral operators

$$R(f)\varphi(x) = \int_G f(g)\varphi(xg)\,dg, \quad f \in L^1(G).$$

In particular, if Γ is a uniform lattice then $R(f)$ is of trace class for $f \in C_c^\infty(G)$ and $L^2(\Gamma\backslash G)$ decomposes discretely as $\bigoplus_{\pi\in\hat{G}} m(\pi)\pi$. Computing the trace in two different ways, Selberg established the *trace formula* identity

$$\sum_{[\gamma]} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\,dg = \sum_{\pi\in\hat{G}} m(\pi)\,\mathrm{tr}\,\pi(f),$$

where γ ranges over the conjugacy classes of Γ [Sel56]. This identity easily extends to the space of smooth functions on G whose derivatives are all in $L^1(G)$.

If Γ is a non-uniform lattice then the trace has to be regularized in order to derive the trace formula. In the case where $G = \mathrm{SL}_2(\mathbb{R})$, this was carried out by Selberg as well. The case of a general adelic quotient $G(F)\backslash G(\mathbb{A})$ of a reductive group G over a number field F was pursued by Arthur in his lifelong work on the trace formula.

In this paper we begin to examine the geometric side of the trace formula from a functional analytic point of view. Namely, we wish to extend the trace formula to test functions which are not necessarily compactly supported. A natural space is obtained by fixing an open subgroup K of $G(\mathbb{A}_{\mathrm{fin}})$ and considering right K-invariant functions on $G(\mathbb{A})$ (viewed as functions on the differentiable manifold $G(\mathbb{A})/K$) all of whose derivatives are in $L^1(G(\mathbb{A}))$.[1] More precisely, the topological space $\mathcal{C}(G(\mathbb{A}); K)$ is defined by the seminorms

$$\|f * X\|_{L^1(G(\mathbb{A}))}, \quad X \in \mathcal{U}(\mathfrak{g}).$$

[1] For the trace formula only functions which are invariant under conjugation by a maximal compact subgroup $\mathbf{K}$ of $G(\mathbb{A})$ need to be considered. This renders the apparent asymmetry inessential.

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Let $\mathcal{B}_m$ be a basis of $\mathcal{U}(\mathfrak{g})_{\leqslant m}$, where $m = [F : \mathbb{Q}] \dim_F G$. For $\gamma \in G(F)$, let $C_G^0(\gamma)$ be the identity component of the centralizer $C_G(\gamma)$ of γ in G. Let $G(F)_{\mathrm{ell}}$ (respectively $G(F)_{\mathrm{ss}}$) denote the set of elliptic (respectively semisimple) elements in $G(F)$. For any $\gamma \in G(F)_{\mathrm{ss}}$, let $M(\gamma)$ be the centralizer of the split part of the center of $C_G^0(\gamma)$, which is a Levi subgroup of G. Fix a maximal split torus T_0 of G defined over F. For any standard Levi subgroup M, the associated weight factor $v_M(x)$ is given by [Art05, (18.2)] (cf. also § 6 below). Note that $v_{M(\gamma)}$ is $C_G(\gamma, F)$-invariant (assuming that $M(\gamma)$ is a standard Levi subgroup). We refer to § 2.1 for unexplained concepts and notation.

Our main result is the following theorem.

THEOREM 1. *The semisimple part of the trace formula, given by the sum-integral*

$$\sum_{[\gamma]\subseteq G(F)_{\mathrm{ss}}} \int_{A_M C_G(\gamma,F)\backslash G(\mathbb{A})} \int_{A_G} f(zx^{-1}\gamma x) v_{M(\gamma)}(x)\, dz\, dx$$
$$= \sum_{[\gamma]\subseteq G(F)_{\mathrm{ss}}} \frac{\mathrm{vol}(A_M C_G^0(\gamma,F)\backslash C_G^0(\gamma,\mathbb{A}))}{[C_G(\gamma,F) : C_G^0(\gamma,F)]} \int_{C_G^0(\gamma,\mathbb{A})\backslash G(\mathbb{A})} \int_{A_G} f(zx^{-1}\gamma x) v_{M(\gamma)}(x)\, dz\, dx$$

over the semisimple conjugacy classes $[\gamma]$ of $G(F)$, where in each class we take a representative γ such that $M(\gamma)$ is a standard Levi subgroup of G, is absolutely convergent for any $f \in \mathcal{C}(G(\mathbb{A}); K)$. Moreover, there exists a constant c such that

$$\sum_{[\gamma]} \int_{A_M C_G(\gamma,F)\backslash G(\mathbb{A})} \int_{A_G} |f(zx^{-1}\gamma x)| v_{M(\gamma)}(x)\, dz\, dx \leqslant c \sum_{X\in\mathcal{B}_m} \|f * X\|_1. \tag{1}$$

In particular, this holds for the elliptic part of the trace formula

$$\int_{A_G G(F)\backslash G(\mathbb{A})} \int_{A_G} \sum_{\gamma\in G(F)_{\mathrm{ell}}} f(zx^{-1}\gamma x)\, dz\, dx$$
$$= \sum_{[\gamma]\subseteq G(F)_{\mathrm{ell}}} \mathrm{vol}(A_G C_G(\gamma,F)\backslash C_G(\gamma,\mathbb{A})) \int_{C_G(\gamma,\mathbb{A})\backslash G(\mathbb{A})} \int_{A_G} f(zx^{-1}\gamma x)\, dz\, dx,$$

where the sum is over the elliptic conjugacy classes of $G(F)$.

Implicit in the theorem is the choice of a maximal compact subgroup $\mathbf{K}$ of $G(\mathbb{A})$. For the analogous result on the spectral side (with a larger value of m), cf. [FLM09].

The essence of the argument is the convergence of the elliptic part, which will be proved in § 5. The general semisimple case is easily reduced to the elliptic case (§ 6). Roughly speaking, there are two main themes in the proof. The first is simply to bound sums over lattices by integrals. The second is to linearize the non-compact parameters of the problem. For the space $A_G G(F)\backslash G(\mathbb{A})$, this is done by choosing a Siegel set. For the set $G(F)_{\mathrm{ell}}$, we use the Bruhat decomposition. The ellipticity condition is used in a subtle, but crucial, way to chop off one dimension from the unipotent part of the Bruhat decomposition. Using Mellin inversion, the contribution from each cell is controlled by intertwining operators of principal series, whose properties are well understood. The argument somewhat resembles the one used by Langlands in his work on the Tamagawa number [Lan66]. It is different in flavor from the analysis of [Art78]. In particular, we do not use Arthur's partition lemma [Art78, Lemma 6.4].

Ultimately, we would like to extend Theorem 1 to the non-semisimple terms as well and obtain an expansion for the entire geometric side of the trace formula which is valid for any $f \in \mathcal{C}(G(\mathbb{A}); K)$. In the case where $G = \mathrm{GL}(2)$, this was carried out in [FL11].[2] In the general case, when using Arthur's fine geometric expansion [Art85, Art86] one faces two difficulties. First, it only applies to functions of the form $f_S \otimes \mathbf{1}^S$, where S is a finite set of places containing the Archimedean ones and $\mathbf{1}^S$ is the characteristic function of the maximal compact $\mathbf{K}^S$ outside S (rather than an arbitrary bi-$\mathbf{K}^S$-invariant function outside S). The second and more serious difficulty is that while the local distributions appearing in the expansion are explicit and fairly well understood, their coefficients are left unspecified, and they depend on S in a complicated way. For the problem at hand, it would be imperative to bound them in a uniform way.

2. Preliminaries

2.1 The setup

Let G be a reductive group over a number field F. By passing to the restriction of scalars, we will assume without loss of generality that $F = \mathbb{Q}$.[3] Let $\mathcal{O} = \prod_{p<\infty} \mathbb{Z}_p \subseteq \mathbb{A}_{\mathrm{fin}}$.

For a linear algebraic group H over $\mathbb{Q}$, we denote by δ_H the modulus function of $H(\mathbb{A})$. More generally, if H and X are subgroups of G and H normalizes X, we denote by $\delta_{H;X}$ the modulus function of the conjugation action of $H(\mathbb{A})$ on $X(\mathbb{A})$.

As usual, we denote Lie algebras by small Gothic letters. For example, Lie $G = \mathfrak{g}$, Lie $P = \mathfrak{p}$ etc. The universal enveloping algebra of $\mathfrak{g}$ with the usual grading will be denoted by $\mathcal{U}(\mathfrak{g})$. We denote by $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ the adjoint representation. The centralizer of x in H is denoted by $C_H(x)$, its identity component by $C_H^0(x)$ and its Lie algebra by $\mathfrak{c}_{\mathfrak{h}}(x)$.

Fix a maximal split torus T_0 defined over $\mathbb{Q}$ and a minimal parabolic subgroup P_0 defined over $\mathbb{Q}$ and containing T_0. We have a Levi decomposition $P_0 = M_0U_0$, where $M_0 = C_G(T_0)$ and U_0 is the unipotent radical of P_0. By a *standard parabolic*, we will always mean a parabolic subgroup containing P_0 and defined over $\mathbb{Q}$. Any standard parabolic admits a unique Levi subgroup M containing T_0, and moreover M is defined over $\mathbb{Q}$. Such a Levi subgroup is called *standard*. When P is standard, we always write $P = MU$ for its standard Levi decomposition.

Let $W = W^G$ be the Weyl group $N_G(T_0)(\mathbb{Q})/M_0(\mathbb{Q})$ of G and fix representatives $n_w \in N_G(T_0)(\mathbb{Q})$ for each $w \in W$. We have the Bruhat decomposition

$$G(\mathbb{Q}) = \coprod_{w \in W} \mathfrak{B}_w^G,$$

where

$$\mathfrak{B}_w^G = \mathfrak{B}_w = (U_0(\mathbb{Q})/U_w(\mathbb{Q}))n_w P_0(\mathbb{Q}) = P_0(\mathbb{Q})n_w P_0(\mathbb{Q})$$

and $U_w = U_0 \cap wU_0w^{-1}$. We write δ_w for the modulus function of $M_0(\mathbb{A})$ on $U_0(\mathbb{A})/U_w(\mathbb{A})$. In particular, for the longest element w_0 of W, $\delta_0 = \delta_{w_0}$ is the modulus function of $P_0(\mathbb{A})$, and $\delta_w \delta_{M_0;U_w} = \delta_0$ for all w. We write A_0 for the identity component of $T_0(\mathbb{R})$.

We let $\mathfrak{a}_0^*$ be the vector space $X^*(T_0) \otimes \mathbb{R} = X^*(M_0) \otimes \mathbb{R}$ which is isomorphic to the group of continuous homomorphisms $T_0(\mathbb{Q})\backslash T_0(\mathbb{A}) \to \mathbb{R}_{>0}$. (Each such homomorphism extends uniquely to $M_0(\mathbb{Q})\backslash M_0(\mathbb{A})$.) The homomorphism corresponding to $\chi \in X^*(T_0)$ is $|\chi|_{\mathbb{A}}$. Complexifying, to any $\lambda \in \mathfrak{a}_{0,\mathbb{C}}^*$ we attach a continuous homomorphism $M_0(\mathbb{A}) \to \mathbb{C}^*$, denoted $m \mapsto m^\lambda$, which we can of course pull back to $P_0(\mathbb{A})$. Let $\mathfrak{a}_0 = X_*(T_0) \otimes \mathbb{R}$ be the dual space of $\mathfrak{a}_0^*$. The Weyl group W acts naturally on $X^*(T_0)$, $\mathfrak{a}_0^*$ and $\mathfrak{a}_0$. Let $\rho_0 \in \mathfrak{a}_0^*$ correspond to $\delta_0^{\frac{1}{2}}$.

[2] We would also like to mention a related work by Werner Hoffmann in this direction [Hof08].
[3] This is only used for ease of notation.

Let $R(T_0,U_0)\subseteq X^*(T_0)$ denote the set of roots of T_0 on U_0, $R_{\mathrm{red}}(T_0,U_0)$ the set of reduced roots and Δ_0 the set of simple roots. The standard parabolic subgroups of G correspond to subsets of Δ_0 by attaching to $P=MU$ the set of simple roots $\Delta_0^P=\Delta_0^M$ of T_0 on $U_0^M:=U_0\cap M$. Let T_M be the maximal split torus in the center of M and let A_M be the identity component of $T_M(\mathbb{R})$, which we identify with a Euclidean space. We also write $P_0^M=P_0\cap M$, $T_0^M=T_0\cap M^{\mathrm{der}}$, $\mathfrak{a}_0^M=X^*(T_0^M)\otimes\mathbb{R}$ and $A_0^M=A_0\cap M^{\mathrm{der}}(\mathbb{A})$, where M^{der} is the derived group of M. The Weyl group W^M of M is identified with a subgroup of W. If M_1, M_2 are two standard Levis, then $M_1\cap M_2$ is also a standard Levi, $\Delta_0^{M_1\cap M_2}=\Delta_0^{M_1}\cap\Delta_0^{M_2}$ and $W^{M_1\cap M_2}=W^{M_1}\cap W^{M_2}$.

Let $\Delta_0^\vee\subseteq X_*(T_0)\subseteq\mathfrak{a}_0$ denote the set of simple co-roots, similarly for $(\Delta_0^M)^\vee$. Note that $\mathfrak{a}_0^M$ is spanned by $(\Delta_0^M)^\vee$. Denote by $\widehat{\Delta}_0=\{\varpi_\alpha\}_{\alpha\in\Delta_0}$ the dual basis of $\Delta_0^\vee$ in $(\mathfrak{a}_0^G)^*$. Similarly, let $\widehat{\Delta}_0^\vee=\{\varpi_\alpha^\vee\}_{\alpha\in\Delta_0}$ denote the dual basis of Δ_0 in $\mathfrak{a}_0^G$.

Lemma 2.1. *We have*

$$\delta_0^{\frac{1}{2}}(awa^{-1}w^{-1})=\delta_w(waw^{-1})^{-1}=\delta_{w^{-1}}(a)$$

for all $a\in A_0$.

Proof. Let U_0' be the unipotent radical of the parabolic subgroup opposite to P_0 containing T_0. Since $U_0=U_w(U_0\cap wU_0'w^{-1})$, we have

$$\delta_w=\delta_{M_0;U_0\cap wU_0'w^{-1}}$$

and

$$\delta_0=\delta_{M_0;U_w}\delta_{M_0;U_0\cap wU_0'w^{-1}}.$$

Using these equalities for waw^{-1} and noting that $wU_{w^{-1}}w^{-1}=U_w$, we get

$$\delta_w(waw^{-1})=\delta_{M_0;w^{-1}U_0w\cap U_0'}(a)=\delta_{M_0;w^{-1}U_0'w\cap U_0}(a)^{-1}=\delta_{w^{-1}}(a)^{-1}$$

and

$$\delta_0(waw^{-1})=\delta_{M_0;U_{w^{-1}}}(a)\delta_{M_0;w^{-1}U_0w\cap U_0'}(a)=\delta_{M_0;U_{w^{-1}}}(a)\delta_w(waw^{-1}).$$

Similarly, $U_0=U_{w^{-1}}(U_0\cap w^{-1}U_0'w)$, so that

$$\delta_0(a)=\delta_{M_0;U_{w^{-1}}}(a)\delta_{M_0;U_0\cap w^{-1}U_0'w}(a)=\delta_{M_0;U_{w^{-1}}}(a)\delta_w(waw^{-1})^{-1}.$$

Together, we get

$$\delta_0(awa^{-1}w^{-1})=\delta_w(waw^{-1})^{-2},$$

as required. □

For any $w\in W$, let $Q(w)=L(w)V(w)$ be the smallest standard parabolic subgroup containing n_w. It is defined by

$$\Delta_0^{Q(w)}=\{\alpha\in\Delta_0:\exists\beta\in\Phi_w\text{ such that }\langle\varpi_\alpha,\beta^\vee\rangle>0\},$$

where $\Phi_w=\{\alpha\in R_{\mathrm{red}}(T_0,U_0):w\alpha<0\}$. Note that $U_w\supseteq V(w)$ and therefore

$$U_0^{L(w)}/U_w^{L(w)}\simeq U_0/U_w. \tag{2}$$

An alternative characterization of $Q(w)$ and $L(w)$ is obtained by noting that n_w belongs to the standard parabolic Q_α with $\Delta_0^{Q_\alpha}=\Delta_0\backslash\{\alpha\}$ (or, equivalently, to the corresponding standard Levi M_α) if and only if w preserves the set $\{\beta\in R(T_0,U_0):\langle\beta,\varpi_\alpha^\vee\rangle>0\}$, which is in turn equivalent to $w\varpi_\alpha^\vee=\varpi_\alpha^\vee$. This implies that

$$\Delta_0^{Q(w)}=\{\alpha\in\Delta_0:w\varpi_\alpha^\vee\neq\varpi_\alpha^\vee\}.$$

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The following lemma is an easy consequence.

Lemma 2.2. *Let $w \in W$ and $L = L(w)$. Suppose that λ_0 is in the positive Weyl chamber of $\mathfrak{a}_0^*$. Then $(1-w)\lambda_0 = \sum_{\alpha \in \Delta_0^L} c_\alpha \alpha$ with $c_\alpha > 0$.*

Proof. We have $(1-w)\lambda_0 = \sum_{\alpha \in \Delta_0} c_\alpha \alpha$, where

$$c_\alpha = \langle (1-w)\lambda_0, \varpi_\alpha^\vee \rangle = \langle \lambda_0, (1-w^{-1})\varpi_\alpha^\vee \rangle.$$

It is well known that $(1-w^{-1})\varpi_\alpha^\vee$ is a non-negative combination of co-roots. Hence, $\langle \lambda_0, (1-w^{-1})\varpi_\alpha^\vee \rangle \geqslant 0$ and equality holds if and only if $(1-w^{-1})\varpi_\alpha^\vee = 0$. The lemma follows. □

2.2 Reduction theory

Fix a maximal compact subgroup $\mathbf{K} = \prod_{p \leqslant \infty} K_p$ of $G(\mathbb{A})$ such that K_p is special for all $p < \infty$ and hyperspecial for almost all p. Let $c > 0$ and set

$$\mathcal{S}_c = \{pk : p \in P_0(\mathbb{A}), k \in \mathbf{K}, |\alpha|(p) > c \text{ for all } \alpha \in \Delta_0\}.$$

This set is left $P_0(\mathbb{Q})$-invariant and right $\mathbf{K}$-invariant.[4] By reduction theory, $G(\mathbb{Q})\mathcal{S}_c = G(\mathbb{A})$ provided that c is chosen sufficiently small. Let

$$A_c = \{a \in A_G \backslash A_0 : \alpha(a) > c \text{ for all } \alpha \in \Delta_0\}.$$

Using the Iwasawa decomposition, for any left $A_G G(\mathbb{Q})$-invariant non-negative measurable function φ on $G(\mathbb{A})$ we have

$$\begin{aligned}\int_{A_G G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi(g)\, dg &\leqslant \int_{A_G P_0(\mathbb{Q})\backslash \mathcal{S}_c} \varphi(g)\, dg \\ &= \int_{\mathbf{K}} \int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} \int_{M_0(\mathbb{Q})\backslash M_0(\mathbb{A})^1} \int_{A_c} \varphi(uamk)\delta_0(a)^{-1}\, da\, dm\, du\, dk \quad (3)\end{aligned}$$

provided that the right-hand side converges.

2.3 Elliptic elements

Recall that a semisimple element $s \in G(\mathbb{Q})$ is called *elliptic* if $Z(C_G^0(s))/Z(G)$ is anisotropic. We denote by $G(\mathbb{Q})_{\text{ell}}$ the set of elliptic elements of $G(\mathbb{Q})$.

Lemma 2.3. *Let s be a semisimple element of $G(\mathbb{Q})$. The following conditions are equivalent.*

(i) *s is elliptic.*

(ii) *$C_G^0(s)$ is not contained in any proper parabolic subgroup of G defined over $\mathbb{Q}$.*

(iii) *$\mathfrak{c}_{\mathfrak{g}}(s)$ is not contained in any proper parabolic subalgebra of $\mathfrak{g}$ defined over $\mathbb{Q}$.*

(iv) *$(1 - \operatorname{Ad} s)(\mathfrak{g})$ does not contain the nilradical of any proper parabolic subalgebra defined over $\mathbb{Q}$.*

Moreover, if we replace 'parabolic' by 'standard parabolic', the conditions (ii)–(iv) are still equivalent.

Proof. If s is not elliptic, then the maximal split torus T of $Z(C_G^0(s))$ strictly contains T_G. Hence, $C_G(T)$ is a proper Levi subgroup defined over $\mathbb{Q}$ containing $C_G^0(s)$. Conversely, if $C_G^0(s)$ is contained in a proper parabolic P defined over $\mathbb{Q}$, then by the results of [Mos56] we can find a

[4] A fundamental domain for $P_0(\mathbb{Q})\backslash\mathcal{S}_c$ is called a Siegel set.

Levi subgroup M of P defined over $\mathbb{Q}$ containing $C_G^0(s)$. Thus, $Z(C_G^0(s)) \supseteq Z(M)$ and therefore s is not elliptic.

The equivalence of (ii) and (iii) is immediate.

Finally, to show the equivalence of (iii) and (iv), we note that if $\mathfrak{u}$ is the nilradical of a parabolic subalgebra $\mathfrak{p}$ then its orthogonal complement $\mathfrak{u}^\perp$ with respect to the Killing form is $\mathfrak{p}$ while $[(1 - \operatorname{Ad} s)(\mathfrak{g})]^\perp = \mathfrak{c}_\mathfrak{g}(s)$. $\square$

Definition 2.4. We denote by $G(\mathbb{Q})_{\mathrm{well}}$ the set of semisimple elements γ of $G(\mathbb{Q})$ such that $C_G^0(s)$ is not contained in any proper standard parabolic subgroup. Thus, $G(\mathbb{Q})_{\mathrm{ell}} \subseteq G(\mathbb{Q})_{\mathrm{well}}$ and $G(\mathbb{Q})_{\mathrm{well}}$ is stable under conjugation by $P_0(\mathbb{Q})$.

For any $w \in W$ with $Q(w) = LV$, we set

$$\mathfrak{B}_w^L = L(\mathbb{Q}) \cap \mathfrak{B}_w = P_0^L(\mathbb{Q}) n_w P_0^L(\mathbb{Q}) = (U_0^L(\mathbb{Q})/U_w^L(\mathbb{Q})) n_w P_0^L(\mathbb{Q})$$

and

$$\mathfrak{E}_w = G(\mathbb{Q})_{\mathrm{well}} \cap \mathfrak{B}_w^L.$$

Recall that if $P = MU$ is a Levi decomposition of a parabolic subgroup of G and $m \in P(\mathbb{Q})$ is semisimple, then there exists $u \in U(\mathbb{Q})$ such that $umu^{-1} \in M(\mathbb{Q})$ (e.g. [Mos56]). Thus, denoting by $\kappa_s : G \to G$ the commutator map

$$\kappa_s(x) = s^{-1} x s x^{-1},$$

we infer that

$$G(\mathbb{Q})_{\mathrm{well}} \cap \mathfrak{B}_w = \bigcup_{m \in \mathfrak{E}_w} m \kappa_m(V(\mathbb{Q})). \tag{4}$$

3. The space $\mathcal{C}(G(\mathbb{A}); K)$

3.1 Fourier analysis

We begin with an elementary and well-known identity from harmonic analysis of the real line. Let $f \in C_c^\infty(\mathbb{R})$ and $\hat{f}$ its Fourier–Laplace transform

$$\hat{f}(s) = \int_\mathbb{R} e^{-sx} f(x)\, dx.$$

Then for any $s_0 > 0$ we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} s = s_0} \frac{\hat{f}(s)}{s}\, ds = \int_{-\infty}^0 f(x)\, dx. \tag{5}$$

This follows by applying the isometry relation of the Fourier transform

$$\int_\mathbb{R} f_1(x) \overline{f_2(x)}\, dx = \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} s = 0} \hat{f}_1(s) \overline{\hat{f}_2(s)}\, ds$$

to the L^2-functions $f_1(x) = f(x) e^{-s_0 x}$ and $f_2(x) = e^{s_0 x} \chi_{[-\infty, 0]}(x)$, observing that $\hat{f}_1(s) = \hat{f}(s_0 + s)$ and $\hat{f}_2(s) = 1/(s_0 - s)$.

It follows that the left-hand side of (5) which *a priori* makes sense only for $f \in C_c^\infty(\mathbb{R})$ extends to a continuous linear functional on $L^1(\mathbb{R})$.

Consider the higher-dimensional case. Let V be a Euclidean space and V^* its dual space. We consider the Frechet space $\mathcal{C}(V)$ consisting of smooth functions f on V such that $\|f * D\|_{L^1(V)} < \infty$ for any invariant differential operator D on V.

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We say that a function f on a subset A of $V_{\mathbb{C}}^*$ is of *moderate growth* if there exist n and c such that $|f(\lambda)| \leqslant c(1+\|\lambda\|)^n$. Similarly, f is *rapidly decreasing* if for all n there exists c_n such that $|f(\lambda)| \leqslant c_n(1+\|\lambda\|)^{-n}$ on A. For $f \in C_c^\infty(V)$, let

$$\hat{f}(\lambda) = \int_V e^{-\langle \lambda, v\rangle} f(v)\, dv, \quad \lambda \in V_{\mathbb{C}}^*.$$

It is rapidly decreasing for $\operatorname{Re}\lambda$ in a bounded set. Let I be a linearly independent set in V and V_I its linear span. Let $\lambda_0 \in V^*$ be such that $\langle \lambda_0, u\rangle > 0$ for all $u \in I$. As before, for any $f \in C_c^\infty(V)$ we have

$$\nu_I \int_{\operatorname{Re}\lambda=\lambda_0} \frac{\hat{f}(\lambda)}{\prod_{u\in I}\langle \lambda, u\rangle}\, d\lambda = \int_{V_I} f(x)\chi_I(x)\, dx, \tag{6}$$

where χ_I is the characteristic function of

$$\left\{\sum_{u\in I} \alpha_u u : \alpha_u \leqslant 0 \text{ for all } u\right\},$$

$\nu_I = \operatorname{vol}_{V_I}(\{\sum_{u\in I}\alpha_u u : 0 \leqslant \alpha_u \leqslant 1 \text{ for all } u\})$ and the measure on $\operatorname{Re}\lambda = \lambda_0$ is obtained by identifying it with V^* via $v \mapsto \operatorname{Im} v$ and taking the Haar measure on V^* dual to that on V. To prove (6), we first reduce to the case $V_I = V$ by taking the restriction g of f to V_I and noting that $\hat{g}(\lambda) = \int_{\mathrm{i}V_I^\perp} \hat{f}(\lambda+\mu)\, d\mu$. In the case $V_I = V$, the identity (6) follows by applying the isometry relation of the Fourier transform to the inner product of $f_1(v) = f(v)e^{-\langle \lambda_0, v\rangle}$ and $f_2(v) = e^{\langle \lambda_0, v\rangle}\chi_I(v)$, observing that $\hat{f}_1(\lambda) = \hat{f}(\lambda_0+\lambda)$ and $\hat{f}_2(\lambda) = \nu_I/(\prod_{u\in I}\langle \lambda_0 - \lambda, u\rangle)$.

Proposition 3.1. *Let $r > 0$, $\mu_0 \in V^*$ and h be a holomorphic function of moderate growth on the tube $\|\operatorname{Re}\lambda - \mu_0\| < r$. Let S be a linearly independent set in V and let $\lambda_0 \in V^*$ be such that $\|\lambda_0 - \mu_0\| < r$ and $\langle \lambda_0 - \mu_0, u\rangle > 0$ for all $u \in S$. Then*

$$f \mapsto \int_{\operatorname{Re}\lambda=\lambda_0} \frac{\hat{f}(\lambda-\mu_0)h(\lambda)}{\prod_{u\in S}\langle \lambda - \mu_0, u\rangle}\, d\lambda \tag{7}$$

extends to a continuous functional on $\mathcal{C}(V)$.

Proof. Without loss of generality, we may assume that $\mu_0 = 0$. The statement is clear for $S = \emptyset$. Indeed, if $h(\lambda) \leqslant c(1+\|\lambda\|)^n$ then the integral is majorized by $\sum_D \|\widehat{f * D}\|_\infty \leqslant \sum_D \|f * D\|_1$, where D ranges over a basis of the invariant differential operators of degree $\leqslant n + \dim V + 1$.

Consider the general case. Fix elements $\varpi_u \in V^*$, $u \in S$ such that $\langle \varpi_u, u'\rangle = \delta_{u,u'}$ for all $u, u' \in S$. For any $I \subseteq S$, let $\lambda^I = \lambda - \sum_{u\in I}\langle \lambda, u\rangle \varpi_u$. Then $\lambda \mapsto \lambda^I$ is a projection on the annihilator $I^\perp$ of I whose kernel is the span of ϖ_u, $u \in I$. The annihilator W_I of $\{\varpi_u, u \in I\}$ is a complement of V_I in V. We define

$$h_{S,I}(\lambda) = \frac{\sum_{I\subseteq J\subseteq S}(-1)^{|J|-|I|}h(\lambda^J)}{\prod_{u\in S\setminus I}\langle \lambda, u\rangle}.$$

Then $h_{S,I}(\lambda)$ depends only on λ^I and is a holomorphic function of moderate growth on any tube $\|\operatorname{Re}\lambda\| < r'$, $r' < r$. By inclusion–exclusion, (7) is equal to the sum over $I \subseteq S$ of

$$\int_{\operatorname{Re}\lambda=\lambda_0} \frac{\hat{f}(\lambda)h_{S,I}(\lambda^I)}{\prod_{u\in I}\langle \lambda, u\rangle}\, d\lambda.$$

By splitting the integral and using (6), each summand is equal up to a constant to

$$\int_{\mathrm{i}I^{\perp}} \widehat{f_{\lambda_0}^I}(\lambda) h_{S,I}(\lambda)\, d\lambda,$$

where $f_{\lambda_0}^I \in C_c^\infty(W_I)$ is given by

$$f_{\lambda_0}^I(u) = \int_{V_I} \chi_I(v) f(u+v)\, dv, \quad u \in W_I,$$

and we identify $I^\perp$ with the dual space of W_I. The proposition follows from the previous case and the fact that the map $f \mapsto f_{\lambda_0}^I$ defines a continuous operator from $\mathcal{C}(V)$ to $\mathcal{C}(W_I)$. □

3.2 Sobolev estimates

Let now H be an affine algebraic group over $\mathbb{Q}$ and let K be an open subgroup of $H(\mathbb{A}_{\mathrm{fin}})$. Define $\mathcal{C}(H(\mathbb{A}); K)$ to be the Frechet space of smooth functions on $H(\mathbb{A})$ which are right K-invariant and such that $\|f * X\|_{L^1(H(\mathbb{A})/K)} < \infty$ for any $X \in \mathcal{U}(\mathfrak{h})$. (We usually write $\|f\|_1$ for the L^1-norm if the measure space is clear from the context.) In particular, let $\mu_0(f) = \sum_{X \in \mathcal{B}_H} \|f * X\|_1$, where $\mathcal{B}_H$ is a basis of $\mathcal{U}(\mathfrak{h})_{\leqslant \dim H}$.

The following lemma is clear.

Lemma 3.2. *Let $C \subseteq H(\mathbb{A})$ be a compact set. Then there exists an open subgroup K' of $H(\mathbb{A}_{\mathrm{fin}})$ such that for any continuous seminorm μ' of $\mathcal{C}(H(\mathbb{A}); K')$ there exists a continuous seminorm μ of $\mathcal{C}(H(\mathbb{A}); K)$ such that for all $f \in \mathcal{C}(H(\mathbb{A}); K)$ and $x \in C$, the function $f^x(g) = f(x^{-1}gx)$ belongs to $\mathcal{C}(H(\mathbb{A}); K')$ and $\mu'(f^x) \leqslant \mu(f)$.*

We will also need the following.

Lemma 3.3. *Let C be a compact neighborhood of the identity element of $H(\mathbb{A})$. Then there exists c such that for any $f \in \mathcal{C}(H(\mathbb{A}); K)$ and any $h \in H(\mathbb{A})$, we have*

$$|f(h)| \leqslant c \sum_{X \in \mathcal{B}_H} \int_C |(f * X)(hx)|\, dx. \tag{8}$$

Moreover,

$$\sum_{\gamma \in H(\mathbb{Q})} |f(h\gamma)| \leqslant c\mu_0(f)$$

for all $h \in H(\mathbb{A})$.

Proof. The second part follows from the first one by choosing a compact neighborhood C of e such that $C \cap \gamma C = \emptyset$ for any $1 \neq \gamma \in H(\mathbb{Q})$ and summing (8) over $h\gamma$, $\gamma \in H(\mathbb{Q})$.

To show (8), we can assume upon left translation that $h = e$. We can also assume that $C = C'K'$, where C' is a neighborhood of e in $H(\mathbb{R})$ and K' is an open subgroup of K. Using local coordinates and letting $n = \dim H$, the inequality reduces to the inequality

$$|f(0)| \leqslant \sum_{I \subseteq \{1,\dots,n\}} \int_{[0,1]^n} \left| \frac{\partial^{|I|} f}{\prod_{i \in I} \partial x_i}(x) \right| dx$$

for any smooth function on $\mathbb{R}^n$, which is turn follows from the identity

$$f(0) = \sum_I \int_{[0,1]^n} \frac{\partial^{|I|} f}{\prod_{i \in I} \partial x_i}(x) \prod_{i \in I}(x_i - 1)\, dx,$$

which can be shown by induction on n using integration by parts. □

From now on, we fix a compact open subgroup K of $G(\mathbb{A}_{\mathrm{fin}})$.

Lemma 3.4. *There exist constants c_X, $X \in \mathcal{U}(\mathfrak{g})$ such that for any $f \in \mathcal{C}(G(\mathbb{A}); K)$ there exists $\tilde{f} \in \mathcal{C}(G(\mathbb{A}); K)$ satisfying:*

(i) *$\tilde{f}$ is right $\mathbf{K}$-invariant;*

(ii) *$\tilde{f}(x) \geqslant |f(x)|$ for all $x \in G(\mathbb{A})$;*

(iii) *$\|\tilde{f} * X\|_1 \leqslant c_X \mu_0(f)$ for all $X \in \mathcal{U}(\mathfrak{g})$.*

Moreover, if $f \in C_c^\infty(G(\mathbb{A}))$, then we can choose $\tilde{f} \in C_c^\infty(G(\mathbb{A}))$.

Proof. Choose a compact symmetric neighborhood C of e in $G(\mathbb{A})$ and let c be as in the previous lemma for $H = G$. Fix a non-negative right $\mathbf{K}$-invariant function $\varphi \in C_c^\infty(G(\mathbb{A}))$ such that $\varphi \geqslant c$ on C. Let

$$\tilde{f} = \sum_{X \in \mathcal{B}_G} |f * X| * \varphi.$$

By construction, $\tilde{f}$ is right $\mathbf{K}$-invariant. Moreover, it follows from (8) that $\tilde{f}(g) \geqslant |f(g)|$. Finally,

$$\|\tilde{f} * Y\|_1 \leqslant \mu_0(f)\|\varphi * Y\|_1$$

for any $Y \in \mathcal{U}(\mathfrak{g})$. The lemma follows. □

Corollary 3.5. *Suppose that ν is a Radon measure on $G(\mathbb{A})$. Then $\int_{G(\mathbb{A})} f(x)\, d\nu(x)$ is continuous on $\mathcal{C}(G(\mathbb{A}); K)$ if and only if there exists a continuous seminorm μ on $\mathcal{C}(G(\mathbb{A}); K)$ such that $\int_{G(\mathbb{A})} f(x)\, d\nu(x) \leqslant \mu(f)$ for all right $\mathbf{K}$-invariant non-negative $f \in C_c^\infty(G(\mathbb{A}); K)$. In this case, $\int_{G(\mathbb{A})} |f(x)|\, d\nu(x)$ is a continuous seminorm with respect to μ_0.*

3.3 Principal series and intertwining operators

Consider the principal series representation $I(\lambda)$, $\lambda \in \mathfrak{a}_{0,\mathbb{C}}^*$, which is parabolically induced from the character p^λ on $P_0(\mathbb{A})$ (normalized induction). Explicitly,

$$I(\lambda) = \{\varphi : G(\mathbb{A}) \to \mathbb{C} \text{ smooth} \mid \varphi(pg) = p^{\lambda+\rho_0}\varphi(g) \text{ for all } p \in P_0(\mathbb{A}), g \in G(\mathbb{A})\}.$$

We can construct $I(\lambda)$ as follows. Let $f \in C_c^\infty(G(\mathbb{A}))$ and set

$$F_f(g) = \int_{P_0(\mathbb{A})^1} f(pg)\, dp.$$

Then for any $\lambda \in \mathfrak{a}_{0,\mathbb{C}}^*$

$$\varphi(\lambda)(g) = \int_{A_0} F_f(ag) a^{-(\lambda+\rho_0)}\, da$$

belongs to $I(\lambda)$.

Lemma 3.6. *Let $\tilde{F}_f = \delta_0^{-1} F_f|_{A_0}$. If f is right $\mathbf{K}$-invariant, then $\|\tilde{F}_f * X\|_1 \leqslant \|f * X\|_1 + 2|\langle \rho_0, X\rangle| \|f\|_1$ for any $X \in \operatorname{Lie} A_0$. If $f \in C_c^\infty(G(\mathbb{A}))$, then $\varphi(\lambda)(e) = \widehat{\tilde{F}_f}(\lambda - \rho_0)$ for $\lambda \in \mathfrak{a}_{0,\mathbb{C}}^*$.*

Proof. Note that $\tilde{F}_f(a) = \int_{P_0(\mathbb{A})^1} f(ap)\, dp$. Thus, $\tilde{F}_f * X = \tilde{F}_{f*X} - \langle 2\rho_0, X\rangle \tilde{F}_f$ for any $X \in \operatorname{Lie} A_0$. If f is right $\mathbf{K}$-invariant, then we have

$$\|\tilde{F}_f\|_1 = \int_{A_0} |F_f(a)|\delta_0(a)^{-1}\, da \leqslant \int_{A_0} \int_{P_0(\mathbb{A})^1} |f(ap)|\, dp\, da = \|f\|_1.$$

The last part is immediate. □

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Remark 1. The integral defining F_f converges for any $f \in \mathcal{C}(G(\mathbb{A}); K)$. However, already for $G = \mathrm{GL}(2)$ the integral defining $\varphi(\lambda)$ will not converge for $\mathrm{Re}\,\lambda \neq \rho_0$ for all $f \in \mathcal{C}(G(\mathbb{A}); K)$ since $\tilde{F}_f$ can be an arbitrary function in $\mathcal{C}(A_0)$ and its Fourier–Laplace transform is in general only defined on the imaginary line.

The intertwining operators

$$M(w, \lambda) : I(\lambda) \to I(w\lambda)$$

are given by the meromorphic continuation of

$$M(w, \lambda)\varphi(g) = \int_{U_w(\mathbb{A})\backslash U_0(\mathbb{A})} \varphi(n_w^{-1} ug)\, du.$$

This integral converges if $\mathrm{Re}\,\lambda - \rho_0$ lies in the positive Weyl chamber. If φ is right $\mathbf{K}$-invariant, then

$$[M(w, \lambda)\varphi](e) = m(w, \lambda)\varphi(e),$$

where we can write

$$m(w, \lambda) = \prod_{\alpha \in \Phi_w} m_\alpha(\langle \lambda, \alpha^\vee \rangle).$$

For any α, let M_α be the Levi subgroup such that $\mathfrak{a}_0^{M_\alpha}$ is the line spanned by $\alpha^\vee$. Let ρ_0^α be the element in $(\mathfrak{a}_0^{M_\alpha})^*$ corresponding to $\delta_{P_0 \cap M_\alpha}^{\frac{1}{2}}$. Each function $m_\alpha(s)$ is holomorphic for $\mathrm{Re}\, s > s_\alpha := \langle \rho_0^\alpha, \alpha^\vee \rangle$, has finitely many poles $s_1, \ldots, s_k$ for $\mathrm{Re}\, s > 0$ which are all simple and real [MW95, Proposition IV.1.11] and $\prod_{i=1}^k ((s - s_i)/(s + s_i)) m_\alpha(s)$ is holomorphic and bounded on the strip $|\mathrm{Re}\, s - s_\alpha| \leqslant s_\alpha$ [HC68, Lemma 101]. Note that $s_\alpha \leqslant \langle \rho_0, \alpha^\vee \rangle$ with equality if and only if $\alpha \in \Delta_0$. In particular, we have the following.

Lemma 3.7. *For $r > 0$ sufficiently small, the function*

$$\prod_{\alpha \in \Delta_0 \cap \Phi_w} \langle \lambda - \rho_0, \alpha^\vee \rangle m(w, \lambda)$$

is holomorphic and of moderate growth on $\|\mathrm{Re}\,\lambda - \rho_0\| < r$.

4. Lattice sums and integrals

The following elementary lemma is essentially contained in the proof of [Hum75, Theorem 18.3]. For completeness, we provide some details. Recall the commutator map $\kappa_s(x) = s^{-1} x s x^{-1}$.

Lemma 4.1. *Let U be a unipotent subgroup of G defined over $\mathbb{Q}$, let Z be its center and let $s \in G(\mathbb{Q})$ be a semisimple element normalizing U. Suppose that $\kappa_s(u_2)\kappa_s(u_1)^{-1} \in Z(\mathbb{Q})$ for some $u_1, u_2 \in U(\mathbb{Q})$. Then $\kappa_s(u_2)\kappa_s(u_1)^{-1} \in \kappa_s(Z(\mathbb{Q}))$.*

Proof. Let $\overline{U} = U/Z$ and let $x \mapsto \overline{x}$ denote the canonical projection $N_G(U) \to N_G(U)/Z$. By the condition on u_1, u_2 we have $\overline{u_1^{-1} u_2} \in C_{\overline{U}}(\overline{s})(\mathbb{Q})$. In the proof of [Hum75, Theorem 18.3], it is shown that $C_{\overline{U}}(\overline{s}) = \overline{C_U(s)}$. Since $\overline{C_U(s)}(\mathbb{Q}) = C_U(s)(\mathbb{Q})/Z(\mathbb{Q})$, there exist $x \in C_U(s)(\mathbb{Q})$ and $z \in Z(\mathbb{Q})$ such that $u_1^{-1} u_2 = xz$. Thus,

$$\kappa_s(u_2) = \kappa_s(z u_1 x) = \kappa_s(z)\kappa_s(u_1 x) = \kappa_s(z)\kappa_s(u_1),$$

as required. □

Lemma 4.2. *Let U and s be as before. Let $U_0 = 0 \subseteq U_1 = Z(U) \subseteq U_2 \subseteq \cdots \subseteq U_d = U$ be the ascending central series of U, where d is the nilpotency class of U. Suppose that $\mathfrak{u} = \bigoplus_{j=1}^{d} V_j$ is a decomposition of $\mathfrak{u}$ into $\operatorname{Ad} s$-invariant subspaces defined over $\mathbb{Q}$ such that $\mathfrak{u}_i = \bigoplus_{j \leqslant i} V_j$, $i = 1, \ldots, d$. Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{u}$ over $\mathbb{Q}$ such that $e_{\dim U_{i-1}+1}, \ldots, e_{\dim U_i}$ is a basis of V_i for all $i = 1, \ldots, d$. Let B be a compact set of $U(\mathbb{A})$. Suppose that $0 \leqslant k \leqslant n$ is such that $e_k \notin (1 - \operatorname{Ad} s)(\mathfrak{u})$. (For $k = 0$, there is no condition.) Then there exists a constant c such that for any $a \in G(\mathbb{R})$ which normalizes $U(\mathbb{A})$ and satisfies $\operatorname{Ad} a(e_i) = a_i e_i$, $i = 1, \ldots, n$, with $a_i > 0$ and any $u \in U(\mathbb{A})$, we have*

$$|aBa^{-1} \cap uU(\mathbb{Q})| \leqslant c \prod_{i=1}^{n} (a_i + 1)$$

and

$$|aBa^{-1} \cap u\kappa_s(U(\mathbb{Q}))| \leqslant c \prod_{i \neq k} (a_i + 1).$$

Proof. We prove the statements by induction on the nilpotency class d of U. Let

$$Y = aBa^{-1} \cap u\kappa_s(U(\mathbb{Q})).$$

Suppose first that U is abelian. We identify U with $\mathfrak{u}$ through the exponential map and identify κ_s with the linear transformation $\operatorname{Ad}(s^{-1}) - 1$ on $\mathfrak{u}$. In particular, $\kappa_s(U)$ is a vector subspace of U defined over $\mathbb{Q}$. Clearly,

$$|aBa^{-1} \cap uU(\mathbb{Q})| \leqslant |aB'a^{-1} \cap U(\mathbb{Q})|$$

and

$$|Y| \leqslant |aB'a^{-1} \cap \kappa_s(U(\mathbb{Q}))|,$$

where $B' = BB^{-1}$. Identify $\mathfrak{u}(\mathbb{Q})$ with $\mathbb{Q}^n$ through $e_1, \ldots, e_n$. Without loss of generality, we may assume that $B' = [-\frac{1}{2}, \frac{1}{2}]^n + \mathcal{O}^n$, since in any case B' is contained in the union of finitely many translates of the latter. Trivially,

$$|U(\mathbb{Q}) \cap aB'a^{-1}| = \left| \mathbb{Q}^n \cap \prod_{i=1}^{n} ([-a_i/2, a_i/2] + \mathcal{O}^n) \right| \leqslant \prod_i (a_i + 1).$$

This proves the first inequality and hence also the case $k = 0$. Suppose that $k > 0$ and let $p_k : \mathfrak{u} \to \mathfrak{u}$ be the projection defined by

$$p_k\left(\sum_i x_i e_i\right) = \sum_{i \neq k} x_i e_i.$$

By the condition on k, p_k is injective on $\kappa_s(U)$. Thus,

$$\begin{aligned} |\kappa_s(U(\mathbb{Q})) \cap aB'a^{-1}| &\leqslant \left| p_k(\kappa_s(U(\mathbb{Q}))) \cap \prod_{i \neq k} ([-a_i/2, a_i/2] + \mathcal{O}^{n-1}) \right| \\ &\leqslant \left| \mathbb{Q}^{n-1} \cap \prod_{i \neq k} ([-a_i/2, a_i/2] + \mathcal{O}^{n-1}) \right| \leqslant \prod_{i \neq k} (a_i + 1). \end{aligned}$$

For the induction step, let Z be the center of U and let $\overline{U} = U/Z$. As before, denote the image of an element x (respectively a subset X) of $N_G(U)$ in $N_G(U)/Z$ by $\overline{x}$ (respectively $\overline{X}$). We clearly have

$$|Y| \leqslant |\overline{Y}| \max_{v \in U(\mathbb{A})} |Y \cap vZ(\mathbb{A})|$$

and

$$|\overline{Y}| \leqslant |\overline{u}\kappa_{\overline{s}}(\overline{U}(\mathbb{Q})) \cap \overline{a}\overline{B}\overline{a}^{-1}|.$$

Furthermore, let $\iota : \mathfrak{u} \to V_{>1} := \bigoplus_{j>1} V_j$ be the projection. Then ι is $\mathrm{Ad}\, s$-equivariant and induces an $\mathrm{Ad}\, s$-equivariant isomorphism $\overline{\iota} : \overline{\mathfrak{u}} = \mathfrak{u}/\mathfrak{z} \to V_{>1}$. Thus, if $k > \dim Z$ and $\overline{e_k} \in (1 - \mathrm{Ad}\, \overline{s})(\overline{\mathfrak{u}})$ then applying $\overline{\iota}$ we get $e_k \in (1 - \mathrm{Ad}\, s)(V_{>1})$. Therefore, by the induction hypothesis,

$$|\overline{Y}| \leqslant c \prod_{\{i > \dim Z : i \neq k\}} (a_i + 1).$$

On the other hand, by the previous lemma, $u\kappa_s(U(\mathbb{Q})) \cap vZ(\mathbb{A})$, if non-empty, is a coset of $\kappa_s(Z(\mathbb{Q}))$. Therefore,

$$\max_{v \in U(\mathbb{A})} |Y \cap vZ(\mathbb{A})| \leqslant \max_{v \in U(\mathbb{A})} |aBa^{-1} \cap v\kappa_s(Z(\mathbb{Q}))|.$$

Once again, this is majorized by $|aB'a^{-1} \cap \kappa_s(Z(\mathbb{Q}))|$. By the abelian case, this is bounded by $c \prod_{\{1 \leqslant i \leqslant \dim Z : i \neq k\}} (a_i + 1)$. Altogether we get the second inequality. The first inequality is proved in a similar vein. $\square$

COROLLARY 4.3. *Let U, s, $e_1, \ldots, e_n$ and k be as before. Let K_U be an open compact subgroup of $U(\mathbb{A}_{\mathrm{fin}})$. Then there exists a continuous seminorm μ on $\mathcal{C}(U(\mathbb{A}); K_U)$ such that for any $f \in \mathcal{C}(U(\mathbb{A}); K_U)$ and a as above we have*

$$\sum_{\gamma \in U(\mathbb{Q})} |f(a^{-1}\gamma a)| \leqslant \mu(f) \prod_{i=1}^{n} (a_i + 1)$$

and

$$\sum_{\gamma \in \kappa_s(U(\mathbb{Q}))} |f(a^{-1}\gamma a)| \leqslant \mu(f) \prod_{i \neq k} (a_i + 1).$$

Proof. Let B be a compact neighborhood of the identity in $U(\mathbb{A})$. By Lemma 3.3,

$$|f(x)| \leqslant c \sum_{X \in \mathcal{B}_U} \int_B |f * X(xu)| \, du$$

for any $f \in \mathcal{C}(U(\mathbb{A}); K_U)$ and $x \in U(\mathbb{A})$. Therefore,

$$\sum_{\gamma \in U(\mathbb{Q})} |f(a^{-1}\gamma a)| \leqslant c \sum_{X \in \mathcal{B}_U} \int_{U(\mathbb{A})} |f * X(u)| |a^{-1}U(\mathbb{Q})a \cap uB^{-1}| \, du$$

and

$$\sum_{\gamma \in \kappa_s(U(\mathbb{Q}))} |f(a^{-1}\gamma a)| \leqslant c \sum_{X \in \mathcal{B}_U} \int_{U(\mathbb{A})} |f * X(u)| |a^{-1}\kappa_s(U(\mathbb{Q}))a \cap uB^{-1}| \, du.$$

We now appeal to the lemma above. $\square$

We specialize to the case where $P = MU$ is a proper standard parabolic. For any $\alpha \in \Delta_0 \backslash \Delta_0^P$, let Q_α be the maximal parabolic of G (containing P) determined by α and let V_α be its unipotent radical. The roots $R(T_0, V_\alpha)$ are exactly the roots in $R(T_0, U_0)$ whose α-coordinate with respect to the basis Δ_0 is positive. Let

$$\Xi_P = \left\{ \frac{1}{|\Delta_0 \backslash \Delta_0^P|} \sum_{\alpha \in \Delta_0 \backslash \Delta_0^P} \beta_\alpha : \beta_\alpha \in R(T_0, V_\alpha) \text{ for all } \alpha \in \Delta_0 \backslash \Delta_0^P \right\}.$$

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Thus, Ξ_P is a finite subset of

$$\mathfrak{a}_{0,P+}^* := \left\{ \sum_{\alpha\in\Delta_0} c_\alpha \alpha : c_\alpha \geqslant 0 \text{ for all } \alpha \in \Delta_0^P \text{ and } c_\alpha > 0 \text{ for all } \alpha \in \Delta_0 \backslash \Delta_0^P \right\}.$$

Using the previous corollary, we obtain the following crucial estimate.

Proposition 4.4. *Let $P = MU$ be a proper standard parabolic. There exists a continuous seminorm μ on $\mathcal{C}(U(\mathbb{A}); K_U)$ such that for any $f \in \mathcal{C}(U(\mathbb{A}); K_U)$ and $a \in A_c$ we have*

$$\sum_{\gamma\in U(\mathbb{Q})} |f(a^{-1}\gamma a)| \leqslant \mu(f)\delta_P(a)$$

and

$$\sum_{\gamma\in\kappa_s(U(\mathbb{Q}))} |f(a^{-1}\gamma a)| \leqslant \mu(f)\delta_P(a) \max_{\xi\in\Xi_P} a^{-\xi} \tag{9}$$

for any $s \in M(\mathbb{Q}) \cap G(\mathbb{Q})_{\mathrm{well}}$.

Proof. Let $\mathfrak{u} = \bigoplus_{i=1}^d V_j$ be a decomposition of $\mathfrak{u}$ into $\mathrm{Ad}\, M$-invariant subspaces such that $\mathfrak{u}_i := \bigoplus_{j=1}^i V_j$, $i = 1, \ldots, d$, are the Lie algebras of the ascending central series $U_1 = Z(U) \subseteq \cdots \subseteq U_d = U$ of U. Each V_j further decomposes according to the roots of T_0. Choose bases of V_j consisting of T_0-eigenvectors and let $e_1, \ldots, e_n$ be the ensuing basis of $\mathfrak{u}$. Let $\mathfrak{u} = \bigoplus_{\beta\in R(T_M,U)} \mathfrak{u}_\beta$ be the decomposition of $\mathfrak{u}$ into eigenspaces of T_M. For each $\beta \in R(T_M, U)$, the space $\mathfrak{u}_\beta$ is spanned by the basis vectors e_i contained in it. The first inequality follows immediately from the previous corollary. For the second part, let $\alpha \in \Delta_0 \backslash \Delta_0^P$. By Lemma 2.3, $\mathfrak{v}_\alpha \not\subseteq (1 - \mathrm{Ad}\, s)(\mathfrak{u})$ and hence there exists $\beta \in R(T_0, V_\alpha)$ such that $\mathfrak{u}_\beta \not\subseteq (1 - \mathrm{Ad}\, s)(\mathfrak{u})$. Thus, $e_i \in \mathfrak{u}_\beta \backslash (1 - \mathrm{Ad}\, s)(\mathfrak{u})$ for some i. It follows from the previous corollary that

$$\sum_{\gamma\in\kappa_s(U(\mathbb{Q}))} |f(a^{-1}\gamma a)| \leqslant \mu(f)\delta_P(a)a^{-\beta}.$$

Since $\alpha \in \Delta_0 \backslash \Delta_0^P$ was arbitrary, the estimate (9) follows. □

5. The elliptic contribution

In this section we will prove Theorem 1 for the elliptic part. Let A be a closed connected subgroup of A_G and set $f_A(g) = \int_A f(zg)\, dz$. The essential step is the following assertion.

Proposition 5.1. *Let $w \in W$, $Q(w) = Q = LV$ and $\xi \in \mathfrak{a}_{0,Q+}^*$. Then*

$$\int_{U_0(\mathbb{A})/U_w(\mathbb{A})} \int_{U_0(\mathbb{A})} \int_{A_c} \int_{M_0(\mathbb{A})^1} f_A(a^{-1}u_2 n_w a u_1 m) a^{-\xi}\, dm\, da\, du_1\, du_2 \tag{10}$$

is a continuous linear form on $\mathcal{C}(G(\mathbb{A}); K)$.

Proof. By Corollary 3.5, it suffices to show that there exists a continuous seminorm μ on $\mathcal{C}(G(\mathbb{A}); K)$ such that (10) is bounded by $\mu(f)$ for any non-negative right $\mathbf{K}$-invariant $f \in C_c^\infty(G(\mathbb{A}); K)$. Applying the argument of Lemma 3.3 to A_G/A, it is enough to consider the case $A = A_G$. We write (10) as

$$\int_{U_0(\mathbb{A})/U_w(\mathbb{A})} \int_{U_{w^{-1}}(\mathbb{A})\backslash U_0(\mathbb{A})} \int_{U_{w^{-1}}(\mathbb{A})} \int_{A_c} \int_{M_0(\mathbb{A})^1} f_{A_G}(a^{-1}u_2 n_w a u_2' v m) a^{-\xi}\, dm\, da\, du_2'\, dv\, du_2.$$

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Conjugating u_2' over $n_w a$, we get

$$\int_{U_0(\mathbb{A})/U_w(\mathbb{A})}\int_{U_{w^{-1}}(\mathbb{A})\backslash U_0(\mathbb{A})}\int_{U_w(\mathbb{A})}\int_{A_c}\int_{M_0(\mathbb{A})^1} f_{A_G}(a^{-1}u_2u_2'n_wavm)a^{-\xi}\delta_{M_0;U_{w^{-1}}}(a)^{-1}\,dm\,da\,du_2'\,dv\,du_2$$
$$=\int_{U_0(\mathbb{A})}\int_{U_{w^{-1}}(\mathbb{A})\backslash U_0(\mathbb{A})}\int_{A_c}\int_{M_0(\mathbb{A})^1} f_{A_G}(a^{-1}un_wavm)a^{-\xi}\delta_{M_0;U_{w^{-1}}}(a)^{-1}\,dm\,da\,dv\,du$$
$$=\int_{U_{w^{-1}}(\mathbb{A})\backslash U_0(\mathbb{A})}\int_{A_c}(F_f)_{A_G}(a^{-1}n_wav)a^{-\xi}\delta_{w^{-1}}(a)\,da\,dv.$$

Using Mellin inversion, we can write this as

$$\int_{A_c}\int_{U_{w^{-1}}(\mathbb{A})\backslash U_0(\mathbb{A})}\int_{\lambda\in(\mathfrak{a}_0^G)^*_{\mathbb{C}}:\mathrm{Re}\,\lambda=\lambda_0}\varphi(\lambda)(n_wv)a^{-\xi}\delta_{w^{-1}}(a)\delta_0(a^{-1}waw^{-1})^{\frac{1}{2}}a^{(w^{-1}-1)\lambda}\,d\lambda\,dv\,da$$
$$=\int_{A_c}\int_{U_{w^{-1}}(\mathbb{A})\backslash U_0(\mathbb{A})}\int_{\lambda\in(\mathfrak{a}_0^G)^*_{\mathbb{C}}:\mathrm{Re}\,\lambda=\lambda_0}\varphi(\lambda)(n_wv)a^{-(1-w^{-1})\lambda-\xi}\,d\lambda\,dv\,da$$

for any $\lambda_0\in(\mathfrak{a}_0^G)^*$, where we used Lemma 2.1. Suppose that $\lambda_0-\rho_0$ lies in the positive Weyl chamber. We claim that the integral is then absolutely convergent as a triple integral. Indeed, replacing $\varphi(\lambda)$ by $|\varphi(\lambda)|\in I(\lambda_0)$ and interchanging the order of integration, we get

$$\int_{\mathrm{Re}\,\lambda=\lambda_0}\int_{A_c} m(w^{-1},\lambda_0)|\varphi(\lambda)(e)|a^{-(1-w^{-1})\lambda_0-\xi}\,da\,d\lambda.$$

Since $\varphi(\lambda)(e)$ is rapidly decreasing, the integral over λ converges. On the other hand, by Lemma 2.2 and the assumption on ξ, $(1-w^{-1})\lambda_0+\xi$ is a positive linear combination of the elements of Δ_0. Hence, the integral over a converges as well, justifying our claim.

Returning to the original integral, we can now rewrite it by interchanging the order of integration as

$$\int_{\mathrm{Re}\,\lambda=\lambda_0}\int_{A_c} m(w^{-1},\lambda)\varphi(\lambda)(e)a^{-(1-w^{-1})\lambda-\xi}\,da\,d\lambda$$
$$=\mathrm{vol}(\mathfrak{a}_0^G/\mathbb{Z}\widehat{\Delta}_0^\vee)\int_{\mathrm{Re}\,\lambda=\lambda_0}\prod_{\varpi^\vee\in\widehat{\Delta}_0^\vee}\frac{c^{-\langle\xi+(1-w^{-1})\lambda,\varpi^\vee\rangle}}{\langle\xi+(1-w^{-1})\lambda,\varpi^\vee\rangle}m(w^{-1},\lambda)\varphi(\lambda)(e)\,d\lambda.$$

Our assertion follows now from Lemmas 3.7, 3.6 and Proposition 3.1 (with $V=\mathfrak{a}_0^G$, $\mu_0=\rho_0$ and $S=\Delta_0\cap\Phi_{w^{-1}}$). □

Remark 2. Note that the argument above does not apply directly to $f\in\mathcal{C}(G(\mathbb{A});K)$ (cf. Remark 1). For $G=\mathrm{GL}(2)$, a direct argument valid for any $f\in\mathcal{C}(G(\mathbb{A});K)$ was given in [FL11]. However, it seems harder to pursue this approach in the higher rank case.

We are now ready to prove the elliptic part of Theorem 1. Our point of departure will be Proposition 5.1. Fix $w\in W$ and let $Q=Q(w)=LV$ and $\xi\in\mathfrak{a}^*_{0,Q+}$ be as above. We proceed in several steps. In what follows, we will constantly use Corollary 3.5 so that we can assume that $f\geqslant 0$.

We first apply Lemma 3.3 to M_0 to conclude that

$$\int_{U_0(\mathbb{A})/U_w(\mathbb{A})}\int_{U_0(\mathbb{A})}\int_{A_c}\sum_{m\in M_0(\mathbb{Q})} f_A(a^{-1}u_2n_wau_1m)a^{-\xi}\,da\,du_1\,du_2,$$

obtained from (10) by replacing the integral over $M_0(\mathbb{A})^1$ by the sum over $M_0(\mathbb{Q})$, is also a continuous linear form on $\mathcal{C}(G(\mathbb{A}); K)$. Using (2), we can write this as

$$\int_{U_0^L(\mathbb{A})/U_w^L(\mathbb{A})} \int_{U_w^L(\mathbb{A})/U_w^L(\mathbb{Q})} \int_{U_0(\mathbb{A})} \int_{A_c} \sum_{m\in M_0(\mathbb{Q})} f_A(a^{-1}u_2 v m n_w a u_1) a^{-\xi}\, da\, du_1\, dv\, du_2,$$

since the integral over a and u_1 does not depend on v. Combining v and u_2, we write this as

$$\int_{U_0^L(\mathbb{A})/U_w^L(\mathbb{Q})} \int_{U_0(\mathbb{A})} \int_{A_c} \sum_{m\in M_0(\mathbb{Q})} f_A(a^{-1}u m n_w a u_1) a^{-\xi}\, da\, du_1\, du$$

or, as

$$\int_{U_0^L(\mathbb{A})/U_0^L(\mathbb{Q})} \int_{U_0(\mathbb{A})} \int_{A_c} \sum_{m\in M_0(\mathbb{Q})} \sum_{u_2\in U_0^L(\mathbb{Q})/U_w^L(\mathbb{Q})} f_A(a^{-1}u u_2 m n_w a u_1) a^{-\xi}\, da\, du_1\, du,$$

or, if we wish, as

$$\int_{U_0^L(\mathbb{Q})\backslash U_0^L(\mathbb{A})} \int_{U_0^L(\mathbb{A})} \int_{V(\mathbb{A})} \int_{A_c} \sum_{m\in M_0(\mathbb{Q})} \sum_{u_2\in U_0^L(\mathbb{Q})/U_w^L(\mathbb{Q})} f_A(a^{-1}u^{-1}u_2 m n_w a u_1 u_3) a^{-\xi}\, da\, du_1\, du_3\, du.$$

We write $a = a_L a^L$, where $a_L \in A_L$ and $a^L \in A_0^L$. Conjugating u_1 over a^L, we obtain

$$\int_{U_0^L(\mathbb{Q})\backslash U_0^L(\mathbb{A})} \int_{U_0^L(\mathbb{A})} \int_{V(\mathbb{A})} \int_{A_c} \sum_{m\in M_0(\mathbb{Q})} \sum_{u_2\in U_0^L(\mathbb{Q})/U_w^L(\mathbb{Q})} f_A(a^{-1}u^{-1}u_2 m n_w a_L u_1 a^L u_3) a^{-\xi}\, da\, du_1\, du_3\, du.$$

Next, we claim that

$$\int_{U_0^L(\mathbb{Q})\backslash U_0^L(\mathbb{A})} \int_{V(\mathbb{A})} \int_{A_c} \sum_{u_3\in U_0^L(\mathbb{Q})} \sum_{m\in M_0(\mathbb{Q})} \sum_{u_2\in U_0^L(\mathbb{Q})/U_w^L(\mathbb{Q})} f_A(a^{-1}u^{-1}u_2 m n_w a_L u_1 u_3 u a^L) a^{-\xi} \delta_{P_0^L}(a)^{-1}\, da\, du_1\, du,$$

obtained by replacing the integral over $u_3 \in U_0^L(\mathbb{A})$ by a sum over a translate of $(a^L)^{-1}U_0^L(\mathbb{Q})a^L$ and dividing by $\delta_{P_0^L}(a)$, is continuous. This follows from Proposition 4.4 applied to the parabolic subgroup $P_0^L = M_0 U_0^L$ of L, the element $a^L \in A_c^L$ and the functions $f(g\tilde{u}^{-1}\cdot\tilde{u})$ on $U_0^L(\mathbb{A})$, where $\tilde{u} = (a^L)^{-1}u a^L$. For this, we use Lemma 3.2 and the fact that $\tilde{u}$ ranges in a compact set because $a^L \in A_c^L$ and u can be integrated over a compact fundamental domain for $U_0^L(\mathbb{Q})\backslash U_0^L(\mathbb{A})$. Conjugating u_1 back, we obtain

$$\int_{U_0^L(\mathbb{Q})\backslash U_0^L(\mathbb{A})} \int_{V(\mathbb{A})} \int_{A_c} \sum_{u_3\in U_0^L(\mathbb{Q})} \sum_{m\in M_0(\mathbb{Q})} \sum_{u_2\in U_0^L(\mathbb{Q})/U_w^L(\mathbb{Q})} f_A(a^{-1}u^{-1}u_2 m n_w a_L u_3 u a^L u_1) a^{-\xi} \delta_{P_0^L}(a)^{-1}\, da\, du_1\, du$$

and, since a_L commutes with $U_0^L(\mathbb{A})$, we obtain

$$\int_{U_0^L(\mathbb{Q})\backslash U_0^L(\mathbb{A})} \int_{V(\mathbb{A})} \int_{A_c} \sum_{u_3\in U_0^L(\mathbb{Q})} \sum_{m\in M_0(\mathbb{Q})} \sum_{u_2\in U_0^L(\mathbb{Q})/U_w^L(\mathbb{Q})} f_A(a^{-1}u^{-1}u_2 m n_w u_3 u a u_1) a^{-\xi} \delta_{P_0^L}(a)^{-1}\, da\, du_1\, du.$$

We can rewrite this as

$$\int_{U_0^L(\mathbb{Q})\backslash U_0^L(\mathbb{A})} \int_{V(\mathbb{A})} \int_{A_c} \sum_{m \in \mathfrak{B}_w^L} f_A(a^{-1}u^{-1}muau_1)a^{-\xi}\delta_{P_0^L}(a)^{-1}\, da\, du_1\, du.$$

We conclude that

$$\int_{U_0^L(\mathbb{Q})\backslash U_0^L(\mathbb{A})} \int_{V(\mathbb{Q})\backslash V(\mathbb{A})} \int_{V(\mathbb{A})} \int_{A_c} \sum_{m \in \mathfrak{B}_w^L} f_A(a^{-1}u^{-1}v^{-1}mvuau_1)a^{-\xi}\delta_{P_0^L}(a)^{-1}\, da\, du_1\, dv\, du$$

is continuous, since the integral over a and u_1 is independent of v. In particular,

$$\int_{U_0^L(\mathbb{Q})\backslash U_0^L(\mathbb{A})} \int_{V(\mathbb{A})} \int_{V(\mathbb{Q})\backslash V(\mathbb{A})} \int_{A_c} \sum_{m \in \mathfrak{E}_w} f_A(a^{-1}u^{-1}v^{-1}mvuau_1)a^{-\xi}\delta_{P_0^L}(a)^{-1}\, da\, dv\, du_1\, du$$

is continuous. The next step is to show the continuity of

$$\begin{aligned}\int_{U_0^L(\mathbb{Q})\backslash U_0^L(\mathbb{A})} \int_{V(\mathbb{Q})\backslash V(\mathbb{A})} \int_{A_c} \sum_{m \in \mathfrak{E}_w} \sum_{u_1 \in \kappa_m(V(\mathbb{Q}))} f_A(a^{-1}u^{-1}v^{-1}mu_1vua)\delta_0(a)^{-1}\, da\, dv\, du \\ = \int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} \int_{A_c} \sum_{m \in \mathfrak{E}_w} \sum_{u_1 \in \kappa_m(V(\mathbb{Q}))} f_A(a^{-1}u^{-1}mu_1ua)\delta_0(a)^{-1}\, da\, du.\end{aligned}$$

In other words, we want to replace the integral over $u_1 \in V(\mathbb{A})$ by the sum over a translate of $a^{-1}\kappa_m(V(\mathbb{Q}))a$ and divide by $\delta_Q(a)a^{-\xi}$. For this, we apply Proposition 4.4 and Lemma 3.2 to the functions $f(g\tilde{u}^{-1} \cdot \tilde{u})$, where $\tilde{u} = a^{-1}vua$, taking into account that $\tilde{u}$ ranges in a compact set because $a \in A_c$ and u and v can be integrated over compact fundamental domains. By Lemma 3.2, we conclude the continuity of

$$\int_{\mathbf{K}} \int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} \int_{M_0(\mathbb{Q})\backslash M_0(\mathbb{A})^1} \int_{A_c} \sum_{\substack{m \in \mathfrak{E}_w \\ u_1 \in \kappa_m(V(\mathbb{Q}))}} f_A((uask)^{-1}mu_1(uask))\delta_0(a)^{-1}\, da\, ds\, du\, dk$$

by considering $f^{sk}(\cdot) = f((sk)^{-1} \cdot sk)$ for any $s \in M_0(\mathbb{Q})\backslash M_0(\mathbb{A})^1$ and $k \in \mathbf{K}$ and using the compactness of $M_0(\mathbb{Q})\backslash M_0(\mathbb{A})^1$.

Finally, by the Bruhat decomposition and (4), the sum of these expressions for all $w \in W$ is

$$\begin{aligned}\int_{\mathbf{K}} \int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} \int_{M_0(\mathbb{Q})\backslash M_0(\mathbb{A})^1} \int_{A_c} \sum_{\gamma \in G(\mathbb{Q})_{\mathrm{well}}} f_A((uask)^{-1}\gamma(uask))\delta_0(a)^{-1}\, da\, ds\, du\, dk \\ = \int_{A_G P_0(\mathbb{Q})\backslash \mathcal{S}_c} \sum_{\gamma \in G(\mathbb{Q})_{\mathrm{well}}} f_A(g^{-1}\gamma g)\, dg, \qquad (11)\end{aligned}$$

which is therefore continuous.

Taking $A = A_G$, the elliptic part of Theorem 1 now follows from the reduction-theoretic inequality (3) and the fact that $G(\mathbb{Q})_{\mathrm{ell}} \subseteq G(\mathbb{Q})_{\mathrm{well}}$.

Remark 3. For the regular elliptic contribution, the argument simplifies: only w for which $L = G$ contribute and we do not need to use (9).

6. The semisimple terms

For any semisimple $\gamma \in G(\mathbb{Q})$, let $M(\gamma)$ be the centralizer of the split part of the center of $C_G^0(\gamma)$. Thus, $M(\gamma)$ is the smallest Levi subgroup (not necessarily standard) defined over $\mathbb{Q}$

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containing $C_G^0(\gamma)$. It is clear that $M(g^{-1}\gamma g)=g^{-1}M(\gamma)g$ for any $g\in G(\mathbb{Q})$. Also, since $C_G^0(\gamma)$ is reductive, any parabolic subgroup containing $C_G^0(\gamma)$ will also contain $M(\gamma)$. Denote by $P(\gamma)$ the smallest standard parabolic containing $C_G^0(\gamma)$. By the proof of Lemma 2.3, $P(\gamma)$ is equivalently the smallest standard parabolic $P=MU$ containing γ such that $C_U(\gamma)=1$. It follows that

$$\{u^{-1}\gamma u : u\in U(\mathbb{Q})\}=\gamma U(\mathbb{Q}). \tag{12}$$

Clearly, $P(p^{-1}\gamma p)=P(\gamma)$ for any $p\in P_0(\mathbb{Q})$. For any standard parabolic P, let $M(\mathbb{Q})_{\mathrm{rell}}=\{\gamma\in M(\mathbb{Q}) : P(\gamma)=P\}\subseteq M(\mathbb{Q})_{\mathrm{well}}$. We recall the notation $\mathfrak{a}_P$, $H_P : G(\mathbb{A})\to\mathfrak{a}_P$ and the characteristic functions τ_P, $\hat{\tau}_P$ (e.g. [Art05, §§ 4–6]).

Observe that if $f\geqslant 0$ then for any standard parabolic $P=MU$ we have

$$\int_{\mathcal{S}_c} f(g)\,dg\leqslant\int_{\mathbf{K}}\int_{\mathcal{S}_c^M}\int_{U(\mathbb{A})} f(muk)\tau_P(H_P(m)-T'_P)\,du\,dm\,dk, \tag{13}$$

where $T'\in\mathfrak{a}_0$ is such that $\langle\alpha,T'\rangle=\log c$ for all $\alpha\in\Delta_0$.

Lemma 6.1. *Let $T\in\mathfrak{a}_0$. Then*

$$f\mapsto\int_{A_GP_0(\mathbb{Q})\backslash\mathcal{S}_c}\sum_{\gamma\in G(\mathbb{Q})_{\mathrm{ss}}} f_{A_G}(x^{-1}\gamma x)\hat{\tau}_{P(\gamma)}(T_{P(\gamma)}-H_{P(\gamma)}(x))\,dx \tag{14}$$

is continuous on $\mathcal{C}(G(\mathbb{A});K)$.

Proof. As always, we can assume that $f\geqslant 0$. Using (12), we can write (14) as the sum over standard parabolic subgroups $P=MU$ of

$$\begin{aligned}
&\int_{A_GP_0(\mathbb{Q})\backslash\mathcal{S}_c}\sum_{\gamma\in G(\mathbb{Q})_{\mathrm{ss}}:P(\gamma)=P} f_{A_G}(x^{-1}\gamma x)\hat{\tau}_P(T_P-H_P(x))\,dx\\
&=\int_{A_GP_0(\mathbb{Q})\backslash\mathcal{S}_c}\sum_{\gamma\in M(\mathbb{Q})_{\mathrm{rell}}}\sum_{u\in U(\mathbb{Q})} f_{A_G}(x^{-1}\gamma ux)\hat{\tau}_P(T_P-H_P(x))\,dx\\
&=\int_{A_GP_0(\mathbb{Q})\backslash\mathcal{S}_c}\sum_{\gamma\in M(\mathbb{Q})_{\mathrm{rell}}}\sum_{u\in U(\mathbb{Q})} f_{A_G}(x^{-1}u^{-1}\gamma ux)\hat{\tau}_P(T_P-H_P(x))\,dx\\
&=\int_{A_GP_0^M(\mathbb{Q})\backslash\mathcal{S}_c}\sum_{\gamma\in M(\mathbb{Q})_{\mathrm{rell}}} f_{A_G}(x^{-1}\gamma x)\hat{\tau}_P(T_P-H_P(x))\,dx.
\end{aligned}$$

By (13), this is bounded by the product of $\int_{A_G\backslash A_M}\hat{\tau}_P(T_P-X)\tau_P(X-T'_P)\,dX$ and

$$\begin{aligned}
&\int_{\mathbf{K}}\int_{A_MP_0^M(\mathbb{Q})\backslash\mathcal{S}_c^M}\sum_{\gamma\in M(\mathbb{Q})_{\mathrm{rell}}}\int_{U(\mathbb{A})} f_{A_G}(k^{-1}u^{-1}m^{-1}\gamma muk)\,du\,dm\,dk\\
&=\int_{A_MP_0^M(\mathbb{Q})\backslash\mathcal{S}_c^M}\sum_{\gamma\in M(\mathbb{Q})_{\mathrm{rell}}}(f_P)_{A_G}(m^{-1}\gamma m)\,dm\\
&\leqslant\int_{A_MP_0^M(\mathbb{Q})\backslash\mathcal{S}_c^M}\sum_{\gamma\in M(\mathbb{Q})_{\mathrm{well}}}(f_P)_{A_G}(m^{-1}\gamma m)\,dm,
\end{aligned}$$

where

$$f_P(m)=\int_{\mathbf{K}}\int_{U(\mathbb{A})} f(k^{-1}muk)\,du\,dk.$$

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It remains to invoke the continuity of (11) with respect to M and $A = A_G$ and the easy fact that the map $f \mapsto f_P$, $\mathcal{C}(G(\mathbb{A}); K) \to \mathcal{C}(M(\mathbb{A}); \bigcap_{k\in\mathbf{K}} K^k \cap M(\mathbb{A}))$ is continuous. □

We can now finish the proof of Theorem 1. Let T_0 be as in [Art05, (9.4)]. Suppose that $M(\gamma)$ is standard and that $\alpha(T - T_0) \geqslant 0$ for all $\alpha \in \Delta_0$. For any $x \in G(\mathbb{A})$, let $\mathfrak{C}^T_{M(\gamma)}(x)$ be the convex hull in $\mathfrak{a}_{M(\gamma)}$ of $w^{-1}(T_{P'} - H_{P'}(n_w x))$, where w ranges over the set $W(M(\gamma))$ of right $W^{M(\gamma)}$-reduced Weyl group elements such that $M' = wM(\gamma)w^{-1}$ is standard and P' is the standard parabolic subgroup with Levi M'. Set

$$\psi^T_{M(\gamma)}(x) = \begin{cases} 1 & \text{if } 0 \in \mathfrak{C}^T_{M(\gamma)}(x), \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively,

$$\psi^T_{M(\gamma)}(x) = \prod_{w\in W(M(\gamma))} \hat{\tau}_{P'}(T_{P'} - H_{P'}(n_w x))$$

(cf. the proof of [Art05, Lemma 17.2]). Then the weight function $v_{M(\gamma)}(x)$ is equal to the volume of the subset $\mathfrak{C}^{T_0}_{M(\gamma)}(x)$ of $\mathfrak{a}^G_{M(\gamma)}$ (cf. [Art05, pp. 62–63, 102]) and the semisimple part of the trace formula can be written as

$$\begin{aligned} &\sum_{[\gamma]\subseteq G(\mathbb{Q})_{\mathrm{ss}}} \int_{A_M C_G(\gamma,\mathbb{Q})\backslash G(\mathbb{A})} f_{A_G}(x^{-1}\gamma x) v_{M(\gamma)}(x)\, dx \\ &= \sum_{[\gamma]\subseteq G(\mathbb{Q})_{\mathrm{ss}}} \int_{C_G(\gamma,\mathbb{Q})\backslash G(\mathbb{A})} f_{A_G}(x^{-1}\gamma x) \psi^{T_0}_{M(\gamma)}(x)\, dx. \end{aligned}$$

We claim that for any T as above we have

$$\begin{aligned} &\sum_{[\gamma]\subseteq G(\mathbb{Q})_{\mathrm{ss}}} \int_{C_G(\gamma,\mathbb{Q})\backslash G(\mathbb{A})} f_{A_G}(x^{-1}\gamma x) \psi^{T}_{M(\gamma)}(x)\, dx \\ &= \int_{A_G G(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{[\gamma]\subseteq G(\mathbb{Q})_{\mathrm{ss}}} \sum_{\delta\in C_G(\gamma,\mathbb{Q})\backslash G(\mathbb{Q})} f_{A_G}(x^{-1}\delta^{-1}\gamma\delta x) \psi^{T}_{M(\gamma)}(\delta x)\, dx \\ &\leqslant \int_{A_G P_0(\mathbb{Q})\backslash \mathcal{S}_c} \sum_{\gamma\in G(\mathbb{Q})_{\mathrm{ss}}} f_{A_G}(x^{-1}\gamma x) \hat{\tau}_{P(\gamma)}(T_{P(\gamma)} - H_{P(\gamma)}(x))\, dx. \end{aligned}$$

Indeed, suppose that $\psi^T_{M(\gamma)}(\delta x) = 1$ for some $\delta \in G(\mathbb{Q})$ and let $Q = P(\delta^{-1}\gamma\delta) = LV$. Then $\delta^{-1}M(\gamma)\delta = M(\delta^{-1}\gamma\delta) \subseteq Q$ and, hence, since $M(\gamma)$ and Q are standard, there exists $w \in W(M(\gamma))$ such that $\delta \in n_w^{-1}Q(\mathbb{Q})$ and $M' = wM(\gamma)w^{-1} \subseteq L$. Thus, $\hat{\tau}_Q(T_Q - H_Q(x)) = \hat{\tau}_Q(T_Q - H_Q(n_w\delta x)) = 1$, since $\hat{\tau}_{P'}(T_{P'} - H_{P'}(n_w\delta x)) = 1$ by the assumption on δx.

Theorem 1 now follows from Lemma 6.1 upon substituting $T = T_0$.

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