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A NOTE ON NATURALLY ORDERED SEMIGROUPS OF TRANSFORMATIONS WITH INVARIANT SET

LEI SUN✉ and JUNLING SUN

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Abstract

In this short note, we describe all the elements in the semigroup

$$S(X, Y) = \{f \in \mathcal{T}_X : f(Y) \subseteq Y\}$$

which are left compatible with respect to the so-called natural partial order. This result corrects an error in a paper by Sun and Wang ['Natural partial order in semigroups of transformations with invariant set', *Bull. Aust. Math. Soc.* **87** (2013), 94–107].

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Let $\mathcal{T}_X$ be the full transformation semigroup on the nonempty set X and fix a nonempty subset Y of X. Endow the semigroup

$$S(X, Y) = \{f \in \mathcal{T}_X : f(Y) \subseteq Y\}$$

with the so-called natural partial order [1], that is, for $f, g \in S(X, Y)$,

$$f \leq g \quad \text{if and only if} \quad f = kg = gh \text{ and } f = kf \text{ for some } k, h \in S(X, Y).$$

Sun and Wang [2] gave a characterisation of this partial order $\leq$, namely, $f \leq g$ if and only if the following statements hold:

(C1) $\pi(g)$ refines $\pi(f)$ and $\pi_Y(g)$ refines $\pi_Y(f)$;
(C2) if $g(x) \in f(X)$ for some $x \in X$, then $f(x) = g(x)$;
(C3) $f(X) \subseteq g(X)$ and $f(Y) \subseteq g(Y)$.

A transformation $h \in S(X, Y)$ is said to be *strictly left compatible* (*left compatible*) with the partial order if $hf < hg$ ($hf \leq hg$) whenever $f < g$ ($f \leq g$).

THEOREM 1 [2]. *Let $h \in S(X, Y)$. Then h is strictly left compatible if and only if h is an injection.*

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From the proof of [2, Theorem 2.3], the necessity of Theorem 1 holds, but the sufficiency may not be true. Now let us show a counterexample.

EXAMPLE 2. Let $X = \{1, 2, 3, \ldots\}$ and $Y = \{4, 5, 6, \ldots\}$. Let

$$h(x) = \begin{cases} 1 & \text{if } x = 1, \\ x + 2 & \text{if } x \in X - \{1\}. \end{cases}$$

It is clear that $h \in S(X, Y)$ and h is an injection. Take

$$f(x) = \begin{cases} 1 & \text{if } x \in \{1, 2, 3\}, \\ 4 & \text{if } x \in Y \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x = 1, \\ 2 & \text{if } x \in \{2, 3\}, \\ 4 & \text{if } x \in Y. \end{cases}$$

Then $f, g \in S(X, Y)$ and $f < g$. Now we assert that h is not strictly left compatible. Indeed, if h is strictly left compatible, then $hf < hg$. By (C1), $\pi_Y(hg)$ refines $\pi_Y(hf)$. However, $hg(2) = hg(3) = 4 \in Y$ and $hf(2) = hf(3) = 1 \in X - Y$ which implies that $\pi_Y(hg)$ does not refine $\pi_Y(hf)$, a contradiction. So h is not strictly left compatible.

Our main purpose in this short note is to correct an error in Theorem 1 and give a necessary and sufficient condition for all the left compatible elements of the semigroup $S(X, Y)$ with respect to this partial order. First, we give a necessary condition for the strictly left compatible elements.

LEMMA 3. *Let $h \in S(X, Y)$. If h is strictly left compatible, then either $h^{-1}(Y) = X$ or $h^{-1}(Y) \subseteq Y$.*

PROOF. There are two cases to consider.

Case 1. $|X - Y| = 1$. Obviously, for each $h \in S(X, Y)$, we have $h^{-1}(Y) = X$ or $h^{-1}(Y) \subseteq Y$, as required.

Case 2. $|X - Y| \geq 2$. Suppose that $h^{-1}(Y) \neq X$. Let $a \in X - Y$ be such that $h(a) \in X - Y$. Now we assert that $h^{-1}(Y) \subseteq Y$. Indeed, if $h(b) = c \in Y$ for some $b \in X - Y$ ($b \neq a$). Then define $f, g : X \to X$ by

$$f(x) = \begin{cases} a & \text{if } x \in X - Y, \\ x & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} a & \text{if } x \in X - Y - \{b\}, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $f, g \in S(X, Y)$ and $f < g$. Noting that h is left compatible, we have $hf < hg$. However, on the one hand, $hf(b) = h(a) \in X - Y$; on the other hand, $hf(b) = khg(b) = kh(b) = k(c) \in Y$ for some $k \in S(X, Y)$, a contradiction. So $h^{-1}(Y) \subseteq Y$ and the conclusion follows. □

So Theorem 1 should be corrected as follows.

THEOREM 1′. *Let $h \in S(X, Y)$. Then h is strictly left compatible if and only if h is an injection with either $h^{-1}(Y) = X$ or $h^{-1}(Y) \subseteq Y$.*

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Proof. The necessity follows from the proof of [2, Theorem 2.3] and Lemma 3, and now we show the sufficiency. Let $f, g \in S(X, Y)$ and $f < g$. We verify that $hf < hg$. Assume that $hg(x) = hg(y)$ for some $x, y \in X$. Noting that h is an injection, we have $g(x) = g(y)$ and $f(x) = f(y)$ since $\pi(g)$ refines $\pi(f)$. So $hf(x) = hf(y)$ and $\pi(hg)$ refines $\pi(hf)$. Moreover, if $hg(x) = hg(y) \in Y$, then either $g(x) = g(y) \in Y$ or $g(x) = g(y) \in X - Y$. If the former case occurs, then $f(x) = f(y) \in Y$ and $hf(x) = hf(y) \in Y$. If the latter case occurs, then $h^{-1}(Y) = X$. So $f(x) = f(y)$ and $hf(x) = hf(y) \in Y$. Hence $\pi_Y(hg)$ refines $\pi_Y(hf)$ and the transformations hf, hg satisfy (C1). It is routine to show that the transformations hf, hg satisfy (C2) and (C3). Therefore, $hf < hg$ and h is strictly left compatible. □

We see that, in Example 2, h is an injection and $h^{-1}(Y) = X - \{1\}$ but neither $h^{-1}(Y) = X$ nor $h^{-1}(Y) \subseteq Y$, so h is not strictly left compatible.

We point out that Lemma 3 also holds for the case where h is left compatible, that is, if h is left compatible, then h is an injection with either $h^{-1}(Y) = X$ or $h^{-1}(Y) \subseteq Y$. In what follows we consider a sufficient condition for all the left compatible elements in the semigroup $S(X, Y)$.

Lemma 4. *Suppose that $h \in S(X, Y)$ is not a constant transformation. Then the following statements hold:*

(1) *if $h|_{X-Y}$ is not injective, then h is not left compatible;*
(2) *if $h(X - Y) \cap h(Y) \neq \emptyset$, then h is not left compatible;*
(3) *if $h|_Y$ is not injective and $|h(Y)| \geq 2$, then h is not left compatible.*

Proof. (1) Let $h(a) = h(b) \neq h(c)$ for some distinct $a, b \in X - Y$ and $c \in X$. Suppose to the contrary that h is left compatible. Define $f : X \to X$ by

$$f(x) = \begin{cases} c & \text{if } x = a, \\ x & \text{otherwise.} \end{cases}$$

Clearly $f \in S(X, Y)$ and $f \leq \mathrm{id}_X$. So $hf \leq h\,\mathrm{id}_X = h$ and $\pi(h)$ refines $\pi(hf)$. However, on the one hand, $h(a) = h(b)$; on the other hand, $hf(a) = h(c)$ and $hf(b) = h(b)$. It readily follows from $h(c) \neq h(b)$ that $hf(a) \neq hf(b)$, a contradiction. Hence h is not left compatible.

(2) Let $h(a) = h(b) \neq h(c)$ for some distinct $a \in X - Y$, $b \in Y$ and $c \in X$. Take f as in (1) so that $f \leq \mathrm{id}_X$. Then $hf \leq h$ and $\pi(h)$ refines $\pi(hf)$, which also leads to a contradiction. Thus h is not left compatible.

(3) Let $h(a) = h(b) \neq h(c)$ for some distinct $a, b, c \in Y$. Take f as in (1) so that $f \leq \mathrm{id}_X$. Then $hf \leq h$ and $\pi(h)$ refines $\pi(hf)$ which also leads to a contradiction. It follows that h is not left compatible. □

It is routine to verify the following lemma.

Lemma 5. *Let $h \in S(X, Y)$. If $h|_{X-Y}$ is injective, $h(X - Y) \cap h(Y) = \emptyset$ and $|h(Y)| = 1$, then the following statements hold:*

(1) *if $h^{-1}(Y) = X$, then h is left compatible;*
(2) *if $h^{-1}(Y) \subseteq Y$, then h is left compatible.*

Now we obtain the main result of this short note.

THEOREM 6. *Let X be a set and Y be the subset with $Y \neq X$. Suppose that $|X| \geq 3$ and $|Y| \geq 2$. Then $h \in S(X, Y)$ is left compatible if and only if one of the following statements holds:*

(1) *h is a constant transformation;*
(2) *h is an injection with either $h^{-1}(Y) = X$ or $h^{-1}(Y) \subseteq Y$;*
(3) *$h|_{X-Y}$ is injective, $h(X - Y) \cap h(Y) = \emptyset$ and $|h(Y)| = 1$ with either $h^{-1}(Y) = X$ or $h^{-1}(Y) \subseteq Y$.*

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LEI SUN, School of Mathematics and Information Science,
Henan Polytechnic University, Henan, Jiaozuo, 454003, PR China
e-mail: sunlei97@163.com

JUNLING SUN, School of Mathematics and Information Science,
Henan Polytechnic University, Henan, Jiaozuo, 454003, PR China