DECOMPOSITION THEOREMS FOR THE GENERALIZED METAHARMONIC EQUATION IN SEVERAL INDEPENDENT VARIABLES

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(Received 1 August 1969)

Communicated by J. P. O. Silberstein

In this paper solutions of the generalized metaharmonic equation in several independent variables

$$(1) \qquad L^{(n)}_{\lambda, s}[u] = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} + \frac{\partial^2 u}{\partial \rho^2} + \frac{s}{\rho}\frac{\partial u}{\partial \rho} + \lambda^2 u = 0$$

where $\lambda > 0$ are uniquely decomposed into the sum of a solution regular in the entire space and one satisfying a generalized Sommerfeld radiation condition. Due to the singular nature of the partial differential equation under investigation it is shown that the radiation condition in general must hold uniformly in a domain lying in the space of several complex variables. This result indicates that function theoretic methods are not only the correct and natural avenue of approach in the study of singular ordinary differential equations, but are basic in the investigation of singular partial differential equations as well.

The techniques employed in the analytic theory of partial differential equations in $n > 2$ variables are in general quite different than in the case of two independent variables since one now needs to study analytic functions of several complex variables instead of a single complex variable (c.f. [9]). This point is aptly illustrated in the present work since although for $n = 1$ the above mentioned decomposition theorem has been previously obtained in [2] ,the methods used there do not immediately generalize to the several variable case considered here. This is due to the fact that in [2] rather explicit evaluations of certain contour integrals over the Riemann surface of multivalued analytic functions were required, and for functions of several complex variables this becomes prohibitively difficult. Hence an entirely different approach is employed ,namely the use of differential recursion relations similar to those first used in [3] and [4] to investigate the analytic theory and uniqueness problems for a class of singular equations closely related to 1). Although the use of contour integration is avoided the approach remains function theoretic in nature.

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For the special case $n = 2$, $s = 0$, (i.e. the nonsingular case) a particularly good discussion of the decomposition problem under consideration (including its application to scattering theory) can be found in [7] pp. 312–320. The results presented in [7] were first obtained by John in [13].

1. Appell series and generalized metaharmonic functions in several variables

An Appell series (c.f. [10]) is a series of the form

$$\sum a_M V_M^{(s)}(\xi) = \sum_{\mu=0}^{\infty} \sum_{|M|=\mu} a_M V_M^{(s)}(\xi) \tag{2}$$

where the polynomials

$$V_M^{(s)}(\xi) = V_M(\xi_1, \cdots, \xi_n); \quad M = (m_1, \cdots, m_n) \tag{3}$$

are uniquely defined by the generating function

$$(1-2(\alpha, \xi)+||\alpha||^2)^{-(n+s-1)/2} = \sum \alpha^M V_M^{(s)}(\xi). \tag{4}$$

Here

$$(\alpha, \xi) = \sum_{i=1}^{n} \alpha_i \xi_i, \ ||\alpha|||^2 = (\alpha, \alpha), \alpha^M = \alpha_1^{m_1} \cdots \alpha_n^{m_n},$$

$|M| = m_1+m_2+\cdots+m_n$, and the summation in equations (2), (4), and in what follows is meant to be an n-fold sum over all indices from zero to infinity. A related set of polynomials

$$U_M^{(s)}(\xi) = U_M^{(s)}(\xi_1, \cdots, \xi_n) \tag{5}$$

is defined by the generating function

$$\{[(\alpha, \xi)-1]^2+||\alpha||^2(1-||\xi||^2)\}^{-s/2} = \sum \alpha^M U_M^{(s)}(\xi). \tag{6}$$

For $s > -1$, $s \neq 0$, these polynomials satisfy the biorthogonality relation

$$\begin{aligned}\int_{S(0;1)} &(1-||\xi||^2)^{(s-1)/2} V_M^{(s)}(\xi)\, U_L^{(s)}(\xi)\, d^n\xi \\ &= \delta_{LM} \frac{2\pi^{n/2}\Gamma(s/2+1)(s)_{|M|}}{(2|M|+n+s-1)\Gamma\left(\frac{n}{2}+\frac{s}{2}-\frac{1}{2}\right)M!}\end{aligned} \tag{7}$$

where $\delta_{LM} = \delta_{l_1 m_1} \cdots \delta_{l_n m_n}$, $M! = m_1! \cdots m_n!$ and $S(0; 1)$ is the real solid n dimensional ball $\{\xi \,|\, ||\xi|| \leqq 1\}$. A basic result concerning Appell series is the following theorem obtained by the author and R. P. Gilbert in [5] and [6]:

THEOREM 1.1. *Let $f(\xi)$ be an analytic function of n complex variables in some neighbourhood $\tilde{\eta}$ of the unit ball $S(0; 1)$. Then $f(\xi)$ can be expanded in an Appell series*

$$f(\xi) = \sum a_M^{(s)} V_M^{(s)}(\xi);\ s > -1,\ s \neq 0 \tag{8}$$

which converges uniformly for $\xi \in \eta\ \{[S(0;1)\} \cap \eta^*\{S(0;1)\}$ *with* $S(0;1) \subset \eta\{S(0;1)\} \subset \tilde{\eta}$, $\eta^* = \{\xi | \xi^* \in \eta\}$ (* *denotes complex conjugation*), *and where the coefficients are given by the formula*

$$\begin{aligned} a_M^{(s)} &= h_M^s \int_{S(0;1)} (1-||\xi||^2)^{(s-1)/2} f(\xi) U_M^{(s)}(\xi)\, d^n\xi \\ &\text{with} \\ h_M^s &= \frac{(2|M|+n+s-1)\Gamma\left(\frac{n}{2}+\frac{s}{2}-\frac{1}{2}\right) M!}{2\pi^{n/2}\Gamma\left(\frac{s}{2}+1\right)(s)_{|M|}}. \end{aligned} \tag{9}$$

There exists a neighbourhood η *of the unit ball* $S(0;1)$ *such that the series* (8) *converges uniformly to a holomorphic function for* $\xi \in \eta$ *if and only if the function*

$$F(\xi) = \sum a_M^{(s)} \xi^M \tag{10}$$

can be continued to a holomorphic function on

$$\overline{\Delta(0;1)} = \{\xi | \sum_{i=1}^{n} |\xi_i|^2 \leqq 1\}.$$

Theorem 1.1 leads to an expansion theorem for solutions to equation (1) which are analytic functions of $(x, \rho) = (x_1, \cdots, x_n, \rho)$. This can be seen as follows. The plane $\rho = 0$ is a singular surface of the regular type with indices 0 and $1-s$ ([11]). Hence there always exist solutions of equation 1) which are analytic on some portion of the plane $\rho = 0$, and if $s \neq 0, -2, -4, \cdots$ such solutions can be continued across this plane as even functions of ρ. Each such regular solution is analytic in a domain D that is symmetric with respect to the plane $\rho = 0$ and can therefore be expressed as $u(x, \rho) = \tilde{u}(r, \xi)$ where r, $\xi = (\xi_1, \cdots, \xi_n)$ are the hyper-zonal coordinates defined as

$$\begin{aligned} x_1 &= r\xi_1 \\ x_2 &= r\xi_2 \\ &\vdots \\ x_n &= r\xi_n \\ \rho &= r(1-\sum_{i=1}^{n} \xi_i^2)^{\frac{1}{2}} \\ r^2 &= x_1^2 + \cdots + x_n^2 + \rho^2. \end{aligned} \tag{11}$$

In these coordinates the differential equation (1) becomes ([9], p. 229)

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$$r^2(L^{(n)}_{\lambda,s}[\tilde{u}]) = \frac{1}{r^{n+s-2}}\frac{\partial}{\partial r}\left(r^{n+s}\frac{\partial\tilde{u}}{\partial r}\right) + \lambda^2 r^2\tilde{u} + n(s-1)\tilde{u} \tag{12}$$
$$+\sum_{j=1}^{n}\frac{\partial}{\partial\xi_j}\left\{\frac{\partial\tilde{u}}{\partial\xi_j} - \xi_j\left(\sum_{k=1}^{n}\xi_k\frac{\partial\tilde{u}}{\partial\xi_k} + (s-1)\tilde{u}\right)\right\} = 0.$$

From the above discussion and theorem 1.1 it is seen that for each r (where the sphere of radius r is contained in D) it is possible to expand $\tilde{u}(r, \xi)$ in an Appell series

$$\tilde{u}(r, \xi) = \sum a^{(s)}_M(r)V^{(s)}_M(\xi) \tag{13}$$

and then use equations (9), (12) and the differential equation satisfied by the $U^{(s)}_M(\xi)$ ([10] p. 278) to conclude that

$$a^{(s)}_M(r) = r^{-\frac{1}{2}(n+s-1)}[c^{(s)}_M J_{|M|+(n+s-1)/2}(\lambda r) + d^{(s)}_M H^{(1)}_{|M|+(n+s-1)/2}(\lambda r)] \tag{14}$$

where J_μ is a Bessel function of order μ, $H^{(1)}_\mu$ is a Hankel function of the first kind of order μ, and $c^{(s)}_M$, $d^{(s)}_M$ are constants. Hence we have the following theorem ([5]. [6]):

THEOREM 1.2. *Let $\tilde{u}(r, \xi)$ be a regular solution of $L^{(s)}_{\lambda,s}$ $[\tilde{u}] = 0$ in the exterior of a bounded domain D let and $s > -1$, $s \neq 0$. Then for $r \geqq a$ (where a is such that $r = a$ contains D) $\tilde{u}(r, \xi)$ can be expanded as*

$$\tilde{u}(r, \xi) = \sum a^{(s)}_M(r)V^{(s)}_M(\xi) \tag{15}$$

where the coefficients $a^{(s)}_M(r)$ are given by equation (14). *For each fixed r the series* (15) *converges uniformly for ξ contained in some complex neighbourhood of* $S(0; 1)$.

The following theorem can be proved directly from the differential equation (12) satisfied by $\tilde{u}(r, \xi)$. The reader is referred to [3] for details in the case of two independent variables.

THEOREM 1.3. *Let $\tilde{u}(r, \xi)$ be a regular solution of $L^{(n)}_{\lambda,s}[\tilde{u}] = 0$ in a domain D. Then for $1 \leqq i \leqq n$,*

$$\tilde{u}^+(r, \xi) = \frac{1}{r}\frac{\partial\tilde{u}(r, \xi)}{\partial\xi_i}$$

is a regular solution of $L^{(n)}_{\lambda,s+2}[\tilde{u}] = 0$ in D.

By using the relationships ([10], p. 176, 275)

$$\frac{dC^\nu_m(x)}{dx} = 2\nu C^{\nu+1}_{m-1}(x), \qquad m \geqq 1 \tag{16}$$

$$||b||^m C^{\frac{1}{2}(n+s-1)}_m\left[\frac{(b, \xi)}{||b||}\right] = \sum_{m_1+\cdots+m_n=m} b_1^{m_1}\cdots b_n^{m_n} V^{(s)}_{m_1,\cdots,m_n}(\xi_1, \cdots \xi_n) \tag{17}$$

where $C_m^\nu(x)$ denotes Gegenbauer's polynomial of index ν, one can verify that for $m_i \geqq 1, i \leqq l \leqslant n$,

$$\frac{\partial V^{(s)}_{m_1,\cdots,m_n}}{\partial \xi_i}(\xi_1,\cdots,\xi_n) = (n+s-1)V^{(s+2)}_{m_1,\cdots,m_i-1,\cdots,m_n}(\xi_1,\cdots,\xi_n). \tag{18}$$

Equation (18) shows that for $s \neq 0, -1, -2, \cdots$ there exist no nontrivial representations of zero of the form

$$\sum a_M^{(s)} V_M^{(s)}(\xi) = 0 \tag{19}$$

with the series 19) converging uniformly in a complex neighbourhood of $S(0; 1)$. This follows by observing that if such a representation existed the series could be differentiated termwise with respect ot ξ_i, $1 \leqq i \leqq n$, as often as desired, resulting in a series of the form (19) with $s > 0$. Use of the biorthogonality property shows that all the coefficients of this latter series are zero and hence the original series (19) consists of only a finite number of terms. Since for $\nu \neq 0, -1, -2, \cdots$ the Appell polynomials are of degree exactly m, (c.f.[10], p. 274) each of the coefficients in this finite series must be identically zero, i.e. $a_M^{(s)} = 0$ for every M.

Theorem 1.3 now enables us to extend the result of theorem 1.2 to include the cases $s < -1, s \neq -2, -3, \cdots$.

THEOREM 1.4. *If $\tilde{u}(r, \xi)$ is a regular solution of $L^{(n)}_{\lambda,s}[\tilde{u}] = 0$ in the exterior of a bounded domain D and $s \neq 0, -1, -2, \cdots$ then for r sufficiently large $\tilde{u}(r, \xi)$ can be expanded as*

$$\tilde{u}(r, \xi) = \Sigma\, a_M^{(s)}(r)\, V_M^{(s)}(\xi) \tag{20}$$

where the coefficients $a_M^{(s)}(r)$ are given by equation (14). *For each fixed r the series converges uniformly for ξ contained in some complex neighbourhood of* $S(0; 1)$.

PROOF. For $s > -1$ this result is given in theorem 1.2. Let $-2 < s < -1$. Then by theorem 1.3 $r^{-1}\operatorname{grad}_\xi \tilde{u}(r, \xi)$ is a vector whose components $\tilde{u}_i(r, \xi)$ are regular solutions of $L^{(n)}_{\lambda,s+2}[\tilde{u}] = 0$ in the exterior of D. Hence by theorem 1.2

$$\tilde{u}_i(r, \xi) = \textstyle\sum_i a_M^{(s+2)}(r)\, V_M^{(s+2)}(\xi) \tag{21}$$

where the subscript i denotes the dependence of the coefficient on $\tilde{u}_i(r, \xi)$, We furthermore have

$$\frac{\partial u_j}{\partial \xi_i} = \frac{\partial u_i}{\partial \xi_j}. \tag{22}$$

Since the series (21) converges uniformly for ξ contained in a complex neighbourhood of $S(0; 1)$ it is possible to differentiate (21) termwise. Doing this and using equations (18), (22) and the observation following equation (18) we have for $m_i, m_j \geqq 1$

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$$(23)\qquad {}_j a^{(s+2)}_{m_1,\cdots,m_{j-1},\cdots m_n}(r) = {}_i a^{(s+2)}_{m_1,\cdots,m_{i-1},\cdots m_n}(r).$$

For $|M| \geqq 1$ let

$$(24)\qquad a^{(s)}_M(r) = a^{(s)}_{m_1,\cdots m_n}(r) = (n+s-1)^{-1} r\, {}_i a^{(s+2)}_{m_1,\cdots,m_{i-1},\cdots m_n}(r)$$

where m_i is a non zero index. By equation (23) the $a^{(s)}_M(r)$ are well defined. Now let $\tilde{u}^+(r, \xi)$ be defined by the formal series

$$(25)\qquad \tilde{u}^+(r, \xi) = \sum_{\mu=1}^{\infty} \sum_{|M|=\mu} a^{(s)}_M(r) V^{(s)}_M(\xi).$$

Since each $\tilde{u}_i(r, \xi)$, $1 \leqq i \leqq n$, is regular for $r > a$ (where the sphere $r = a$ contains D) the results of [5] and [6] show that the associated functions $F_i(\xi) \equiv F_i(r, \xi)$ defined in theorem 1.1 are analytic for $r > a$, $\xi \in \overline{\Delta(0; 1)}$. Equations (21) and (24) now show that

$$(26)\qquad F(\xi) = \sum_{\mu=1}^{\infty} \sum_{|M|=\mu} a^{(s)}_M(r) \xi^M$$

defines a holomorphic function of r, ξ for $r > a$, $\xi \in \overline{\Delta(0; 1)}$. Hence from Cauchy's theorem for several complex variables (c.f. [9], p. 5) there exists a vector $\beta = (\beta_1, \cdots, \beta_n)$ where $\beta_i > 0$ for $1 \leqq i \leqq n$ and $||\beta|| < 1$ such that for r on compact subsets of (a, ∞)

$$(27)\qquad |a^{(s)}_M(r)| \leqq C\beta^M$$

where C, B_i, $1 \leqq i \leqq n$ are positive constants which depend on the size of the compact subset of (a, ∞). By considering the generating function (4) as a power series in $\alpha_1, \cdots, \alpha_n$ it is seen that for fixed $\gamma < 1$ the series in equation (4) converges absolutely and uniformly for $\alpha \in \overline{\Delta(0, \gamma)}$ and ξ lying in some complex neighbourhood of $S(0; 1)$ whose size depends on γ. From the several variable analogue of the Weierstrass comparison theorem and equation (4) it is now possible to conclude that the series (25) converges uniformly for r on compact subsets of (a, ∞) and ξ contained in some complex neighbourhood of $S(0; 1)$ provided γ is chosen close enough to one, i.e. γ and hence the size of the complex neighbourhood of $S(0; 1)$ will depend on the particular compact subset of (a, ∞) that is chosen. A similar analysis shows that termwise differentiation with respect to $r, \xi_1, \cdots, \xi_n$ is permissible for $\xi \in S(0; 1)$ and r on compact subsets of (a, ∞) with the resulting series being uniformly convergent. Hence $\tilde{u}^+(t, \xi)$ defines a regular solution of $L^{(n)}_{\lambda,s}[\tilde{u}] = 0$ for $r > a$, $\xi \in S(0; 1)$. Termwise differentiation now gives

$$(28)\qquad \operatorname{grad}_\xi[\tilde{u}(r, \xi) - \tilde{u}^+(r, \xi)] = 0.$$

Hence $\tilde{u}(r, \xi) - \tilde{u}^+(r, \xi)$ is a solution of equation 12) which depends only on r which implies that

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(29) $$\tilde{u}(r,\xi)-\tilde{u}^{+}(r,\xi) = a_0^{(s)}(r)$$

with $a_0^{(s)}(r)$ being of the form defined in equation (14). Hence for $-2 < s < -1$

(30) $$\tilde{u}(r,\xi) = \sum a_M^{(s)}(r)V_M^{(s)}(\xi).$$

This proves the theorem for $-2 < s < -1$ and the complete theorem follows by induction on s.

2. Decomposition theorems

We now proceed to derive decomposition theorems for generalized metaharmonic functions in several independent variables which are analogous to those obtained in [2] for two independent variables. We begin with the case when $s > -1, s \neq 0$.

THEOREM 2.1. *Assume* $s > -1$, $s \neq 0$. *Let* $\tilde{u}(r,\xi)$ *be a regular solution of equation* (12) *in the exterior of a bounded domain* D. *Then* $\tilde{u}(r,\xi)$ *can be uniquely decomposed as*

(31) $$u(r,\xi) = U(r,\xi)+V(r,\xi)$$

where $U(r,\xi)$ *is a regular solution of equation* (12) *in the entire* $n+1$ *dimensional Euclidean space* R^{n+1} *and* $V(r,\xi)$ *is a regular solution of equation* 12) *in the exterior of* D *which satisfies the radiation condition*

(32) $$\lim_{r\to\infty} r^{\frac{1}{2}(n+s)}\left(\frac{\partial V}{\partial r}-i\lambda V\right) = 0$$

uniformly for $\xi \in S(0;1)$.

PROOF. Let $0 < a < r$ and consider the function $\Omega(a,\zeta;r,\xi)$ defined by the formal series

(33) $$\Omega(a,\zeta;r,\xi) = (ar)^{-\frac{1}{2}(n+s-1)}\,\Sigma\,(h_M^s)^{-1}J_{|M|+(n+s-1)/2}(\lambda a)\cdot H^{(1)}_{|M|+(n+s-1)/2}(\lambda r)U_M^{(s)}(\zeta)V_M^{(s)}(\xi).$$

By using theorem 1.1 and the asymptotic formulae ([10]. p. 4,8)

(34) $$\Gamma(\mu)\left(\frac{\lambda a}{2}\right)^{-\mu}J_\mu(\lambda a) = 1+o(1); \quad \mu\to\infty$$

(35) $$-\frac{\pi}{i}\,\frac{\left(\frac{\lambda r}{2}\right)^{\mu}}{\Gamma(\mu)}\,H_\mu^{(1)}(\lambda r) = 1+o(1); \quad \mu\to\infty$$

(which hold uniformly for a, r on compact subsets of the positive real axis) it is seen from the several variable analogue of the Weierstrass comparison theorem

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and Hartog's theorem that for $0 < a < r$ the series (33) converges uniformly to a holomorphic function of the $2n$ variables ζ, ξ for (ζ, ξ) contained in some complex neighbourhood of $S(0;1)\times S(0;1)$. (Cauchy's formula for several complex variables is applied to equation 6) in order to obtain bounds for $U_M^{(s)}(\zeta)$.) A similar analysis shows that for a, r, ζ, ξ as indicated above, termwise differentiation is permissible and $\Omega(a, \zeta; r, \xi)$ converges uniformly to a solution of equation 12) both as a function of (a, ζ) and of (r, ξ). Now let a be chosen such that D is contained in the sphere of radius a and consider the solution to equation 12) defined for $r > a$ by

$$V(r,\xi) = \frac{a^{n+s}\pi}{2i}\int_{S(0;1)} r^{n+s}(1-\|\zeta\|^2)^{(s-1)2}\left[\tilde{u}(a,\zeta)\frac{\partial\Omega(a,\zeta;r,\xi)}{\partial a} - \Omega(a,\zeta;r,\xi)\frac{\partial\tilde{u}(a,\zeta)}{\partial a}\right]d\zeta^n. \tag{36}$$

Using the relation ([10], p. 80)

$$\frac{2i}{\pi a} = J_\mu(\lambda a)\frac{dH_\mu^{(1)}(\lambda a)}{da} - H_\mu^{(1)}(\lambda a)\frac{dJ_\mu(\lambda a)}{da} \tag{37}$$

equations (7), (15), (33), and the uniform convergence of the series under consideration, we have

$$V(r,\xi) = r^{-\frac{1}{2}(n+s-1)}\sum d_M^{(s)}H^{(1)}_{|M|+(n+s-1)/2}(\lambda r)V_M^{(s)}(\xi) \tag{38}$$

where the series (38) is uniformly convergent for each fixed $r > a$, ξ contained in some complex neighbourhood of $S(0;1)$. (Details of this last calculation for the case $n = 1$ are provided in [2]). By using the Lommel polynomials to express $H^{(1)}_{|M|+(n+s-1)/2}(\lambda r)$ in terms of $H^{(1)}_{(n+s-1)/2}(\lambda r)$ and $H^{(1)}_{(n+s-3)/2}(\lambda r)$, substituting this relationship into the series (38), and then rearranging terms, it can be seen that the solution $V(r, \xi)$ satisfies the radiation condition (32) uniformly for ξ contained in some complex neighbourhood of $S(0; 1)$. The details of this last operation are identical to the case when $n = 1$ and the reader is referred to [1] and [14] for more information. From equation (15) we therefore can write

$$\tilde{u}(r,\xi) = U(r,\xi)+V(r,\xi);\ r > a \tag{39}$$

where

$$U(r,\xi) = r^{-\frac{1}{2}(n+s-1)}\sum c_M^{(s)}J_{|M|+(n+s-1)/2}(\lambda r)V_M^{(s)}(\xi) \tag{40}$$

is uniformly convergent for each fixed $r > a$, ξ contained in some complex neighbourhood of $S(0; 1)$. From the results of [8] and [9] it is seen that $U(r, \xi)$ can be continued analytically into all of R^{n+1}, and this fact along with equation (39) shows that $U(r, \xi)$ is an everywhere regular solution of equation 12). (It is now clear that $V(r, \xi)$ is regular in the exterior of D and not: only for $r > a$.) The decomposition (31) is unique since if $U(r, \xi)$ is a generalized metaharmonic function which is regular in the entire plane and also satisfies the radiation condition (32), then the biorthogonality property (7) and the series representation (15) shows that $U(r, \xi)$ must be identically zero.

THEOREM 2.2. *Assume* $s < -1$, $s \neq -2, -3, -4, \cdots$. *Let* $\tilde{u}(r, \xi)$ *be a regular solution of equation* (12) *in the exterior of a bounded domain* D. *Then* $\tilde{u}(r, \xi)$ *can be uniquely decomposed as*

$$\tilde{u}(r, \xi) = U(r, \xi) + V(r, \xi) \tag{41}$$

where $U(r, \xi)$ *is a regular solution of equation* (12) *in the entire* $n+1$ *dimensional Euclidean space* R^{n+1} *and* $V(r, \xi)$ *is a regular solution of equation* (12) *in the exterior of* D *which satisfies the radiation condition*

$$\lim_{r \to \infty} r^{(n+s)/2} \left(\frac{\partial V}{\partial r} - i\lambda V \right) = 0 \tag{42}$$

uniformly for ξ *contained in some complex domain inclosing* $S(0; 1)$ *in its interior.*

PROOF. First let $-2 < s < -1$. Let the sphere of radius a contain D in its interior. Using theorem 1.4 $\tilde{u}(r, \xi)$ can be expressed as

$$\tilde{u}(r, \xi) = \sum a_M^{(s)}(r) V_M^{(s)}(\xi); \; r > a \tag{43}$$

where the coefficients are given by equation (14) and for each fixed $r > a$ the series (43) can be differentiated with respect to ξ_i, $1 \leqq i \leqq n$. Using this fact, theorem 1.3, and equation 18) we have that $r^{-1} \operatorname{grad}_\xi \tilde{u}(r, \xi)$ is a vector whose components $\tilde{u}_i(r, \xi)$ are regular solutions of $L_{\lambda, s+2}^{(n)}[\tilde{u}] = 0$ for $r > a$ and have the expansion

$$\tilde{u}_i(r, \xi) = \sum {}_i a_M^{(s+2)}(r) V_M^{(s+2)}(\xi); \; r > a \tag{44}$$

where

$${}_i a_M^{(s+2)}(r) = {}_i a_{m_1, \cdots, m_n}^{(s+2)}(r) = r^{-1}(n+s-1) {}_i a_{m_1, \cdots, m_{i+1}, \cdots, m_n}^{(s)}(r). \tag{45}$$

By theorem 2.1 it is possible to conclude that for each i, $1 \leqq i \leqq n$, the series

$$r^{-\frac{1}{2}(n+s-1)} \Sigma_i d_M^{(s+2)} H_{|M|+(n+s+1)/2}^{(1)}(\lambda r) V_M^{(s+2)}(\xi) \tag{46}$$

where

$${}_i d_M^{(s+2)} = {}_i d_{m_1, \cdots, m_n}^{(s+2)} = (n+s-1) d_{m_1, \cdots, m_{i+1}, \cdots m_n}^{(s)} \tag{47}$$

is uniformly convergent for each fixed $r > a$ and ξ contained in some complex neighbourhood of $S(0; 1)$. By using this fact and arguments similar to those used in theorem 1.4 it can be concluded that

$$V(r, \xi) = r^{-\frac{1}{2}(n+s-1)} \Sigma \, d_M^{(s)} H_{|M|+(n+s-1)/2}^{(1)}(\lambda r) V_M^{(s)}(\xi) \tag{48}$$

converges to a solution of $L_{\lambda,s}^{(n)}[\tilde{u}] = 0$ for $r > a$ and ξ contained in some complex neighbourhood of $S(0; 1)$. By arguements analogous to those used in theorem 2.1 it is seen that if $U(r, \xi)$ is defined by

$$(49) \qquad U(r, \xi) = r^{-\frac{1}{2}(n+s-1)} \sum c_M^{(s)} J_{|M|+(n+s-1)/2}(\lambda r)\, V_M^{(s)}(\xi)$$

then $V(r, \xi)$ and $U(r, \xi)$ have the properties ascribed to them in the theorem. We now come to the question of the uniqueness of the decomposition, and this is where it is necessary to require that the radiation condition (42) be valid in a complex domain instead of simply for the closed unit ball $S(0; 1)$. For suppose the decomposition (41) is not unique. Then there exists a nontrivial solution $U(r, \xi)$ of $L_{\lambda,s}^{(n)}[\tilde{u}] = 0$ which is regular in R^{n+1} and also satisfies the complex radiation condition (42). From Vitali's theorem for several complex variables ([12]), and the radiation condition (42), we have that for $1 \leqq i \leqq n$

$$(50) \qquad \lim_{r\to\infty} r^{\frac{1}{2}(n+s+2)} \left(\frac{1}{r} \frac{\partial^2 U}{\partial r \partial \xi_i} - i \frac{\lambda}{r} \frac{\partial U}{\partial \xi_i} \right) = 0$$

uniformly for ξ contained in some complex domain inclosing $S(0; 1)$ in its interior. But

$$(51) \qquad \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial U}{\partial \xi_i} \right) = \frac{1}{r} \frac{\partial^2 U}{\partial r \partial \xi_i} - \frac{1}{r^2} \frac{\partial U}{\partial \xi_i}$$

or

$$(52) \qquad \lim_{r\to\infty} r^{\frac{1}{2}(n+s+2)} \left(\frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial U}{\partial \xi_i} \right] + \frac{1}{r} \left[\frac{1}{r} \frac{\partial U}{\partial \xi_i} \right] - i \left[\frac{\lambda}{r} \frac{\partial U}{\partial \xi} \right] \right) = 0$$

uniformly for ξ contained in the above mentioned complex domain. By theorem 1.3 we have that $U_i = 1/r\, \partial U/\partial \xi_i$ is a regular solution of $L_{\lambda,s+2}^{(n)}[\tilde{u}] = 0$ in R^{n+1}. Hence $U_i(r, \xi)$ can be represented as ([8])

$$(53) \qquad U_i(r, \xi) = r^{-\frac{1}{2}(n+s+1)} \Sigma_i c_M^{(s+2)} J_{|M|+(n+s+1)/2}(\lambda r)\, V_M^{(s+2)}(\xi).$$

By using the biorthogonality property 7) and equation (52) it is seen that ${}_i c_M^{(s+2)} = 0$ for all M and hence $U_i(r, \xi)$ is identically zero for each i, $1 \leqq i \leqq n$. Hence $U(r, \xi)$ is a function of r alone i.e.

$$(54) \qquad U(r, \xi) = \text{const.}\, r^{-\frac{1}{2}(n+s-1)} J_{(n+s-1)|2}(\lambda r).$$

The radiation condition (42) and the asymptotic expansion (which can be differentiated with respect to r)

$$(55) \qquad J_\mu(\lambda r) = \sqrt{\frac{2}{\pi \lambda r}} \cos \left(\lambda r - \frac{\mu \pi}{2} - \frac{\pi}{4} \right) + 0 \left(\frac{1}{r^{\frac{3}{2}}} \right); \qquad r \to \infty$$

now shows that $U(r, \xi)$ must be identically zero, which is the desired contradiction.

For $s < -2, s \neq -3, -4, \cdots$ the decomposition follows by induction on s. The uniqueness of the decomposition follows by repeated application of theorem 1.3 in the manner just completed and by observing that if a finite series of the form

$$(56) \qquad r^{-\frac{1}{2}(n+s-1)} \sum_{\mu=0}^{n} \sum_{|M|=\mu} c_M^{(s)} J_{|M|+(n+s-1)/2}(\lambda r) V_M^{(s)}(\xi)$$

satisfies the radiation condition (42) and $s \neq 0, -1, -2, \cdots$ then $c_M^{(s)} = 0$ for each M. This can be seen from the asymptotic expression (55) and the discussion following equation (18).

EXAMPLE 2.1. For $s < -1$ the radiation condition (42) must hold for ξ lying in a complex domain containing $S(0; 1)$ in its interior and cannot be weakened to hold only for $\xi \in S(0; 1)$. For in this latter case

$$(57) \qquad U(r, \xi) = R^{-\frac{1}{2}(n-2)} J_{(n-2)/2}(\lambda r)$$

where $R = r(\xi_1^2 + \cdots + \xi_n^2)^{\frac{1}{2}}$ is a solution of $L_{\lambda,s}^{(n)}[\tilde{u}] = 0$ which is regular in the entire plane. But equation (55) shows that $U(r, \xi)$ also satisfies the radiation condition if $s < -1$, i.e. the decomposition is no longer unique.

If $V(r, \xi)$ is a solution of equation 12) for $s < -1$, $s \neq -2, -3, -4 \cdots$ and satisfies the complex radiation condition (42), then by theorem 2.2 $V(r, \xi)$ has the representation

$$(58) \qquad V(r, \xi) = r^{-\frac{1}{2}(n+s-1)} \sum d_M^{(s)} H_{|M|+(n+s-1)/2}^{(1)}(\lambda r) V_M^{(s)}(\xi).$$

By using the Lommel polynomials to express $H_{|M|+(n+s-1)2}^{(1)}(\lambda r)$ in terms of $H_{(n+s-1)/2}^{(1)}(\lambda r)$ and $H_{(n+s-3)/2}^{(1)}(\lambda r)$, substituting this relationship into the series (58), and then rearranging terms (c.f. [1], [14]) it is seen that $V(r, \xi)$ can be represented asymptotically as

$$(59) \qquad V(r, \xi) = \frac{f(\xi)}{r^{\frac{1}{2}(n+s)}} e^{i\lambda r} + 0\left(\frac{1}{r^{\frac{1}{2}(n+s)+1}}\right); \qquad r \to \infty$$

where $V(r, \xi)$ is uniquely determined by its 'scattering amplitude' $f(\xi)$. Example 2.1. shows that if a complex radiation condition is not insisted upon, then $f(\xi)$ no longer uniquely determines $V(r, \xi)$, i.e. the inverse scattering problem (c.f. [8], [9]) is improperly posed.

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