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Notes on the Hodge conjecture for Fermat varieties

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Abstract
We review a combinatoric approach to the Hodge conjecture for Fermat varieties and announce new cases where the conjecture is true. We show the Hodge conjecture for Fermat fourfolds X^4_m of degree $m \le 100$ coprime to 6, and also prove the conjecture for X^n_{21} and X^n_{27}, for all n.

1. Introduction

The Hodge conjecture is major open problem in Complex Algebraic Geometry that has been puzzling mathematician for decades now. The modern statement is the following: Let X be smooth complex projective variety, then the (rational) cycle class map is surjective:

$$cl_{\otimes\mathbb{Q}} : CH^p(X)\otimes\mathbb{Q} \to H^{p,p}\cap H^{2p}(X,\mathbb{Q}),$$

where $cl_{\otimes\mathbb{Q}}(\sum a_i X_i) = \sum a_i [X_i]$, $a_i \in \mathbb{Q}$, and $[X_i]$ is the class of the subvariety X_i.

The case $p = 1$ is the only case that it is known to hold in general, which follows from Lefschetz's theorem on (1,1)-classes. Special cases have emerged during the years, but all of them were specific for certain classes of varieties, for example, abelian varieties of prime dimension, unirational and uniruled fourfolds, and hypersurfaces of degree less than 6. For a summary of known cases before 2000, see Lewis (1999), and for more recent approaches, see Arapura (2019), Bergeron et al. (2016), and Markman (2021).

Using hard Lefschetz theorem, Lefschetz hyperplane theorem, and some Hilbert scheme arguments, we can reduce the Hodge conjecture to the case of an even dimensional (>2) variety and primitive middle cohomology classes.

Shioda (1979) gave an interesting characterization of the Hodge conjecture for Fermat varieties, which we now review.

2. Shioda's work

Let $X^n_m \in \mathbb{P}^{n+1}$ denote the Fermat variety of dimension n and degree m, that is, the solution to the equation:

$$x_0^m + x_1^m + \ldots + x_{n+1}^m = 0,$$

and μ_m the group of mth roots of unity. Let G^n_m be quotient of the group $\overbrace{\mu_m \times \ldots \times \mu_m}^{n+2}$ by the subgroup of diagonal elements.

The group G_m^n acts naturally on X_m^n by coordinatewise multiplication; moreover, the character group $\widehat{G}_m^n$ of G_m^n can be identified with the group:

$$\widehat{G}_m^n = \{(a_0,\ldots,a_{n+1}) | a_i \in \mathbb{Z}_m, a_0+\ldots+a_{n+1}=0\}$$

via $(\zeta_0,\ldots,\zeta_{n+1}) \mapsto \zeta_0^{a_0}\ldots\zeta_{n+1}^{a_{n+1}}$, where $(\zeta_0,\ldots,\zeta_{n+1}) \in G_m^n$.

By the previous section, in order to prove the Hodge conjecture, it is enough to prove it for primitive classes; therefore, in this paper, we will focus on primitive cohomology $H^i_{prim}(X_m^n,\mathbb{C}) := \ker(\smile H^{n-i+1})$, where H is a hyperplane class. The action of G_m^n into the primitive cohomology and makes $H^i_{prim}(X_m^n,\mathbb{Q})$ and $H^i_{prim}(X_m^n,\mathbb{C})$ G_m^n-modules. For $\alpha \in \widehat{G}_m^n$, we set:

$$V(\alpha) = \left\{\xi \in H^n_{prim}(X_m^n,\mathbb{C}) | g^*(\xi) = \alpha(g)\xi \text{ for all } g \in G_m^n\right\}.$$

Before stating the characterization of Hodge classes, we need a few notations. Let

$$\mathfrak{U}_m^n := \left\{\alpha = (a_0,\ldots,a_{n+1}) \in \widehat{G}_m^n | a_i \neq 0 \text{ for all } i\right\}.$$

For $\alpha \in \mathfrak{U}_m^n$, we set $|\alpha| = \sum_i \frac{<a_i>}{m}$, where $<a_i>$ is the representative of $a_i \in \mathbb{Z}_m$ between 1 and $m-1$. *Suppose* $n = 2p$: then, we set

$$\mathfrak{B}_m^n := \{\alpha \in \mathfrak{U}_m^n \| t\alpha| = p+1 \text{ for all } t \in \mathbb{Z}_m^*\}.$$

Theorem 2.1 (Ran, 1980; Shioda, 1979). *Let* $Hdg^p := H^{p,p} \cap H^{2p}_{prim}(X,\mathbb{Q})$ *be the group of primitive Hodge cycles. Then*:

(a) $\dim V(\alpha) = 0$ *or* 1, *and* $V(\alpha) \neq 0 \Leftarrow \alpha \in \mathfrak{U}_m^n$,
(b) $Hdg^p = \bigoplus_{\alpha \in \mathfrak{B}_m^n} V(\alpha)$.

Now, let $C(X_m^n)$ denote the subspace of Hdg^p which are classes of algebraic cycles. Then, $C(X_m^n)$ is a G_m^n-submodule, and by the theorem above, there is a subset $\mathfrak{C}_m^n \subseteq \mathfrak{B}_m^n$ such that:

$$C(X_m^n) = \bigoplus_{\alpha \in \mathfrak{C}_m^n} V(\alpha)$$

the Hodge conjecture can then be stated as follows.

Conjecture 1 (Hodge conjecture). *For all n,m, we have* $\mathfrak{C}_m^n = \mathfrak{B}_m^n$.

By the discussion in the previous section, this is true for $n \leq 2$ and all m. The idea to prove this equality for Fermat varieties it to use the fact that X_m^n "contains" disjoint unions of X_m^k with $k < n$; we then blow those subvarieties up to find a relation between the cohomologies and to inductively construct algebraic cycles in X_m^n. More precisely, we have:

Theorem 2.2 (Shioda, 1979). *Let* $n = r+s$ *with* $r,s \geq 1$. *Consider the map* $h: G_m^r \times G_m^s \to G_m^n$, *which sends* $([\zeta_0,\ldots,\zeta_r,1],[\zeta'_0,\ldots,\zeta'_s,1])$ *to* $([\zeta_0,\ldots,\zeta_r,\zeta'_0,\ldots,\zeta'_s])$, *and set* $\mu_m := \ker(h)$. *Then, there is an isomorphism*

$$f: \left[H^r_{prim}(X_m^r,\mathbb{C}) \otimes H^s_{prim}(X_m^s,\mathbb{C})\right]^{\mu_m} \oplus H^{r-1}_{prim}(X_m^{r-1},\mathbb{C}) \otimes H^{s-1}_{prim}(X_m^{s-1},\mathbb{C}) \xrightarrow{\sim} H^n_{prim}(X_m^n,\mathbb{C})$$

with the following properties:

(a) *f is* G_m^n*-equivariant.*
(b) *f is a morphism of Hodge structures of type (1,1) on the first summand and of type* (1,1) *on the second.*

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(c) *If $n = 2p$, then f preserves algebraic cycles; moreover, if*

$$Z_1 \otimes Z_2 \in H^{r-1}_{prim}(X^{r-1}_m, \mathbb{C}) \otimes H^{s-1}_{prim}(X^{s-1}_m, \mathbb{C}),$$

then $f(Z_1 \otimes Z_2) = mZ_1 \wedge Z_2$, where $Z_1 \wedge Z_2$ is the algebraic cycle obtained by joining Z_1 and Z_2 by lines on X^n_m, when Z_1, Z_2 are viewed as cycles in X^n_m.

In light of this theorem, we introduce the following notation:

$$\mathfrak{U}^{r,s}_m = \{(\beta,\gamma) \in \mathfrak{U}^r_m \times \mathfrak{U}^s_m \mid \beta = (b_0,\ldots,b_{r+1}),\ \gamma = (c_0,\ldots,c_{s+1}), \text{ and } b_{r+1} + c_{s+1} = 0\}.$$

For $(\beta,\gamma) \in \mathfrak{U}^{r,s}_m$, we define:

$$\beta \# \gamma = (b_0,\ldots,b_r,c_0,\ldots,c_s) \in \mathfrak{U}^{r+s}_m,$$

and for $\beta' = (b_0,\ldots,b_r) \in \mathfrak{U}^{r-1}_m$ and $\gamma' = (c_0,\ldots,c_s) \in \mathfrak{U}^{s-1}_m$, we set:

$$\beta' * \gamma' = (b_0,\ldots,b_r,c_0,\ldots,c_s) \in \mathfrak{U}^{r+s}_m.$$

Then, we have a bijection

$$\mathfrak{U}^{r,s}_m \coprod (\mathfrak{U}^{r-1}_m \times \mathfrak{U}^{s-1}_m) \leftrightarrow \mathfrak{U}^{r+s}_m.$$

Using the theorem above, we have:

$$V(\beta\#\gamma) = f(V(\beta) \otimes V(\gamma)),$$
$$V(\beta' * \gamma') = f(V(\beta') \otimes V(\gamma')).$$

Corollary 2.3. Suppose $n = 2p = r + s$, where $r, s \geq 1$.

(a) If r, s are odd and $(\beta',\gamma') \in \mathfrak{C}^{r-1}_m \times \mathfrak{C}^{s-1}_m$, then $\beta' * \gamma' \in \mathfrak{C}^n_m$.
(b) If r, s are even and $(\beta,\gamma) \in (\mathfrak{C}^r_m \times \mathfrak{C}^s_m) \cap \mathfrak{U}^{r,s}_m$, then $\beta\#\gamma \in \mathfrak{C}^n_m$.

By the above corollary, the Hodge conjecture can be proved inductively for the Fermat X^n_m if the following conditions are true for every $\alpha \in \mathfrak{B}^n_m$:

(P1) $\alpha \sim \beta' * \gamma'$ for some $(\beta',\gamma') \in \mathfrak{B}^{r-1}_m \times \mathfrak{B}^{s-1}_m (r, s \text{ odd})$,
(P2) $\alpha \sim \beta\#\gamma$ for some $(\beta,\gamma) \in (\mathfrak{B}^r_m \times \mathfrak{B}^s_m) \cap \mathfrak{U}^{r,s}_m (r, s \text{ even and positive})$,

where $\sim$ means equality up to permutation between factors.

In order to make these conditions more explicit, we introduce the additive semigroup M_m of nonnegative solutions $(x_1,\ldots,x_{m-1};y)$ with $y > 0$, to the following system of linear equations:

$$\sum_{i=1}^{m-1} < ti > x_i = my \text{ for all } t \in \mathbb{Z}^*_m.$$

Furthermore, define $M_m(y)$ as those solutions, where y is fixed. Note that by Gordan's lemma, M_m is finitely generated.

Definition 2.4. An element $a \in M_m$ is called *decomposable* if $a = c + d$ for some $c, d \in M_m$; otherwise, it is called *indecomposable*. An element b is called *quasi-decomposable* if $a + b = c + d$ for some $a \in M_m(1)$ and $c, d \in M_m$ with $c, d \neq b$.

With this notation, we can identify elements of $\mathfrak{B}^n_m$ with elements of M_m using the map:

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$$\{\}:\alpha=(a_0,\dots,a_{n+1})\in\mathfrak{B}^n_m\mapsto\{\alpha\}=\left(x_1(\alpha),\dots,x_{m-1}(\alpha),\frac{n}{2}+1\right)\in M_m\left(\frac{n}{2}+1\right),$$

where $x_k(\alpha)$ is the number of i's such that $<a_i> =k$.

Note that α satisfies $(P1)$ above if and only if $\{\alpha\}$ is decomposable. If α satisfies $(P2)$, then $\{\alpha\}$ is quasi-decomposable. Conversely, if the latter is true, then α satisfies $(P1)$ or $(P2)$. So it makes sense to introduce the following conditions:

(P^n_m) Every indecomposable element of $M_m(y)$ with $3\leq y\leq \frac{n}{2}+1$, if any, is quasi-decomposable.
(P_m) Every indecomposable element of $M_m(y)$ with $y\geq 3$ is quasi-decomposable.

By the results above, we conclude:

Theorem 2.5 (Shioda, 1979). *If condition (P_m) is satisfied, then the Hodge conjecture is true for X^n_m, for any n. If (P^n_m) is satisfied, then the Hodge conjecture is true for X^n_m.*

For m prime or $m = 4$, M_m is generated by $M_m(1)$, which gives:

Theorem 2.6 (Ran, 1980; Shioda, 1979). *If m is prime or $m = 4$, the Hodge conjecture is true for X^n_m, for all n. Shioda manually verified condition (P_m) for m 20 and concluded:*

Theorem 2.7 (Shioda, 1979). *If m 20, the Hodge conjecture is true for X^n_m, for all n.*

Starting at $m=21$, the number of indecomposables and the length of elements of M_m are very large, so it is hard to verify (P_m) by hand for unknown cases, unless $m=p^2$ is a square of a prime. In the latter case, condition (P_m) is not always true; it is false for $m=25$, for example. However, Aoki (1987) explicitly constructed the algebraic cycles that generate each $V(\alpha)$; such cycles are called *standard* cycles.

Theorem 2.8 (Aoki, 1987). *If $m = p^2$, the Hodge conjecture is true for X^n_m, for all n, even though condition (P_m) may be false.*

3. New cases of the Hodge conjecture

A natural question is whether or not the Hodge conjecture can always be proved using condition (P_m). As described above, there are false negatives, that is, (P_m) is false, but the Hodge conjecture is still true. This is due to the fact that there are cycles not coming from the induced structure (see Aoki, 1987).

The next obvious question is then for which values of m, if any, the condition (P_m) is false, besides $m = p^2$. In such cases, one expects (if one believes the Hodge conjecture) that there are cycles, not of standard type as in Aoki (1987), such that they too do not come from the induced structure. Alternatively, they are candidates for a counterexample to the Hodge conjecture.

We used SAGE math to answer that question by investigating when condition (P_m) is true. All the code used in this section can be found in the Appendix.

In the case of Fermat fourfolds, we computed, first, all the indecomposable elements with length 3, because the (2,2) cycles have length exactly 3. So the idea was to find values of m for which there were none of them.

Proposition 3.1. If $m\leq 100$ is an integer coprime to 6, then the Hodge conjecture is true for all Fermat fourfolds X^4_m.

This is a strong evidence that the Hodge conjecture should be true for fourfolds X^4_m where m is coprime to 6. It suggests the structure of $\mathfrak{B}^4_m$, namely, if $3|m$, then there are indecomposables elements of length 3 (see Proposition 3.4 below).

The following corollary is immediate by Shioda (1979).

Corollary 3.2. If $m_i\leq 100$ are integers coprime to 6, then the Hodge conjecture is true for arbitrary products of Fermat fourfolds $X^4_{m_1}\times\ldots\times X^4_{m_k}$.

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We slightly extended Shioda's work by verifying condition (P_m) for $m = 21, 27$. A computational proof can be found in the Appendix.

Theorem 3.3. *The Hodge conjecture is true for Fermats X_{21}^n and X_{27}^n, for all n.*

An interesting case is $m = 33$, where condition (P_m) is false, because we have explicitly found a Hodge class that is not quasi-decomposable and is not of standard type either.

Proposition 3.4. Condition P_{33} is false. More precisely, the following cycle

$$(0,0,0,0,0,0,1,0,0,1,0,0,1,0,0,0,0,0,1,0,0,1,0,0,0,0,0,1,0,0,0,0,3)$$

supported on X_{33}^4 is not quasi-decomposable in M_{33}.

This proposition confirms that starting at $n = 4$, there are cycles not coming from the induced structure. Therefore, we cannot prove the Hodge conjecture only using this approach. One thing that can be done is to find explicitly the algebraic cycles whose class project nontrivially to $V(\alpha)$ for each $\alpha \in \mathfrak{B}_m^n$ (see Aoki, 1987).

In the particular case where $m = 3d$ and $3 \nmid d$, as above, we have a candidate. Consider the following elementary symmetric polynomials in $\mathbf{x} = (x_0, \ldots, x_5)$:

$$\begin{aligned} p_1(\mathrm{x}) &:= x_0 + x_1 + x_2 + x_3 + x_4 + x_5, \\ p_2(\mathrm{x}) &:= x_0x_1 + x_0x_2 + \ldots + x_4x_5, \\ p_3(\mathrm{x}) &:= x_0x_1x_2 + \ldots + x_3x_4x_5. \end{aligned} \tag{3.1}$$

Recall the Newton identity:

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = p_1(\mathrm{x})^3 - 3p_1(\mathrm{x})p_2(\mathrm{x}) + 3p_3(\mathrm{x}). \tag{3.2}$$

Set $\mathrm{x}^d = \left(x_0^d, \ldots, x_5^d\right)$, then:

$$x_0^m + x_1^m + x_2^m + x_3^m + x_4^m + x_5^m = p_1\left(x^d\right)^3 - 3p_1\left(x^d\right)p_2\left(x^d\right) + 3p_3\left(x^d\right). \tag{3.3}$$

Let W denote the following variety in $\mathbb{P}^5$:

$$p_1\left(\mathrm{x}^d\right) = p_2\left(\mathrm{x}^d\right) = p_3\left(\mathrm{x}^d\right) = 0. \tag{3.4}$$

By construction, $W \subseteq X_m^4$ is a subvariety of codimension 2, so $[W] \in Hdg^2\left(X_m^4\right)$.

Question 1. *Can $[W]$ project nontrivially in $V(\alpha)$ for every $\alpha \in \mathfrak{B}_m^4$ which is not quasi-decomposable and not of standard type?*

If the answer is yes, then we would have a positive answer to the Hodge conjecture in this case.

We know by Gordan's lemma that the number of indecomposable elements is finite. Given $m \in \mathbb{Z}_+$, in order to prove the Hodge conjecture for X_m^n and any n, it is enough to prove for all $X_m^n, n \leq n'$, where $n' = 2(m' - 1)$ and m' is the largest length of all the indecomposable elements in M_m.

Let $\mathscr{I}_m$ be set of indecomposable elements of M_m. Define $\phi : \mathbb{Z}_+ \to \mathbb{Z}_+$ by the rule

$$\phi(m) = \{\max y \mid (x_1, \ldots x_{m-1}, y) \in \mathscr{I}_m\}. \tag{3.5}$$

We have the following:

Proposition 3.5. If the Hodge conjecture is true for X_m^n, for all $n \leq 2(\phi(m) - 1)$, then it is true for X_m^n and any n.

Proof. The Hodge classes in X_m^n are parametrized by $\mathcal{B}_m^n$, which can be viewed inside M_m as elements of length $\frac{n}{2} + 1$. Since the indecomposables generate M_m, it is enough that those be classes of algebraic cycles. But that is the case if the Hodge conjecture is true when $\frac{n}{2} + 1 \leq \phi(m)$, by definition of $\phi(m)$.

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Therefore, for Fermat varieties of degree m, we do not need to check the Hodge conjecture in every dimension. It is enough to prove the result for dimension up to $2(\phi(m)-1)$.

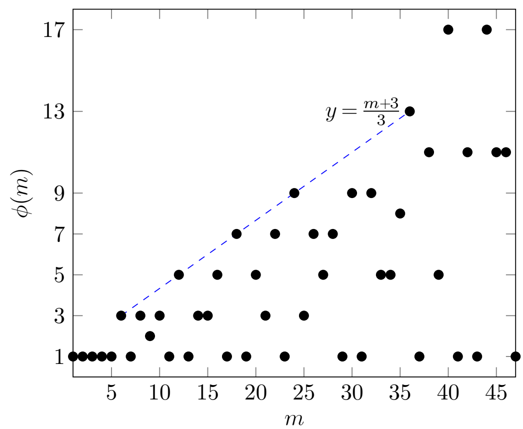
Clearly then, it is desirable to find an expression for the function $\phi(m)$. For m prime or $m = 4$, we know already that $\phi(m) = 1$. Furthermore, by Aoki (1987), we know that for $p > 2$ prime, $\phi(p^2) = \frac{p+1}{2}$. Here is a table with the a few values of $\phi(m)$:

m	$\phi(m)$	m	$\phi(m)$	m	$\phi(m)$	m	$\phi(m)$
20	5	26	7	32	9	38	11
21	3	27	5	33	5	39	5
22	7	28	7	34	5	40	17
23	1	29	1	35	8	41	1
24	9	30	9	36	13	42	11
25	3	31	1	37	1	43	1

Based on the values above and the ones already computed, we believe the following is true.

Conjecture 2. *For $p > 2$ prime, we have $\phi\left(p^k\right) = \frac{p^{k-1}+1}{2}$, and $\phi\left(2^l\right) = 2^{l-2} + 1$, for $l > 2$.*

Computing $\phi(m)$ for $m < 48$ gives the following:



For $m < 48$, computations become more time-consuming, and specially if m has a lot of prime powers in its prime decomposition. But the results obtained here give us a glimpse about the structure of M_m and, consequently, the Hodge conjecture in the case of Fermat varieties.

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Appendix: A code implementation of Shioda's approach

The following SAGE code, for a fixed $m > 1$, checks whether or not every indecomposable element is quasi-decomposable. If such condition is true, then the Hodge conjecture holds for X_m^n.

```
from itertools import product
from fractions import gcd
import sys
import numpy as np
#from sage.geometry.polyhedron.backend_normaliz import Polyhedron_normaliz

def how_many_indec(m):
    p = MixedIntegerLinearProgram(base_ring=QQ)
    w = p.new_variable(integer=True, nonnegative=True)
    for k in range(1,m):
        if gcd(k,m) == 1:
            l=0
            for i in range(1,m):
                l += ((i*k) % m)*w[i-1]
            l += -m*w[m-1]
            #print l
            p.add_constraint(l == 0)
    p.add_constraint(w[m-1] >= 1)
    indec = p.polyhedron(backend='normaliz').integral_points_generators()[0]
    indec_less = [ x for x in indec if x[-1]> 2]
    return len(indec_less)

def poly_sol(m):
    p = MixedIntegerLinearProgram(base_ring=QQ)
    w = p.new_variable(integer=True, nonnegative=True)
    for k in range(1,m):
        if gcd(k,m) == 1:
            l=0
            for i in range(1,m):
                l += ((i*k) % m)*w[i-1]
            l += -m*w[m-1]
            p.add_constraint(l == 0)
    p.add_constraint(w[m-1] >= 1)
    return p.polyhedron(backend='normaliz')

def lengthOne(m):
    p = MixedIntegerLinearProgram(base_ring=QQ)
    w = p.new_variable(integer=True, nonnegative=True)
    for k in range(1,m):
        if gcd(k,m) == 1:
            l=0
            for i in range(1,m):
                l += ((i*k) % m)*w[i-1]
            l += -m*w[m-1]
            p.add_constraint(l == 0)
    p.add_constraint(w[m-1] == 1)
    return p.polyhedron(backend='normaliz').integral_points()
```

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```
def get_indec(m):
    p = MixedIntegerLinearProgram(base_ring=QQ)
    w = p.new_variable(integer=True, nonnegative=True)
    #print 'x is %d and m is %d' % (x,m)
    for k in range(1,m):
        if gcd(k,m) == 1:
            l=0
            for i in range(1,m):
                l += ((i*k) % m)*w[i-1]
            l += -m*w[m-1]
            #print l
            p.add_constraint(l == 0)
    p.add_constraint(w[m-1] >= 1)
    return p.polyhedron(backend='normaliz').integral_points_generators()[0]


def get_standard(m,primes):
    result = []
    for p in primes:
        d = m/p
        if p == 2:
            for i in range(1,m):
                if (p*i) % m != 0 :#(d/gcd(i,d))>2:#(p*i) % m != 0 and
                    2*((p*i) % m) != m:##
                    temp = [i,(i+d) % m,(m-2*i) % m,d]
                    #print temp
                    std = []
                    for e in range(1,m):
                        std.append(temp.count(e))
                    std.append(2)
                    if tuple(std) not in result:
                        result.append(tuple(std))
        else:
            for i in range(1,m):
                if (p*i) % m != 0: #and 2*((p*i) % m) != m:#d/gcd(i,d)>2:
                    #print i
                    temp = [0]*(p+1)
                    for k in range(p):
                        temp[k]= (i+k*d) % m
                    temp[p]=(m-p*i) % m
                    #print temp
                    std = []
                    for e in range(1,m):
                        std.append(temp.count(e))
                    std.append((p+1)/2)
                    #print gcd(i,d)
                    #print tuple(std)
                    #print '--'
                    if p%2 == 1:
                        if tuple(std) not in result:
                            result.append(tuple(std))
                    else:
```

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```
                        if tuple(std) not in result:
                            result.append(tuple(2*x for x in std))
    return result

def reverse_to(y,m):
    r=[]
    n = len(y)-2
    for e in range(1,m):
        r.append(y.count(e))
    r = r + [n/2 + 1]
    return tuple(r)

def convert_to_u(x,m):
    last = x[-1]
    n = 2*(last-1)
    r = []
    for k in range(m-1):
        if x[k] != 0:
            r = r + [k+1]*x[k]
    return tuple(r)

def get_points_length_less_m(x,m):
    p = MixedIntegerLinearProgram(base_ring=QQ)
    w = p.new_variable(integer=True, nonnegative=True)
    #print 'x is %d and m is %d' % (x,m)
    for k in range(1,m):
        if gcd(k,m) == 1:
            l=0
            for i in range(1,m):
                l += ((i*k) % m)*w[i-1]
            l += -m*w[m-1]
            #print l
            p.add_constraint(l == 0)
    p.add_constraint(w[m-1] >= 1)
    p.add_constraint(w[m-1] <= x)
    return p.polyhedron(backend='normaliz').integral_points()

arr = []

def get_indec_less(m,prm):
    p = poly_sol(m)
    print 'getting indecomposable elements for |m= %d| ...' % m
    indec = p.integral_points_generators()[0]
    #length_one = [ x for x in indec if x[-1]==1]
    #print 'there are %d length one' % len(length_one)
    standards = [list(x) for x in get_standard(m,prm)]
    print 'there are %d STANDARDS ELEMENTS' % len(standards)
    indec_less = [ x for x in indec if x[-1]>= 3 and list(x) not in
        standards]
    print 'there are %d indec of length>=3' % len(indec_less)
    return indec_less

def prime_factors(n):
```

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```
    i = 2
    factors = []
    while i * i <= n:
        if n % i:
            i += 1
        else:
            n //= i
            if i not in factors:
                factors.append(i)
    if n > 1:
        if n not in factors:
            factors.append(n)
    #print('finished computing primes:',factors)
    return factors
m = 21
primes = prime_factors(m)
quasi = []
dict_ = {}
indec_less = get_indec_less(m,primes)
lasts_ =[]
length_one = lengthOne(m)
for el in indec_less:
    last = el[-1]
    print '---'
    print el
    print 'position: %d' % indec_less.index(el)
    count=0
    if last not in dict_:
        possible = get_points_length_less_m(last,m)
        dict_[last] = possible
        for el2,el3,el4 in product(length_one,possible,possible):
            if el + el2 == el3 + el4 and (el != el3 and el != el4):
                print 'I am quasi'
                quasi.append(el)
                break
            samples=len(possible)*len(possible)*len(length_one)
            count+=1
            sys.stdout.write("Progress: %.2f%% \r" %
                 (float(100*count)/samples))
            sys.stdout.flush()
        if el not in quasi:
            print 'This element is not quasi'
            print 'The HC CAN NOT be predicted for degree %d using this
                method, there are only %d quasi of %d' %
                 (m,len(quasi),len(indec_less))
            break
    else:
        for el2,el3,el4 in product(length_one,dict_[last],dict_[last]):
            if el + el2 == el3 + el4 and (el != el3 and el != el4):
                print 'I am quasi'
                quasi.append(el)
                break
            samples=len(dict_[last])*len(dict_[last])*len(length_one)
```

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```
            count+=1
            sys.stdout.write("Progress: %.2f%% \r" %
                (float(100*count)/samples))
            sys.stdout.flush()
        if el not in quasi:
            print 'This element is not quasi'
            print 'The HC CAN NOT be predicted for degree %d using this
                method, there are only %d quasi of %d' %
                (m,len(quasi),len(indec_less))
            break
print 'The HC is TRUE for degree %d fermats' % m
```

Peer Reviews

Reviewing editor: Dr. Adrian Clingher[1,2]
[1]University of Missouri at Saint Louis, Mathematics and Computer Science, One University Blvd, St. Louis, Missouri, United States, 63121
[2]UMSL

This article has been accepted because it is deemed to be scientifically sound, has the correct controls, has appropriate methodology and is statistically valid, and has been sent for additional statistical evaluation and met required revisions.

doi:10.1017/exp.2021.14.pr1

Review 1: Notes on the Hodge Conjecture for Fermat Varieties

Reviewer: Prof. James Lewis

Date of review: 30 May 2021

Conflict of interest statement. No conflict of interest.

Comments to the Author: This paper provides an excellent overview of what is known about the Hodge conjecture for Fermat hypersurfaces. Additionally, new cases of the Hodge conjecture are obtained experimentally for those Fermat hypersurfaces where the inductive structure of the Fermat variety in question fails to supply all of the algebraic cycles. There are very minor spelling corrections that the author can easily detect using spell-check.

Score Card

Presentation

4.7 /5

Is the article written in clear and proper English? (30%)	4/5
Is the data presented in the most useful manner? (40%)	5/5
Does the paper cite relevant and related articles appropriately? (30%)	5/5

Context

5.0 /5

Does the title suitably represent the article? (25%)	5/5
Does the abstract correctly embody the content of the article? (25%)	5/5
Does the introduction give appropriate context? (25%)	5/5
Is the objective of the experiment clearly defined? (25%)	5/5

Analysis

5.0 /5

Does the discussion adequately interpret the results presented? (40%)	5/5
Is the conclusion consistent with the results and discussion? (40%)	5/5
Are the limitations of the experiment as well as the contributions of the experiment clearly outlined? (20%)	5/5

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doi:10.1017/exp.2021.14.pr2

Review 2: Notes on the Hodge Conjecture for Fermat Varieties

Reviewer: Prof. Matt Kerr

Washington University in Saint Louis, Mathematics and Statistics, 1 Brookings Drive, Campus Box 1146, Saint Louis, Missouri, United States, 63130-4899

Date of review: 17 June 2021

Conflict of interest statement. Reviewer declares none.

Comments to the Author: The author revisits a strategy introduced by T. Shioda to attack the Hodge Conjecture for Fermat hypersurfaces in projective space. The idea is to determine when a Hodge class of higher degree ultimately arises from products and joins of divisors, by solving a combinatorial puzzle regarding generation of a semigroup. Using a computer implementation he is able to push this strategy somewhat further in several different ways, establishing the Hodge Conjecture for a large class of Fermat fourfolds and for all Fermats of degrees 21 or 27, but also making new contributions to the strategy. In particular, he conjectures bounds on degrees of the semigroup generators and proposes a new class of algebraic cycles which should represent certain non-quasi-decomposable generators in degree 3, thus providing algebraic cyclists with a pair of interesting and important problems. This is a very nice article for inclusion in Experimental Results.

Score Card

Presentation

5.0 /5

Is the article written in clear and proper English? (30%)	5/5
Is the data presented in the most useful manner? (40%)	5/5
Does the paper cite relevant and related articles appropriately? (30%)	5/5

Context

5.0 /5

Does the title suitably represent the article? (25%)	5/5
Does the abstract correctly embody the content of the article? (25%)	5/5
Does the introduction give appropriate context? (25%)	5/5
Is the objective of the experiment clearly defined? (25%)	5/5

Analysis

5.0 /5

Does the discussion adequately interpret the results presented? (40%)	5/5
Is the conclusion consistent with the results and discussion? (40%)	5/5
Are the limitations of the experiment as well as the contributions of the experiment clearly outlined? (20%)	5/5