Bull. Austral. Math. Soc. 16D90
Vol. 53 (1996) [455–458]

A CHARACTERISATION OF REFLEXIVE MODULES

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We characterise reflexive modules over the rings R such that each finitely generated submodule of $E({}_RR)$ is torsionless (left QF-$3''$ rings) by means of a suitable linear compactness condition relative to the Lambek torsion theory.

1. Introduction

There are a number of papers in which the reflexive R-modules with respect to the R-dual functors (or, more generally, to the "dual functors" defined by a bimodule) are characterised. The rings considered are usually generalisations of quasi-Frobenius (QF) rings and the characterisations obtained often involve some kind of linear compactness condition, inspired by the result of Müller [**8**] that shows that the reflexive modules in a Morita duality are precisely the linearly compact modules.

In recent times, Masaike [**7**] characterised the reflexive modules over QF-3 rings, and this was later extended in [**4**] and [**2**] to QF-$3'$ rings, that is, to the rings R such that both $E({}_RR)$ and $E(R_R)$ are torsionless. There is a rather more general class of rings which retains a good deal of the satisfactory behaviour of QF-$3'$ rings vis-a-vis duality. It is the class of QF-$3''$ rings, namely, the rings R such that every finitely generated submodule of $E({}_RR)$ and of $E(R_R)$ is torsionless. As it was shown in [**6**], a ring is QF-$3''$ if and only if its left maximal ring of quotients is a QF-$3''$ (two-sided) maximal quotient ring. This shows that QF-$3''$ rings are very abundant: in particular, all integral domains are QF-$3''$. Observe, however, that a QF-$3''$ ring may not be QF-$3'$, even if it is noetherian and its maximal quotient ring is a field.

The duality properties of QF-$3''$ rings have been studied in [**1, 3, 4, 5, 6**]. A left R-module X is called Lambek-linearly compact [**2**] if for every inverse system $\{p_i : X \longrightarrow X_i\}_I$ in $R-\mathrm{Mod}$ such that the X_i are torsionless and Coker p_i is a Lambek-torsion module, Coker $\left(\varprojlim p_i\right)$ is also Lambek-torsion. It is then proved in [**4**, Remark, p.9] and [**2**, Proposition 2.3] that if R is left QF-$3''$ and X is Lambek-linearly compact, then X is reflexive if and only if $R-\mathrm{dom.dim}\, X \geqslant 2$. However, the

Received 27th July, 1995
Work partially supported by the DGICYT (PB93-0515, Spain). The first author was also partially supported by the European Community (Contract CHRX-CT93-0091) and the Xunta de Galicia (XUGA 10502B94), and the second author by the Comunidad Autónoma de Murcia (PIB 94-25).

$\mathbb{Z}$-module $\mathbb{Z}^{(\mathbb{N})}$ provides an example of a reflexive module over a QF-$3''$ ring which is not Lambek-linearly compact (see [**2**, Remark following Corollary 2.6]). The purpose of this note is to show that the reflexive modules over (left) QF-$3''$ rings can still be characterised by a (more general) linear compactness condition.

Throughout this paper R denotes an associative ring with identity and $R-\mathrm{Mod}$ (respectively $\mathrm{Mod}-R$) the category of left (respectively right) R-modules. If X and M are left R-modules, X is said to have M-dominant dimension $\geqslant 2$ ($M-\mathrm{dom.dim}\, X \geqslant 2$) when there exists an exact sequence $0 \to X \longrightarrow Y \longrightarrow Z$, with Y and Z isomorphic to direct products of copies of X. The ring R is said to be left QF-$3''$ (see [1]) when each finitely generated submodule of $E({}_RR)$ is torsionless.

We shall denote by $\mathcal{T}_M$ the localising subcategory of $R-\mathrm{Mod}$ cogenerated by the injective envelope $E(M)$ of M.

2. Reflexive modules

We shall fix a module $M \in R-\mathrm{Mod}$ and write $S = \mathrm{End}({}_RM)$. The M-dual functors $\mathrm{Hom}_R(-,M)$ and $\mathrm{Hom}_S(-,M)$ will be denoted by $(\ )^*$, and their composition in either order by $(\ )^{**}$. For each $X \in R-\mathrm{Mod}$ there is a canonical (evaluation) morphism $\sigma_X : X \longrightarrow X^{**}$. σ_X is a monomorphism precisely when X is M-cogenerated, and when σ_X is an isomorphism, X is said to be M-reflexive (or just reflexive if we take $M = {}_RR$).

An inverse system $\{p_i : X \longrightarrow X_i\}_I$ in $R-\mathrm{Mod}$ will be called a $\mathcal{T}_M$-inverse system whenever the X_i are M-cogenerated and $\mathrm{Coker}\, p_i \in \mathcal{T}_M$ for every $i \in I$. The inverse system will be called M-complete if for every $f : X \longrightarrow M$ there exist an index $i \in I$ and a morphism $f_i : X_i \longrightarrow M$ such that $f = f_i \circ p_i$. We shall say that a module X is $\mathcal{T}_M$-linearly compact when for each $\mathcal{T}_M$-inverse system $\{p_i : X \longrightarrow X_i\}_I$, $\mathrm{Coker}\left(\varprojlim p_i\right) \in \mathcal{T}_M$. (This concept was introduced by Hoshino and Takashima in [4].) If this property holds just for all the M-complete $\mathcal{T}_M$-inverse systems $\{p_i : X \longrightarrow X_i\}_I$, then we say that X is $\mathcal{T}_M$-weakly linearly compact. In the particular case that $M = {}_RR$, we have that $\mathcal{T}_M = \mathcal{L}$ is the Lambek localising subcategory (see [9]), and thus we say that X is Lambek-weakly linearly compact when the cokernel of the inverse limit of every R-complete Lambek-inverse system $\{p_i : X \longrightarrow X_i\}_I$ in $R-\mathrm{Mod}$ is a Lambek-torsion module.

We are now ready to give our main result.

Theorem 2.1. *Let $M \in R-\mathrm{Mod}$ be such that every finitely M-generated submodule of $E(M)$ is M-cogenerated. Then the following conditions are equivalent for any left R-module X:*

(i) *X is M-reflexive.*

(ii) *For every M-complete $\mathcal{T}_M$-inverse system $\{p_i : X \longrightarrow X_i\}_I$, $\varprojlim p_i$ is an isomorphism.*

(iii) *X is $\mathcal{T}_M$-weakly linearly compact and $M - \operatorname{dom.dim} X \geqslant 2$.*

PROOF: (i) $\Rightarrow$ (ii), (iii) Let $\{p_i : X \longrightarrow X_i\}_I$ be an M-complete $\mathcal{T}_M$-inverse system in $R-\text{Mod}$. Since $\operatorname{Coker} p_i \in \mathcal{T}_M$, we have a direct system of monomorphisms in $\text{Mod}-S$, $\{p_i^* : X_i^* \longrightarrow X^*\}_I$. Now, the M-completeness hypothesis implies that, for each $f \in X^*$, there exist $i \in I$ and $f_i \in X_i^*$ such that $f = f_i \circ p_i = p_i^*(f_i)$, so that $\varinjlim p_i^*$ is an epimorphism, and hence an isomorphism. Therefore, $\varprojlim p_i^{**} = \left(\varinjlim p_i^*\right)^*$ is also an isomorphism. On the other hand we have that, for each $i \in I$, $p_i^{**} \circ \sigma_X = \sigma_{X_i} \circ p_i$. On taking inverse limits, we obtain:

$$\varprojlim p_i^{**} \circ \sigma_X = \varprojlim \sigma_{X_i} \circ \varprojlim p_i.$$

Since σ_X is an isomorphism by hypothesis and, as we have just seen, $\varprojlim p_i^{**}$ is also an isomorphism, we see that $\varprojlim \sigma_{X_i} \circ \varprojlim p_i$ is an isomorphism. Since the X_i are M-cogenerated, the σ_{X_i} are monomorphisms and so is $\varprojlim \sigma_{X_i}$. This shows that $\varprojlim p_i$ is an isomorphism and we see that (i)$\Rightarrow$(ii) holds. In particular, X is $\mathcal{T}_M$-weakly linearly compact. It is also clear that $M - \operatorname{dom.dim} X \geqslant 2$, for if $R^{(J)} \longrightarrow R^{(I)} \longrightarrow X^* \to 0$ is a free presentation of X^* in $\text{Mod}-S$, then applying the functor $(\ )^*$ and bearing in mind that X is M-reflexive, we obtain an exact sequence $0 \to X \longrightarrow M^I \longrightarrow M^J$ in $R-\text{Mod}$.

(iii)$\Rightarrow$(ii) Let $\{p_i : X \longrightarrow X_i\}_I$ be an M-complete $\mathcal{T}_M$-inverse system in $R-\text{Mod}$. We see as before that $\varprojlim p_i^{**} = \left(\varinjlim p_i^*\right)^*$ is an isomorphism and hence that, up to an isomorphism, $\sigma_X = \varprojlim \sigma_{X_i} \circ \varprojlim p_i$. Since X is M-cogenerated, σ_X is a monomorphism, and so also is $\varprojlim p_i$. Moreover, since $M - \operatorname{dom.dim} X \geqslant 2$, we see using, for example, [2, Lemma 2.2] that $\operatorname{Coker} \sigma_X$ is M-cogenerated. Now $\varprojlim \sigma_{X_i}$ is a monomorphism and so $\operatorname{Coker}\left(\varprojlim p_i\right)$ embeds in $\operatorname{Coker} \sigma_X$. By hypothesis, $\operatorname{Coker}\left(\varprojlim p_i\right) \in \mathcal{T}_M$ and thus we obtain that $\operatorname{Coker}\left(\varprojlim p_i\right) = 0$ and hence that $\varprojlim p_i$ is an isomorphism.

(ii)$\Rightarrow$(i) Let $\{u_i : Y_i \longrightarrow X^*\}_I$ be the direct system of all the finitely generated submodules of X^* in $\text{Mod}-S$, with u_i the canonical inclusions. By [4, Lemma 2.1], whose proof can be easily adapted to the more general case we are considering here, each $u_i^* \circ \sigma_X$ has $\mathcal{T}_M$-torsion cokernel and so we have a $\mathcal{T}_M$-inverse system in $R-\text{Mod}$ $\{u_i^* \circ \sigma_X : X \longrightarrow Y_i^*\}_I$. Let us show that this inverse system is also M-complete. Indeed, if $f \in X^* = \varinjlim Y_i$, then there exists an $i \in I$ such that $f = u_i(f_i)$ for some $f_i \in Y_i \subseteq X^*$. Since, by adjunction, $\sigma_X^* \circ \sigma_{X^*} = 1_{X^*}$, we see that $f = (\sigma_X^* \circ \sigma_{X^*})(f) =$

$(\sigma_X^* \circ \sigma_{X^*} \circ u_i)(f_i) = (\sigma_X^* \circ u_i^{**} \circ \sigma_{Y_i})(f_i) = (u_i^* \circ \sigma_X)^*(\sigma_{Y_i}(f_i)) = \sigma_{Y_i}(f_i) \circ u_i^* \circ \sigma_X$. Thus our hypothesis implies that $\left(\varprojlim u_i^*\right) \circ \sigma_X$ is an isomorphism. Since $\varprojlim u_i^* = \left(\varinjlim u_i\right)^*$ is also an isomorphism, we see that σ_X is an isomorphism and X is M-reflexive. ∎

If we specialise the preceding theorem to the case in which $M = {}_R R$ is a left QF-3″ ring, we obtain the following characterisation of reflexive modules.

COROLLARY 2.2. *Let R be a left QF-3″ ring and X a left R-module. Then the following conditions are equivalent:*

(i) *X is reflexive.*

(ii) *For every R-complete Lambek-inverse system $\{p_i : X \longrightarrow X_i\}_I$, $\varprojlim p_i$ is an isomorphism.*

(iii) *X is Lambek-weakly linearly compact and $R - \mathrm{dom\,.dim}\, X \geqslant 2$.*

Observe that, as the $\mathbb{Z}$-module $\mathbb{Z}^{(\mathbb{N})}$ and its dual, the Specker group $\mathbb{Z}^{\mathbb{N}}$ show, a Lambek-weakly linearly compact module may contain an infinite direct sum of copies of a nonzero module. The same example shows that, despite the fact that the maximal quotient ring of a QF-3″ ring is also QF-3″, the rational completion of a reflexive module over a QF-3″ ring may not be reflexive as a module over the maximal quotient ring.

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