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SOME ALMOST SIMPLE RINGS

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1. Introduction. Herein, a ring is not required to have an identity. All rings are associative but not necessarily commutative. However, we specialize to the commutative case for some of our results. The paper is concerned primarily with rings having the property that all unbounded ideals or all unbounded homomorphic images are isomorphic to the ring. We say that a ring R is *bounded* if $nR = 0$ for some positive integer n; alternately, R, with or without 1, is said to have *finite characteristic*. Unbounded rings having the property that all proper subrings are bounded were characterized in [**8**]. In related work, R. Gilmer has studied in [**5**; **6**; **7**] rings all of whose proper subrings enjoy different properties P. Chew and Lawn [**2**], as well as Levitz and Mott [**11**], have recently considered rings R such that R/I is not only bounded but is finite for all nonzero ideals I of R. We are able to refine some of the results of [**8**] and [**11**] by a simple analysis of the additive ideals. Let R be a ring and let $G = (R; +)$, the additive group of R. We call a subgroup H of G an *additive ideal* of R if H is an ideal in *every* ring that has G as its additive group, not just R itself. For a determination of the additive ideals, see [**3**].

2. Rings whose unbounded subrings are isomorphic to the ring. In this section, we characterize those rings R that have the following property.

σ: Each unbounded subring S of R is isomorphic to R.

Rings having finite characteristic, of course, have property σ. More generally, rings having the property that each proper subring has finite characteristic certainly enjoy property σ. As we mentioned earlier, the latter class of rings has been characterized recently in [**8**]. This class consists only of bounded rings and null rings on quasi-cyclic p-groups. One might expect the class of rings having property σ to be significantly larger. However, we have the following theorem, which shows that this is not the case.

THEOREM 2.1. *A ring R has property σ if and only if R has finite characteristic or else R is the null ring on either the infinite cyclic group or a quasi-cyclic p-group.*

Proof. The "if" part of the theorem is trivial. Hence suppose that R has property σ and assume that R does not have finite characteristic. We wish to show that the multiplication on R is trivial and that the additive group of R

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is either the infinite cyclic group or a quasi-cyclic p-group. Let

$$T = \{x \in R: x \text{ has finite additive order}\}.$$

Then T is an (additive) ideal of R; in particular, T is a subring of R. Thus T has finite characteristic or else $T \cong R$. One should observe that $T \cong R$ if and only if $T = R$ because $T \cong R$ implies that every element of R has finite additive order. Hence, if $T \neq R$, T must have finite characteristic, so let $nT = 0$. Since R does not have finite characteristic, nR does not have finite characteristic. Therefore, $R \cong nR$. But for our choice of n, we observe that nR has no elements of finite additive order since $nT = 0$. Thus the isomorphism $R \cong nR$ implies that R has no elements of finite additive order, and $T = 0$. We conclude that one of the following must hold:

(i) $T = 0$, or

(ii) $T = R$.

In other words, the additive group of R can not be mixed—it is either torsion or torsion free.

Case 1. $T = 0$: In this case, $S \cong R$ for every subring $S \neq 0$ of R because S can not have finite characteristic. Thus R must be generated as a ring by a single element; in particular, R must be commutative. If the multiplication on R is trivial, then $(R; +)$ is a torsion free group G having the property that every subgroup $H \neq 0$ of G is isomorphic to G. Therefore, G is the infinite cyclic group. Thus we may assume that the multiplication on R is not trivial. Now, we eliminate the opposite extreme. Since any field of infinite characteristic contains a subring that is not a field, we conclude that R is not a field. Therefore, there exists $a \neq 0$ in R such that $aR \neq R$. Thus R must contain a proper subring S.

From the relations $(nR)(nR) \subseteq n(nR)$ and $R \cong nR$, we observe that $R^2 \subseteq nR$ for each positive integer n. Setting

$$R_0 = \bigcap_{n<\omega} nR,$$

we have $R^2 \subseteq R_0$. Since $R^2 \neq 0$, R_0 must be different from zero. However, since $(R; +)$ is torsion free, $(R_0; +)$ is divisible; indeed, $(R_0; +)$ is a sum of copies of the additive rational group Q. Since R_0 is a nonzero ideal of R, we have $R \cong R_0$. In particular, the additive part of R is isomorphic to the additive part of R_0, so $(R; +)$ is also divisible. We shall obtain a contradiction on the rank of $(R; +)$. Recall that R must have a proper subring S, and observe that $(S; +)$ must be divisible since $(S; +) \cong (R; +)$. This implies

$$(R; +) = (S; +) \oplus \sum Q,$$

where at least one Q appears. This, however, implies that $(R; +)$ has infinite rank (because S is isomorphic to R and we can decompose $(S; +)$ the same way as $(R; +)$). In order to show, on the other hand, that $(R; +)$ has finite rank, recall that $R = \langle x \rangle$ is generated by a single element x. Since $(R; +)$ is divisible, there exist integers $n_1, n_2, \ldots, n_{k-1}$ and $n_k \neq 0$ such that

$$2\sum_{i=1}^{k} n_i x^i = x.$$

Observe that $k > 1$ since x has infinite additive order. Now, from the above equation, we observe that x^j, for $j \geqq k$, is dependent on the set $\{x, x^2, \ldots, x^{k-1}\}$, so $(R; +)$ has finite rank. We have obtained our contradiction, and we have proved, in Case 1, that R must be the null ring on the infinite cyclic group.

Case 2. $T = R$: As usual, we argue that $(R; +)$ must be a p-group for some prime p. Set

$$R_p = \{x \in R: x \text{ has order a power of } p\}.$$

Clearly, R_p is an ideal of R. Likewise, $\sum R_p$ is an ideal of R where the summation is over any set of primes. If $R \ncong R_p$, then R_p must have finite characteristic. Moreover, if $R_p \neq 0$ for an infinite set S of primes one of which is q, then

$$R = \sum_{p \in S, p \neq q} R_p$$

leads to a contradiction. Thus we conclude that $R_p = 0$ for all but a finite number of primes p. Now, since R, itself, does not have finite characteristic, R_p fails to have finite characteristic for some prime p, and consequently $R \cong R_p$; whence $R = R_p$.

To conclude the proof of the theorem, we need to prove that if R is a ring satisfying property σ, not having finite characteristic, and having a p-group for its additive structure, then R has trivial multiplication and $(R; +)$ is quasi-cyclic. As in the torsion-free case, we observe that $R^2 \subseteq \bigcap_{n<\omega} p^n R$, but unlike the torsion-free case, in the primary case we can not deduce that $\bigcap_{n<\omega} nR$ is divisible. For simplicity of notation, let $p^\omega R = \bigcap_{n<\omega} p^n R$. As we have mentioned, $R^2 \subseteq p^\omega R$. Moreover, $(p^\omega R)^2 = 0$, for if $x, y \in p^\omega R$, then $p^n z = x$ for some $z \in R$ where $p^n y = 0$. If $p^\omega R$ does not have finite characteristic, then $R \cong p^\omega R$ and $R^2 = 0$, that is, R has trivial multiplication. Therefore, if R does not have trivial multiplication, then $p^\omega R$ has finite characteristic and we have $p^{\omega+n} R = 0$ for some positive integer n. However, this yields

$$(p^n R)^2 \subseteq p^n(R^2) \subseteq p^n(p^\omega R) = p^{\omega+n} R = 0,$$

so R has trivial multiplication after all since $R \cong p^n R$. The proof of the theorem is finished with the observation that the only unbounded abelian p-group G having the property that each subgroup H of G is either bounded or else isomorphic to G is the quasi-cyclic group. In order to verify this statement, first notice that we may assume that G is either reduced or divisible. If G is reduced, take a basic subgroup B of G (see [**4**]) to conclude that G is a direct sum of cyclic groups, which is obviously impossible. Hence G is divisible, and from the well-known structure of divisible p-groups (direct sums of quasi-cyclic p-groups), we see that G must be a single quasi-cyclic group.

We now give an interpretation of Theorem 2.1. Let $\mathscr{S}(P)$ denote the class of rings with the property that every proper subring satisfies a given condition P. Here and throughout, by a proper subring or proper ideal we mean a subring or ideal different from zero and different from the whole ring. Observe that $\mathscr{S}(P \vee P') \supseteq \mathscr{S}(P) \cup \mathscr{S}(P')$, but, in general, equality does not hold even if $\mathscr{S}(P') \subseteq \mathscr{S}(P)$ or, indeed, even if $\mathscr{S}(P')$ is void. However, Theorem 2.1 shows that equality does hold for the two conditions associated with the property of being a σ-ring:

P_1: finite characteristic,

P_2: isomorphic to R.

Thus we have

COROLLARY 2.2. $\mathscr{S}(P_1) \vee P_2) = \mathscr{S}(P_1) \cup \mathscr{S}(P_2)$, *that is, a ring R is a σ-ring if and only if either every proper suvring of R has finite characteristic or else every proper subring of R is isomorphic to R.*

We remark that it is easy (especially in view of Theorem 2.1) to show that a ring R has the property that each proper subring is isomorphic to R if and only if R is the null ring on an infinite cyclic group or R has a prime number of elements.

3. σ^*-rings. In this section, we consider rings R that have the following property.

σ^*: Each unbounded ideal of R is isomorphic to R.

We might think of property σ^* as meaning that each ideal of R is either small (bounded by a positive integer) or else large (isomorphic to the whole ring) there being no middle ground. By an *ideal* we always mean a two-sided ideal. A close analysis of the proof of Theorem 2.1 yields the following result.

PROPOSITION 3.1. *Let R be a σ^*-ring. Then one of the following must hold.*
(1) *R has finite characteristic.*
(2) *R is the null ring on a quasi-cyclic p-group.*
(3) *$(R; +)$ is torsion free.*

The succeeding corollary improves Theorem 2.2 in [**8**].

COROLLARY 3.2. *Suppose that R is a σ^*-ring and that R does not have finite characteristic. If R has a nonzero element of finite additive order, then R is the null ring on a quasi-cyclic p-group.*

In view of Proposition 3.1, our interest in σ^*-rings is restricted to the case that the additive group is torsion free. We call such a ring a *torsion-free* ring. The following lemma enables us to specialize further. First, observe that in a torsion-free σ^*-ring R every nonzero ideal I is isomorphic to R.

LEMMA 3.3. *If R is a torsion-free σ^*-ring, then R is either the null ring on the infinite cyclic group or else $(R; +)$ is the direct sum of additive rational groups, that is,*

$$(R; +) = \sum Q.$$

Proof. As we have already observed, if R has trivial multiplication then $(R; +)$ is infinite cyclic. Thus we may assume that $R^2 \neq 0$. Recall (from the proof of Theorem 2.1) that $(R_0; +) = \sum Q$ (or is zero) where $R_0 = \bigcap_{n<\omega} nR$. Thus it suffices to prove that $R_0 \neq 0$. From the relation $R \cong nR$ we obtain $R^2 \subseteq nR$ and, consequently, $R^2 \subseteq R_0$. Hence $R_0 \neq 0$ since $R^2 \neq 0$.

We remark that the preceding lemma generalizes Corollary 2.6 of [**8**]. By a *nontrivial* torsion-free σ^*-ring we mean a torsion-free σ^*-ring different from the null ring on the infinite cyclic group. We have shown that $(R; +)$ is a rational vector space if R is a nontrivial torsion-free σ^*-ring.

LEMMA 3.4. *If R is a nontrivial torsion-free σ^*-ring, then $RI = I = IR$ for every ideal I of R.*

Proof. First, we observe that $R^2 = R$. If $R^2 \neq R$, then $(R; +) = K \oplus (R^2; +)$ where K, as well as $(R^2; +)$, is a nonzero divisible torsion-free group. Notice that $K_0 \oplus (R^2; +)$ is an ideal of R for any subgroup K_0 of K. In particular, we can take $K_0 \cong Z$, but this contradicts the fact that $(R; +)$ is divisible since $(R; +) \cong Z \oplus (R^2; +)$. Thus we conclude that $R^2 = R$. Since R is a σ^*-ring, $I^2 = I$ for every ideal I of R. Finally, we have $I = I^2 \subseteq RI \cap IR$, which implies that $RI = I = IR$.

COROLLARY 3.5. *Let R be a nontrivial torsion-free σ^*-ring and let I be an ideal of R. If A is an ideal of I, then A is necessarily an ideal of R.*

Proof. From Lemma 3.4, we obtain $A = IAI$, an ideal of R.

LEMMA 3.6. *Any torsion-free σ^*-ring is Noetherian.*

Proof. Let R be a torsion-free σ^*-ring. To show that R is Noetherian, it suffices to show that R is not the union of an ascending chain of proper ideals since every nonzero ideal of R is isomorphic to R. Suppose that A is a proper ideal of R, and choose a nonzero element $a \in A$. Let B be the intersection of all the ideals of R that contain a. In view of Corollary 3.5, no proper ideal of B can contain a. Since $R \cong B$, there exists an element $r \in R$ such that no proper ideal of R contains r. It follows that R is not the union of an ascending chain of proper ideals, and the lemma is proved.

LEMMA 3.7. *A nontrivial torsion-free σ^*-ring is prime.*

Proof. Suppose that $AB = 0$ where A and B are both nonzero ideals. Since $(A \cap B)^2 \subseteq AB = 0$, we see that $(A \cap B)^2$, and therefore $A \cap B$, must be zero. Since $R \cong A \oplus B$, we notice (with a change in notation) that there are ideals I_1 and N of R such that $R = I_1 \oplus N$. Likewise, N decomposes as $N = I_2 \oplus M$ where I_2 and M are ideal of R. Continuing in this way, we obtain a sequence of ideals $I_1, I_2, \ldots, I_n, \ldots$ of R such that $I_n \cap (\sum_{j \neq n} I_j)$ is zero. Hence

$$I_1 \subset I_1 \oplus I_2 \subseteq \ldots \subseteq I_1 \oplus I_2 \oplus \ldots \oplus I_n \subseteq \ldots$$

is an ascending sequence of ideals of R. This contradicts the fact that R is Noetherian. Hence $AB \neq 0$ if $A \neq 0$ and $B \neq 0$, which implies that R is prime [**10**].

THEOREM 3.8. *Any torsion-free σ^*-ring with left (or right) identity is simple.*

Proof. Let e be a left identity for R. Suppose that R has a proper ideal I. Since $R \cong I$, I has a left identity $f \neq e$. Now $(e - f)f = 0$, which implies that $(e - f)I = 0$. Letting A denote the left annihilator of I, we see that $A \neq 0$ since $e - f \in A$. Since $AI = 0$, we have a contradiction that R is prime. The argument is obviously similar for a right identity.

COROLLARY 3.9. *Let R be a nontrivial torsion-free σ^*-ring. If R is commutative, then R is a field.*

Proof. Let e be a nonzero element of R. Denote by I the ideal of R generated by e and Re, that is, let $I = (e, Re)$. Since the additive group of I/Re is cyclic (with generator $e + Re$), we conclude that $I = Re$ because both $(I; +)$ and $(Re; +)$ and, consequently, $(I; +)/(Re; +)$ are direct sums of Q's. Hence $e \in Re$ and $e = fe$. The element f is the identity element of R because R is an integral domain according to Lemma 3.7. By Theorem 3.8, R is simple, that is, R is a field.

Summarizing our results for the commutative case, we state

COROLLARY 3.10. *Let R be a commutative ring with the property that each ideal of R is either isomorphic to R or else has finite characteristic. If R does not have finite characteristic and if R is not the null ring on the infinite cyclic group or a quasi-cyclic p-group, then R is a field.*

4. Rings having the property that every unbounded homomorphic image is isomorphic to the ring. We study the dual of σ-rings in this section. For brevity, call a ring R an η-ring if each unbounded homomorphic image of R is isomorphic to R. The determination of η-rings is more difficult than the determination of σ-rings that was accomplished in § 2. In fact, we are unable to give a complete characterization of η-rings even for the commutative case, but several conditions are imposed on a ring that are necessary for it to be an η-ring. We start with the following reduction theorem, which is beginning to look familiar. This time, however, the proof is more complicated.

THEOREM 4.1. *Suppose that R is an η-ring. Then one of the following must hold.*

(1) *R has finite characteristic.*
(2) *R is the null ring on a quasi-cyclic p-group.*
(3) $(R; +)$ *is torsion free.*

Proof. Let R be an η-ring of infinite characteristic; we shall show that either condition (2) or (3) must hold. Arguments similar to those employed in the proof of Theorem 2.1 immediately yield that $(R; +)$ is either torsion free or p-primary for some prime p. Thus what remains to verify is that R is necessarily the null ring on a quasi-cyclic p-group if $(R; +)$ is p-primary.

For simplicity of notation, let $(R; +)$ be denoted by G. First, assume that $p^\omega G \neq 0$. Since R is an η-ring and since $p^\omega G$ is an ideal of R, we conclude that $G/p^\omega G$ is bounded because $p^\omega(G/p^\omega G) = 0$ and $p^\omega G \neq 0$. However, $G/p^\omega G$ being bounded implies, for any abelian p-group, that $G = B \oplus D$ where B is bounded and D is divisible. Since R has infinite characteristic, D is not zero. But $B = 0$, for if we consider the ideal $G[p^n] = \{x \in G : p^n x = 0\}$ where $p^n B = 0$ we obtain

$$G \cong G/G[p^n] \cong D.$$

Since G is divisible, we know that the multiplication on R is trivial, so every subgroup H of G is an ideal of R. Therefore, G/H is either bounded or is isomorphic to G for each subgroup H of G. It follows quickly that G is a single quasi-cyclic group (since we already know that G is divisible). The proof of the theorem will be completed by showing that it is impossible for $p^\omega G$ to be equal to zero.

It is convenient to introduce the notion of a *large* subgroup of a primary abelian group due to Pierce [**12**]. According to [**12**, Theorem 2.7] any large subgroup L of a p-group G satisfies the equality

$$L = \{x \in G : p^i x \in p^{n(i)}G \text{ for all } i < \omega\},$$

where $n(0) < n(1) < \ldots < n(k) < \ldots$ is a fixed increasing sequence of nonnegative integers depending on the choice of L. Conversely, any such subgroup defined in this way is large. Clearly, in view of the above characterization of a large subgroup, any large subgroup L of $G = (R; +)$ is always an ideal of R. Thus L is an additive ideal of R. Furthermore, G/L is a direct sum of cyclic groups if L is a large subgroup of G; this follows from [**12**, Lemma 2.12] since

$$G/L = \{B, L\}/L \cong B/(B \cap L)$$

whenever B is a basic subgroup of G. If we take $n(i) = 2i$ for each $i \geqq 0$, then G/L is unbounded since G is and since $p^\omega G = 0$. Thus the isomorphism $G \cong G/L$ implies that G is a direct sum of cyclic groups. Next, we observe that not only is G a direct sum of cyclic groups but that

$$G = \sum_{i<\omega} \sum_{m} C(p^i)$$

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where the cardinal m is independent of i, that is, the decomposition of G into cyclic groups involves the same number of summands of order p^i for each i. This follows at once from the fact that, for each i, $G \cong G/G[p^i]$. Now we bring into play the multiplicative structure of R. Let $\mathscr{I}$ denote the collection of ideals I of R that satisfy the following two conditions:

(1) $pI = 0$,

(2) $(I; +)$ is a direct summand of $G = (R; +)$.

Note that $0 \in \mathscr{I}$. Furthermore, the (set-theoretic) union I of a chain $\{I_\alpha\}$ of ideals belonging to $\mathscr{I}$ is again a member of $\mathscr{I}$; all that needs to be checked is that $I = \bigcup I_\alpha$ satisfies condition (2). However, $(I_\alpha; +)$ being a direct summand implies that $(I_\alpha; +)$ is pure in G; recall that a subgroup H of G is *pure* if $nG \cap H = nH$ for each positive integer n. Thus $(I; +)$ is pure in G because purity is an inductive property. Now, condition (1) implies that $(I; +)$ is a direct summand of G because a bounded, pure subgroup is always a direct summand [**4**]. We conclude that $\mathscr{I}$ has a maximal element M according to Zorn's lemma; the order on $\mathscr{I}$ is set-theoretic inclusion. From the ring isomorphism $R \cong R/M$, we shall obtain the fact that M must be zero. If $M \neq 0$, there exists an ideal N of R properly containing M such that $p(N/M) = 0$ and such that $(N/M; +)$ is a direct summand of $(R/M; +)$. Since $(M; +)$ is a direct summand of $G = (R; +)$, $p(N/M) = 0$ implies that $pN = 0$. Likewise, N is a direct summand of G since M is a direct summand of G and since $(N/M; +)$ is a direct summand of $(R/M; +)$. Recall for a bounded group purity is equivalent to being a direct summand. Since the existence of $N \supset M$ with these properties is impossible due to the fact that M was chosen to be a maximal element of $\mathscr{I}$, we have shown that $M = 0$. On the other hand, if we consider the large subgroup L of $G = (R; +)$ defined by

$$L = \{x \in G : p^i x \in p^{2i+1}G \text{ for all } i \geqq 0\}$$

and consider the isomorphism $G \cong G/L$, we can show that $M \neq 0$. We have shown, for a suitable index set J, that

$$G = \sum_{i<\omega} \sum_{j \in J} \langle x_{i,j} \rangle$$

where $x_{i,j}$ has order p^i for each $j \in J$. Observe that $\{x_{1,j}\}_{j \in J}$ is independent mod L and that $\langle \{x_{1,j} + L\}_{j \in J} \rangle$ is an ideal of R/L having the property that $p(\{x_{1,j} + L\}_{j \in J}) = 0$ and $(\langle \{x_{1,j} + L\}_{j \in J} \rangle; +)$ is a direct summand of $(R/L; +)$ [**4**, Theorem 24.1]. Thus $\mathscr{I}$ must contain a nonzero element since $R \cong R/L$. This contradiction denies the existence of an η-ring R having an unbounded primary group without elements of infinite height ($p^\omega R = 0$) for its additive structure, so the theorem is proved.

Let $\mathscr{H}(P)$ denote the class of rings with the property that every proper homomorphic image satisfies a given condition P. We call a homomorphism proper if its kernel is a proper ideal. As in the case with $\mathscr{S}(P)$, we have $\mathscr{H}(P \vee P') \supseteq \mathscr{H}(P) \cup \mathscr{H}(P')$ with equality rarely expected. However, Theorem 4.1 enables us to prove $\mathscr{H}(P_1 \vee P_2) = \mathscr{H}(P_1) \cup \mathscr{H}(P_2)$ where, as before, P_1 and P_2 are the following conditions:

$$P_1\text{: finite characteristic,}$$

$$P_2\text{: isomorphic to } R.$$

This is the content of the next theorem.

THEOREM 4.2. *Let R be an η-ring. Then either R/I is bounded for every nonzero ideal I of R or else $R/I \cong R$ for every ideal $I \neq R$.*

Proof. By Theorem 4.1, we may assume that $(R; +)$ is torsion free. Call an ideal I of R a pure ideal of R if $(I; +)$ is a pure subgroup of $(R; +)$. If R is torsion free, as it is for the case at hand, then an ideal I of R is a pure ideal if and only if R/I is again torsion free. If the η-ring R has no proper pure ideal, then R/I has finite characteristic for every proper ideal I of R because if R/I has infinite characteristic then $R \cong R/I$ but R/I has torsion if I is not pure. Thus there is no loss of generality in assuming that R has a proper pure ideal, and we shall make this assumption.

The next step is to show that $M \neq 0$ where M is the intersection of all the nonzero pure ideals of R. Since R contains a proper pure ideal, $M \neq R$. Further, M itself is a pure ideal since R is torsion free. Assume that $M = 0$. Choose a nonzero element $a \in R$ and let I be maximal among the pure ideals of R not containing a; as we have mentioned earlier, purity is an inductive property. Since $M = 0$, there exists a nonzero pure ideal of R not containing a, so $I \neq 0$. We can quickly obtain a contradiction on the maximality of I by considering the isomorphism $R \cong R/I$. In fact, we can choose an ideal I' of R such that I'/I is a nonzero pure ideal of R/I not containing $a + I$. Since I' is pure in R and does not contain a, we have the contradiction on the maximality of I. Thus we conclude that $M \neq 0$ and that M is the unique minimal nonzero pure ideal of R.

Using the fact that $M \neq 0$, we show that $(R; +)$ is divisible. Suppose that there is a prime p such that $pM \neq M$. Since pM is an ideal of R and since R/pM does not have finite characteristic, $R \cong R/pM$. However, this is impossible since R/pM has torsion consisting of $M/pM \neq 0$. Therefore, $pM = M$ for each prime p, and $(M; +)$ is divisible. If $(R; +)$ is not divisible and if D denotes the (unique) maximal divisible subgroup of $(R; +)$, then D is a proper ideal of R since $D \supseteq M \neq 0$. Hence $R \cong R/D$, which implies that R is reduced, that is, has no nonzero divisible subgroup. We conclude that $(R; +)$ is divisible. Now the fact that $R \cong R/I$ for every ideal $I \neq R$ is immediate because a divisible group can not have a bounded, nonzero homomorphic image. Thus R/I has infinite characteristic for each ideal $I \neq R$, and $R \cong R/I$ since R is an η-ring.

One of the results established in the proof of Theorem 4.2 is worth stating as a separate corollary.

COROLLARY 4.3. *Let R be a torsion-free η-ring. If R/I has infinite characteristic for some proper ideal I, then $(R; +)$ is a rational vector space, $(R; +) = \sum Q$.*

COROLLARY 4.4. *Let R be an η-ring different from the null ring on a quasi-cyclic p-group. If R contains a nonzero ideal I such that R/I has infinite characteristic, then every ideal of R is pure.*

Proof. According to Theorem 4.2, $R \cong R/I$ for every proper ideal I of R. Furthermore, R is torsion free by Theorem 4.1, so I is pure.

The preceding results make it easy to establish the following

THEOREM 4.5. *Let R be an η-ring. If R has a proper unbounded homomorphic image, then R contains a unique minimal ideal M. Furthermore, the proper ideals of R form an ascending chain*

$$M = I_0 \subset I_1 \subset \ldots \subset I_\alpha \subset \ldots, \quad \alpha < \beta,$$

indexed by the ordinals less than some limit ordinal β, with the property that $R = \bigcup I_\alpha$. Moreover, M is simple.

Proof. Since the conclusion is valid for the null ring on a quasi-cyclic p-group, we may assume that R is torsion free by Theorem 4.1 because R obviously can not have finite characteristic under the hypothesis. Corollary 4.4 tells us that every ideal of R is pure. From the proof of Theorem 4.2, we recall that the intersection M of all the nonzero pure ideals of R is different from zero. Thus M is the unique minimal ideal of R. Set $I_0 = M$ and define I_α inductively by letting $I_{\alpha+1}/I_\alpha$ correspond to M under the isomorphism $R/I_\alpha \cong R$ and by taking unions at limit ordinals (until the ascension to R is accomplished). Clearly, any proper ideal of R is I_α for some α. In order to show that M is simple, we first show that $M^2 \neq 0$. Suppose that $M^2 = 0$. Then $I_{\alpha+1}{}^2 \subseteq I_\alpha$ since $I_{\alpha+1}/I_\alpha \cong M$. Suppose that we have shown that $MI_\alpha = 0$. If $MI_{\alpha+1} \neq 0$, then $MI_{\alpha+1} = M$ and

$$M = MI_{\alpha+1} = MI_{\alpha+1}{}^2 \subseteq MI_\alpha = 0.$$

Hence, $M^2 = 0$ implies that $MR = 0$ and, likewise, that $RM = 0$. Since M has a proper subgroup and since M is the smallest proper ideal of R, we conclude that $M^2 \neq 0$. Thus $M^2 = M$. Let $A \neq 0$ be an ideal of M. Observe that $MA = 0$ implies that $M^2 = 0$, so $MA \neq 0$. Likewise $MAM \neq 0$. Thus $MAM = M$, and M is simple.

Since a ring with a left (or right) identity can not be the union of a chain of proper ideals, the following corollary is an immediate consequence of Theorem 4.5.

COROLLARY 4.6. *Let R be an η-ring. If R contains a left (or right) identity, then R/I is bounded for every nonzero ideal I of R.*

THEOREM 4.7. *Let R be a ring with infinite characteristic such that $R \cong R/I$ for every ideal $I \neq R$. If R is commutative, then R must be a field or the null ring on a quasi-cycle p-group.*

Proof. Suppose that R is not a field. Then R has a proper ideal. Let M denote the intersection of all the nonzero ideals of R. Then $M \neq 0$ and M is the unique minimal nonzero ideal of R. Let

$$M = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_\alpha \subset \ldots, \quad \alpha < \beta,$$

be the chain of proper ideals of $R = \bigcup I_\alpha$. If $M^2 = 0$, then R is the null ring on a quasi-cyclic p-group. Thus we may assume $M^2 \neq 0$. In this case, R has no proper zero divisors because $ab = 0$ where a and b are not zero implies that $\{x \in R : ax = 0\} \supseteq M$ and, likewise, it implies that $\{x \in R : xM = 0\} \supseteq M$. This, however, yields $M^2 = 0$. Since R is without zero divisors, every proper ideal I_α is prime in view of the isomorphism $R \cong R/I_\alpha$. This, of course, is impossible; it implies, for example, that $a \in Ra$ if $a \neq 0$ and consequently that R has an identity. Thus the theorem is proved.

We now turn to a consideration of those rings R that have the property that R/I is bounded for each nonzero ideal I. We shall call such a ring a β-ring. In view of Theorem 4.1, our interest is primarily in the torsion-free case. We mention at the outset that a complete determination of rings R having the property that R/I is finite for every nonzero ideal I has not yet been made. The latter class of rings, a subclass of β-rings, has been studied in two recent papers [**2**; **11**].

Our first result about torsion-free β-rings is that the additive group of any such ring is necessarily homogeneous (of special type; see Theorem 4.8). Recall that a torsion-free abelian group is said to be *homogeneous* if all the nonzero elements have the same type; the notion of type and the basic results on homogeneous groups are due to Baer [**1**]. The following theorem improves [**11**, Lemma 1.7].

THEOREM 4.8. *If R is a torsion-free ring with the property that R/I has finite characteristic for every proper ideal I of R, then the additive group of R is homogeneous and the type of the group is represented by a sequence*

$$(n_1, n_2, \ldots, n_i, \ldots)$$

where n_i is either 0 or ∞ for each i.

Proof. Suppose $R \neq 0$ is a torsion-free ring having the property that R/I has finite characteristic for every proper ideal I. Let G denote the additive group of R, and define a (possibly empty) subset S of primes by

$$S = \{p\colon p \text{ is a prime such that } p^\omega G = 0\}.$$

If T is an infinite subset of S and if $H = \bigcap_{p \in T} pG$, then G/H must be unbounded. To verify this, suppose G/H is bounded by n. Choose a prime $p \in T$

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such that p does not divide n. By the choice of n and p, we have $nG \subseteq H \subseteq pG \neq G$, but this is impossible since $(n, p) = 1$, for if $g \in G$ is not contained in pG then $ng \notin pG$. Since G/H is unbounded and since H is an ideal of R, we conclude that H must be zero. Next, we observe that p is contained in S if $pG \neq G$.

Assume $pG \neq G$. Then $p^\omega G \neq G$ but $p^\omega G$ is pure in G so $G/p^\omega G$ is torsion-free. Since $p^\omega G$ is an ideal of R, this is impossible unless $p^\omega G = 0$. Thus what we have proved is that $\bigcap_{p \in T} pG = 0$ if the intersection is over an infinite set T of primes p such that $pG \neq G$. It follows that if x is any nonzero element of G, then $\infty > h_p(x) > 0$ for at most a finite number of primes p (each of which belongs to S); $h_p(x)$ denotes the height of the element x in G at the prime p. Obviously, $h_p(x) = \infty$ for all $p \notin S$ (since $pG = G$ if $p \notin S$) and $h_p(x)$ is finite for all $p \in S$. We conclude that the type of x is represented by the sequence $(n_1, n_2, \ldots, n_i, \ldots)$ where n_i is 0 or ∞, depending on whether the ith prime is in or outside of S. Since x was arbitrary and S depends only on G this completes the proof of the theorem.

The preceding result restricts the addition of a torsion-free β-ring. The next proposition further restricts torsion-free β-rings, and it partially refines [**11**, Proposition 1.1].

PROPOSITION 4.9. *If R is a torsion-free ring with the property R/I is bounded for every nonzero ideal I, then R is a prime ring or the null ring on the infinite cyclic group.*

Proof. Suppose $AB = 0$ where A and B are nonzero ideals of R. Since $R/A \cap B \subseteq R/A \times R/B$, there exists a positive integer n such that $nR \subseteq A \cap B$. Consequently,

$$n^2R^2 \subseteq (A \cap B)(A \cap B) \subseteq AB = 0,$$

which implies that $R^2 = 0$ since R is torsion-free. The only null ring that satisfies the hypothesis is the null ring on the infinite cyclic group.

COROLLARY 4.10. *Let R be a commutative torsion-free ring with the property that R/I has finite characteristic for every proper ideal I. Suppose that R is not the null ring on the infinite cyclic group. Then R is an integral domain (not necessarily with identity) and $(R; +)$ is homogeneous with its type consisting of* 0*'s and* ∞*'s.*

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