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BACKWARD STOCHASTIC DIFFERENCE EQUATIONS ON LATTICES WITH APPLICATION TO MARKET EQUILIBRIUM ANALYSIS

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Abstract

We study backward stochastic difference equations (BSΔEs) driven by a d-dimensional stochastic process on a lattice, whose increments take only $d+1$ possible values that generate the lattice. Interpreting the driving process as a d-dimensional asset price process, we provide applications to an optimal investment problem and to a market equilibrium analysis, where utility functionals are defined via BSΔEs.

Keywords: g-expectation; dynamic risk measure; general equilibrium

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1. Introduction

The theory of backward stochastic differential equations (BSDEs), initiated by Bismut [7] and Pardoux and Peng [32], has been extensively studied over the past three decades, particularly in relation to stochastic control, finance, and insurance (see e.g. [17, 39]). Important applications include dynamic risk measures [4] and g-expectations [16, 33], which generalize classical expectations and martingales to nonlinear settings. Recent applications to financial economics include [6], [8], [20], [24], [28], and [30].

While BSDEs are powerful theoretical tools, their solutions are typically implicit and require discretization for numerical implementation. As discrete analogs, backward stochastic difference equations (BSΔEs) have been widely studied, falling into two main categories. The first focuses on BSΔEs as weak approximations of BSDEs [9, 10, 11, 12, 29, 31, 35, 37]. The second explores the structure of BSΔEs themselves. A general framework is provided in [15], while specific cases involving driving martingales with the predictable representation property are studied in [14] and [19].

This paper falls into the second category and studies a class of BSΔEs including the one introduced in [31] and [37], where a d-dimensional scaled random walk – whose increments take only $d+1$ values – is used to approximate Brownian motion in BSDEs. Such a random walk is minimal among discrete-time processes that converge to d-dimensional Brownian

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motions. The weak convergence of this BSΔE to the BSDE was proved in [31], and the convergence rate in a Markovian setting was given in [37], generalizing the one-dimensional case in [11].

This minimal BSΔE is computationally efficient, as it involves only a $(d+1)$-dimensional problem, in contrast to a 2^d-dimensional problem required when using d-dimensional Bernoulli random walks [12]. Although Cohen and Elliott [15] have developed a general BSΔE theory, we focus on the specific structure of this minimal BSΔE, which exhibits properties not covered in their general framework. In the one-dimensional case, it reduces to a BSΔE on a binomial tree, studied in [19] in the context of dynamic risk measures. We treat the general multi-dimensional setting here.

Our key contribution is to identify a gradient constraint on the BSΔE driver, which endows the solution with certain properties as a generalized conditional expectation. This allows us to link the driver to a measure change for the driving random walk, and apply these insights to market equilibrium analysis.

The g-expectation is part of the solution of a BSDE or BSΔE, and generalizes the expectation and the certainty equivalent of an expected utility. A subclass of them with concavity and translation invariance has been employed as the utility functional for market equilibrium analyses in [3], [13], [25], and [28]. In this paper, we also apply our BSΔE to a market equilibrium analysis. In contrast to the preceding studies, which place an emphasis on incomplete markets, we are interested in explicit computations in a dynamically complete market.

Anderson and Raimondo [2] proved the existence of equilibrium in a continuous-time dynamically complete market by means of non-standard analysis, where an approximation to a Brownian motion by a minimal random walk played a key role. We consider a simpler dynamically complete market to derive explicit conditions for market equilibrium.

Under a unique equivalent martingale measure, our asset price model is a multi-dimensional extension of the recombining binomial tree. In our approach, an asset price process is given as a stochastic process taking values on a lattice. We do not argue the existence of an equilibrium price but characterize the agents' utilities under which the given discrete (in both time and space) price process is to be in general equilibrium. This feature is in contrast to the preceding studies [3], [13], and [25] and similar to [8], [23], [28], and [34] in continuous time.

Our framework includes heterogeneous agents with exponential utilities under heterogeneous beliefs. Their risk-aversion coefficients may be stochastic and time-varying. We observe in particular that under equilibrium with heterogeneous beliefs, agents trade with each other, even in the absence of random endowments to hedge, complementing earlier studies of heterogeneous beliefs [5, 21, 22, 26, 30, 38].

In Section 2.1 we describe a lattice in $\mathbb{R}^d$ where a stochastic process $\{X_n\}$ takes values, and give some elementary linear algebraic lemmas as a preliminary. In Section 2.2 we introduce the process $\{X_n\}$ that is the source of randomness in this paper and generates a filtration. It is minimal in the sense that the increment ΔX_n takes values in a set $\{v_0, \dots, v_d\}$ of $d+1$ points in $\mathbb{R}^d$. Some elementary measure change formulas are also given as a preliminary.

In Section 2.3 our BSΔE

$$\Delta Y_n = -g_n(Z_n) + Z_n^\top \Delta X_n, \qquad Y_N = Y,$$

is formulated. Due to the minimality of $\{X_n\}$, there exists a unique solution $\{(Y_n, Z_n)\}$ to the above equation, without orthogonal martingale terms needed in [10] and [12]. The process $\{X_n\}$ itself takes more than $d+1$ points, so this BSΔE is different from the one studied in [14]. The g-expectation $\mathcal{E}_n^g$ for $g = \{g_n\}$ is defined by $\mathcal{E}_n^g(Y) = Y_n$. Proposition 2.1 concerns the case

$g_n(z) = f_n(X_{n-1}, z)$ and $Y_N = h(X_N)$ for deterministic functions f_n and h to provide a nonlinear Feynman–Kac-type formula, which is a computationally efficient recurrence equation on the lattice for a deterministic function u_n such that $Y_n = u_n(X_n)$.

Section 2.4 is about the aforementioned gradient constraint. First we observe that the g-expectation is a conditional expectation when g_n are linear with slope coefficients included in the convex hull Θ of the set $\{v_0, \ldots, v_d\}$. The importance of this constraint on the slope is a special feature of our BSΔE, and to the best of our knowledge has not been recognized in the preceding studies of multi-dimensional BSΔEs. A balance condition introduced by Cohen and Elliot [15] for a comparison theorem to hold is translated in terms of Θ for our BSΔE. We also prove a robust representation when g_n are concave, where the set Θ again plays an important role. In Section 2.5 we show that a translation-invariant filtration-consistent nonlinear expectation is a g-expectation.

In Section 3 we regard $\{X_n\}$ as a d-dimensional asset price process. In Section 3.1 we consider an optimal investment strategy which maximizes the g-expectation of terminal wealth. By the minimality, the market is complete, extending the well-known binomial tree model for a one-dimensional asset. Our asset price model can be seen as a discrete approximation of the multi-dimensional Bachelier model with constant covariance and general stochastic drift. An advantage of our use of the minimal process as an approximation is that the completeness of the Bachelier model is preserved. Further, the minimality property naturally arises in a variance swap pricing model as illustrated in Example 3.2. In Sections 3.2–3.4 we give a market equilibrium analysis. We consider agents whose utility functionals are g expectations and seek conditions on those g expectations under which $\{X_n\}$ is an equilibrium price process.

Throughout our financial application, we have short maturity problems in mind, and so, for brevity, assume interest rates, dividend rates, and consumption rates to be zero as in [3], [13], [25], and [28].

We use the convention that

$$\sum_{i=m}^{n} a_i = 0$$

for any sequence $\{a_i\}$ if $m > n$.

2. BSΔE on a lattice

2.1. Lattice

We start by describing a lattice. Let $\{v_1, \ldots, v_d\}$ be a basis of $\mathbb{R}^d$. The subset

$$L = \left\{ \sum_{i=1}^{d} z_i v_i, \ ; z_i \in \mathbb{Z}, i = 1, \ldots, d \right\}$$

of $\mathbb{R}^d$ is a d-dimensional lattice generated by the basis. Notice that L admits an alternative expression

$$L = \left\{ \sum_{i=0}^{d} n_i v_i, \ ; n_i \in \mathbb{N}, i = 0, 1, \ldots, d \right\}, \tag{2.1}$$

where $v_0 = -v_1 - \cdots - v_d$ and $\mathbb{N}$ is the set of the non-negative integers. Let

$$\mathbf{v} = [v_0, v_1, \ldots, v_d]$$

be the $d \times (d+1)$ matrix with v_i, $i = 0, \ldots, d$ as its column vectors. Put

$$\mathbf{1} = (1, \ldots, 1)^\top \in \mathbb{R}^{d+1}.$$

The following lemmas will be of repeated use in this paper.

Lemma 2.1. *The* $(d+1) \times (d+1)$ *matrix* $(\mathbf{1}, \mathbf{v}^\top)$ *is invertible.*

Proof. We show that the row vectors of $(\mathbf{1}, \mathbf{v}^\top)$ are linearly independent. Suppose

$$(\alpha_0, \ldots, \alpha_d)(\mathbf{1}, \mathbf{v}^\top) = 0,$$

or equivalently

$$\sum_{j=0}^{d} \alpha_j = 0, \quad \sum_{j=0}^{d} \alpha_j v_j = 0$$

for scalars α_j. By the second equation and the definition of v_0, we have

$$\sum_{j=1}^{d} (\alpha_j - \alpha_0) v_j = 0,$$

from which we can conclude $\alpha_j = \alpha_0$ for all j because $\{v_1, \ldots, v_d\}$ is a basis. Together with the first equation, we then conclude $\alpha_j = 0$ for all j. $\square$

Lemma 2.2. *Let* $y \in \mathbb{R}^{d+1}$. *The unique solution to the equation*

$$y = (\mathbf{1}, \mathbf{v}^\top) z, \quad z = (z_0, z_1, \ldots, z_d)^\top \in \mathbb{R}^{d+1} \tag{2.2}$$

is given by

$$z_0 = a(y) := \frac{1}{d+1} \mathbf{1}^\top y, \quad (z_1, \ldots, z_d)^\top = b(y) := (\mathbf{v}\mathbf{v}^\top)^{-1} \mathbf{v} y. \tag{2.3}$$

Proof. By Lemma 2.1, there exists a unique $z \in \mathbb{R}^{d+1}$ such that (2.2) holds. Since $\mathbf{v}\mathbf{1} = 0$, multiplying both sides of (2.2) by $\mathbf{1}^\top$, we obtain the first equation of (2.3). Also, multiplying both sides of (2.2) by $\mathbf{v}$ and again using $\mathbf{v}\mathbf{1} = 0$, we have $\mathbf{v}y = \mathbf{v}\mathbf{v}^\top (z_1, \ldots, z_d)^\top$. Since $\{v_1, \ldots, v_d\}$ is a basis, $\mathbf{v}$ has rank d. Therefore $\mathbf{v}^\top$ has rank d and so, for any $x \in \mathbb{R}^d \setminus \{0\}$, $x^\top \mathbf{v}\mathbf{v}^\top x = |\mathbf{v}^\top x|^2 \neq 0$. This implies that the $d \times d$ matrix $\mathbf{v}\mathbf{v}^\top$ is invertible and, in turn, the second equation of (2.3) is valid. $\square$

2.2. Probability space

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. For a stochastic process $\{X_n\}_{n \in \mathbb{N}}$, we put $\Delta X_n = X_n - X_{n-1}$. Let $\{X_n\}$ be a d-dimensional stochastic process with ΔX_n taking values in $\{v_0, v_1, \ldots, v_d\}$ for all $n \geq 1$ and $X_0 = 0$. By (2.1), X_n takes values in L for all $n \in \mathbb{N}$. Let $\mathscr{F}_n = \sigma(X_0, \ldots, X_n)$, $n \in \mathbb{N}$, be the natural filtration generated by $\{X_n\}$, and put $P_n = (P_{n,0}, \ldots, P_{n,d})^\top$ for $n \geq 1$, where

$$P_{n,j} = \mathbb{P}(\Delta X_n = v_j \mid \mathscr{F}_{n-1}), \quad j = 0, \ldots, d.$$

Note that $\{P_n\}$ is a Δ_d-valued predictable process, where

$$\Delta_d = \left\{p \in \mathbb{R}^{d+1}\ ; \mathbf{1}^\top p = 1,\ p \geq 0\right\}.$$

We assume $P_{n,j}$ is positive for all $n \geq 1$ and $j = 0, \ldots, d$. Let $N \in \mathbb{N} \setminus \{0\}$. The following lemma will be of repeated use in this paper.

Lemma 2.3. *For any Δ_d-valued predictable process $\{\hat{P}_n\}$, there exists a probability measure $\hat{\mathbb{P}}$ on $(\Omega, \mathscr{F}_N)$ such that $\hat{P}_n = (\hat{P}_{n,0}, \ldots, \hat{P}_{n,d})^\top$, where*

$$\hat{P}_{n,j} = \hat{\mathbb{P}}(\Delta X_n = v_j \mid \mathscr{F}_{n-1}), \quad j = 0, \ldots, d,\ n = 1, \ldots, N. \tag{2.4}$$

Proof. Define $\hat{\mathbb{P}}$ by

$$\hat{\mathbb{P}}(A) = \mathbb{E}[L_N 1_A], \quad L_n = \prod_{k=1}^{n} \left(\sum_{j=0}^{d} \frac{\hat{P}_{k,j}}{P_{k,j}} 1_{\{\Delta X_k = v_j\}} \right) \tag{2.5}$$

for $A \in \mathscr{F}_N$. The measure $\hat{\mathbb{P}}$ is a probability measure because

$$\mathbb{E}\left[\sum_{j=0}^{d} \frac{\hat{P}_{n,j}}{P_{n,j}} 1_{\{\Delta X_n = v_j\}} \middle| \mathscr{F}_{n-1} \right] = \sum_{j=0}^{d} \frac{\hat{P}_{n,j}}{P_{n,j}} P_{n,j} = 1$$

and so L_n is a martingale with $L_0 = 1$. Using Bayes' formula, we derive

$$\hat{\mathbb{P}}(\Delta X_n = v_j \mid \mathscr{F}_{n-1}) = \frac{\mathbb{E}[L_N 1_{\{\Delta X_n = v_j\}} \mid \mathscr{F}_{n-1}]}{\mathbb{E}[L_N \mid \mathscr{F}_{n-1}]} = \mathbb{E}\left[\frac{\hat{P}_{n,j}}{P_{n,j}} 1_{\{\Delta X_n = v_j\}} \middle| \mathscr{F}_{n-1} \right] = \hat{P}_{n,j},$$

by the martingale property of L_n. □

Define a measure $\mathbb{Q}$ on $\mathscr{F}_N$ by

$$\mathbb{Q}(A) = \mathbb{E}[L_N 1_A], \quad L_N = \prod_{n=1}^{N} \left(\frac{1}{d+1} \sum_{j=0}^{d} \frac{1}{P_{n,j}} 1_{\{\Delta X_n = v_j\}} \right).$$

Let $\mathbb{E}_\mathbb{Q}$ denote the integration under $\mathbb{Q}$.

Lemma 2.4. *The measure $\mathbb{Q}$ is the unique probability measure on $\mathscr{F}_N$ under which $\{X_n\}$ is a martingale. Under $\mathbb{Q}$, $\{\Delta X_n\}$ is i.i.d. with*

$$\mathbb{Q}(\Delta X_n = v_j) = \mathbb{Q}(\Delta X_n = v_j \mid \mathscr{F}_{n-1}) = \frac{1}{d+1}$$

for all $n = 1, \ldots, N$ and $j = 0, \ldots, d$. We also have

$$\mathbb{E}_\mathbb{Q}[\Delta X_n \mid \mathscr{F}_{n-1}] = 0, \quad \mathbb{E}_\mathbb{Q}[\Delta X_n (\Delta X_n)^\top \mid \mathscr{F}_{n-1}] = \frac{1}{d+1} \mathbf{v}\mathbf{v}^\top. \tag{2.6}$$

Proof. By Lemma 2.3, $\mathbb{Q}$ is a probability measure with $\mathbb{Q}(\Delta X_n = v_j \mid \mathscr{F}_{n-1}) = 1/(d+1)$, which implies

$$\mathbb{E}_\mathbb{Q}[\Delta X_n \mid \mathscr{F}_{n-1}] = \frac{1}{d+1} \mathbf{v}\mathbf{1} = 0.$$

Therefore $\{X_n\}$ is a martingale with (2.6). There is no other such measure because

$$\sum_{j=0}^{d} \alpha_j = 1, \quad \sum_{j=0}^{d} \alpha_j v_j = 0$$

implies $\alpha_j = 1/(d+1)$ as in the proof of Lemma 2.1. Since the conditional law of ΔX_n given $\mathscr{F}_{n-1}$ is deterministic for every n, $\{\Delta X_n\}$ is i.i.d. □

Remark 2.1. For any positive definite $d \times d$ matrix Σ, we can construct such a lattice L that $\mathbf{v}\mathbf{v}^\top = \Sigma$. Indeed, starting with an arbitrary basis, say, $\bar{v}_j = e_j$ (the standard basis of $\mathbb{R}^d$) with $\bar{v}_0 = -\bar{v}_1 - \cdots - \bar{v}_d$ and $\bar{\mathbf{v}} = [\bar{v}_0, \ldots, \bar{v}_d]$, using the Cholesky decomposition $\Sigma = CC^\top$ and $\bar{\mathbf{v}}\bar{\mathbf{v}}^\top = \bar{C}\bar{C}^\top$, if we take $v_j = C\bar{C}^{-1}\bar{v}_j$, $j = 0, \ldots, d$, then $\mathbf{v} = C\bar{C}^{-1}\bar{\mathbf{v}}$ and so we get $\mathbf{v}\mathbf{v}^\top = C\bar{C}^{-1}\bar{\mathbf{v}}\bar{\mathbf{v}}^\top(\bar{C}^\top)^{-1}C^\top = \Sigma$. In particular, we can construct such v_j that $\mathbf{v}\mathbf{v}^\top$ is the identity matrix. In this case a scaling limit of $\{X_n\}$ under $\mathbb{Q}$ is the d-dimensional standard Brownian motion. Such a set of vectors played an essential role in proving the existence of continuous-time market equilibrium in Anderson and Raimondo [2] by means of non-standard analysis, where the existence of the vectors was proved in a recursive manner. It is also the building block of a d-dimensional diamond in topological crystallography [36].

2.3. Existence, uniqueness and representation

Here we introduce our BSΔE. Let $\mathcal{A}$ denote the set of the sequences $g = \{g_n\}_{n=1}^N$ of $\mathscr{F}_{n-1} \otimes \mathscr{B}(\mathbb{R}^d)$ measurable functions $g_n \colon \Omega \times \mathbb{R}^d \to \mathbb{R}$. Now we state an elementary but fundamental result.

Theorem 2.1. *Let Y_N be an $\mathscr{F}_N$-measurable random variable, and let $g = \{g_n\} \in \mathcal{A}$. Then there exist uniquely an adapted process $\{Y_n\}_{n=0,\ldots,N-1}$ and an $\mathbb{R}^d$-valued predictable process $\{Z_n\}_{n=1,\ldots,N}$ such that*

$$\Delta Y_n = -g_n(Z_n) + Z_n^\top \Delta X_n, \quad n = 1, \ldots, N. \tag{2.7}$$

Further, they admit the following representation:

$$\begin{aligned} Y_{n-1} &= \mathbb{E}_{\mathbb{Q}}[Y_n \mid \mathscr{F}_{n-1}] + g_n(Z_n), \\ Z_n &= (d+1)(\mathbf{v}\mathbf{v}^\top)^{-1}\mathbb{E}_{\mathbb{Q}}[Y_n \Delta X_n \mid \mathscr{F}_{n-1}] = (\mathbf{v}\mathbf{v}^\top)^{-1}\mathbf{v}\bar{Y}_n, \end{aligned}$$

where $\bar{Y}_n = (\bar{Y}_{n,0}, \ldots, \bar{Y}_{n,d})^\top$ and

$$\bar{Y}_{n,j} = \mathbb{E}_{\mathbb{Q}}[Y_n \mid \mathscr{F}_{n-1}, \Delta X_n = v_j] = (d+1)\mathbb{E}_{\mathbb{Q}}[Y_n 1_{\{\Delta X_n = v_j\}} \mid \mathscr{F}_{n-1}].$$

Proof. Since Y_N is $\mathscr{F}_N$-measurable, there exists a function $f \colon L^N \to \mathbb{R}$ such that $Y_N = f(X_1, \ldots, X_N)$. Since $\mathbb{P}_{N,j}$ are positive by the assumption, (2.7) for $n = N$ is equivalent to the system of equations for $\mathscr{F}_{N-1}$-measurable random variables

$$Y := \begin{bmatrix} f(X_1, \ldots, X_{N-1}, X_{N-1} + v_0) \\ \vdots \\ f(X_1, \ldots, X_{N-1}, X_{N-1} + v_d) \end{bmatrix} = (\mathbf{1}, \mathbf{v}^\top) \begin{bmatrix} Y_{N-1} - g_N(Z_N) \\ Z_N \end{bmatrix}.$$

Applying Lemma 2.2, we obtain (2.7) for $n = N$ with

$$Z_N = b(Y), \quad Y_{N-1} = a(Y) + g_N(Z_N),$$

where $a(y)$ and $b(y)$ are defined by (2.3). It is clear that both Z_N and Y_{N-1} are $\mathscr{F}_{N-1}$-measurable. By backward induction, we obtain $\{Y_n\}$ and $\{Z_n\}$. The representation follows from (2.7) and (2.6) by taking the conditional expectation under $\mathbb{Q}$. $\square$

For $g = \{g_n\} \in \mathcal{A}$ fixed, the $\mathscr{F}_n$-measurable random variable Y_n given by Theorem 2.1 is uniquely determined by the $\mathscr{F}_N$-measurable random variable Y_N. We write this mapping as $Y_n = \mathcal{E}_n^g(Y_N)$ and call it the g-expectation of Y_N (with respect to $\mathscr{F}_n$). The stochastic process $\{(Y_n, Z_n)\}$ given by Theorem 2.1 is called the solution of the BSΔE (2.7).

Remark 2.2. In the literature, say, in [15], BSΔE is formulated by decomposing ΔY_n into a predictable part and a martingale difference part. In our formulation (2.7), ΔX_n is not necessarily a martingale difference. It is a minor reparametrization because (2.7) can be rewritten as

$$\Delta Y_n = -\hat{g}_n(Z_n) + Z_n^\top(\Delta X_n - A_n)$$

with $\hat{g}_n(z) = g_n(z) - z^\top A_n$, $A_n = \mathbb{E}[\Delta X_n \mid \mathscr{F}_{n-1}]$.

Example 2.1. Let $\gamma > 0$, $\{(\hat{P}_{n,0}, \dots, \hat{P}_{n,d})^\top\}$ be a Δ_d-valued predictable process, and

$$g_n(z) = -\frac{1}{\gamma}\log\left(\sum_{j=0}^{d} \mathrm{e}^{-\gamma z^\top v_j}\hat{P}_{n,j}\right). \tag{2.8}$$

Then

$$\mathcal{E}_n^g(Y) = -\frac{1}{\gamma}\log\hat{\mathbb{E}}\big[\mathrm{e}^{-\gamma Y} \mid \mathscr{F}_n\big], \quad n = 0, 1, \dots, N, \tag{2.9}$$

for any $\mathscr{F}_N$-measurable random variable Y, where $\hat{\mathbb{E}}$ is the expectation under the measure $\hat{\mathbb{P}}$ on $\mathscr{F}_N$ defined by (2.5). To see this, note that by Lemma 2.3,

$$g_n(z) = -\frac{1}{\gamma}\log\hat{\mathbb{E}}\big[\mathrm{e}^{-\gamma z^\top \Delta X_n} \mid \mathscr{F}_{n-1}\big].$$

Substituting $Y_n = Y_{n-1} - g_n(Z_n) + Z_n^\top \Delta X_n$, we have

$$-\frac{1}{\gamma}\log\hat{\mathbb{E}}\big[\mathrm{e}^{-\gamma Y_n} \mid \mathscr{F}_{n-1}\big] = Y_{n-1} - g_n(Z_n) - \frac{1}{\gamma}\log\hat{\mathbb{E}}\big[\mathrm{e}^{-\gamma Z_n^\top \Delta X_n} \mid \mathscr{F}_{n-1}\big] = Y_{n-1}$$

which implies (2.9) for $n = N-1$. The general case follows by backward induction.

Next we give a discrete analog of the nonlinear Feynman–Kac formula, which is computationally efficient when dealing with large N.

Proposition 2.1. *Let $f_n \colon L \times \mathbb{R}^d \to \mathbb{R}$, $n = 1, \dots, N$ and $h \colon L \to \mathbb{R}$. Define $u_n \colon L \to \mathbb{R}$, $n = 0, 1, \dots, N$ backward inductively by*

$$u_{n-1}(x) = u_n(x) + \mathcal{L}u_n(x) + f_n(x, (\mathbf{v}\mathbf{v}^\top)^{-1}\mathbf{v}\mathcal{N}u_n(x))$$

with $u_N = h$, where

$$\mathcal{L}u_n(x) = \frac{1}{d+1}\sum_{j=0}^{d}(u_n(x+v_j) - u_n(x)),$$

$$\mathcal{N}u_n(x) = (u_n(x+v_0) - u_n(x), \dots, u_n(x+v_d) - u_n(x))^\top.$$

Then the unique solution to (2.7) *with* $g_n = f_n(X_{n-1}, \cdot)$ *and* $Y_N = h(X_N)$ *is given by*

$$Y_{n-1} = u_{n-1}(X_{n-1}), \quad Z_n = (\mathbf{v}\mathbf{v}^\top)^{-1}\mathbf{v}\mathcal{N}u_n(X_{n-1}), \quad n = 1, \ldots, N.$$

Proof. By definition, $Y_N = h(X_N) = u_N(X_N)$. Suppose $Y_n = u_n(X_n)$. Then, by Theorem 2.1, $Z_n = (\mathbf{v}\mathbf{v}^\top)^{-1}\mathbf{v}\bar{Y}_n$, where

$$\bar{Y}_{n,j} = \mathbb{E}_{\mathbb{Q}}[Y_n \mid \mathscr{F}_{n-1}, \Delta X_n = v_j] = u_n(X_{n-1} + v_j).$$

Using $\mathbf{v}\mathbf{1} = 0$, we conclude $Z_n = (\mathbf{v}\mathbf{v}^\top)^{-1}\mathbf{v}\mathcal{N}u_n(X_{n-1})$. Further, again by Theorem 2.1,

$$Y_{n-1} = \mathbb{E}_{\mathbb{Q}}[Y_n \mid \mathscr{F}_{n-1}] + g_n(Z_n) = u_n(X_{n-1}) + \mathcal{L}u_n(X_{n-1}) + g_n(Z_n) = u_{n-1}(X_{n-1}),$$

which concludes the proof. $\square$

2.4. A gradient constraint

Let Θ be the closed convex hull spanned by $\{v_0, v_1, \ldots, v_d\}$, or equivalently

$$\Theta = \{\mathbf{v}p \,;\, p \in \Delta_d\}.$$

In this section we study BSΔEs with the gradient of g being constrained in Θ.

Example 2.2. The triangular lattice of $\mathbb{R}^2$ is generated by

$$v_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 \\ -2 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}.$$

In this case, $\mathbf{v}\mathbf{v}^\top = I$ and Θ is an equilateral triangle.

Proposition 2.2. *Let* $g_n(z) = A_n^\top z + B_n$ *for a* Θ*-valued predictable process* $\{A_n\}$ *and a predictable process* $\{B_n\}$, $n = 1, \ldots, N$. *Then*

$$\mathcal{E}_n^g(Y) = \hat{\mathbb{E}}\left[Y + \sum_{i=n+1}^{N} B_i \middle| \mathscr{F}_n\right], \quad n = 0, 1, \ldots, N,$$

for any $\mathcal{F}_N$*-measurable random variable* Y*, where* $\hat{\mathbb{E}}$ *is the expectation under the measure* $\hat{\mathbb{P}}$ *on* $\mathscr{F}_N$ *defined by* (2.5) *with* $\hat{P}_n = (\hat{P}_{n,0}, \ldots, \hat{P}_{n,d})^\top$ *such that* $A_n = \mathbf{v}\hat{P}_n$.

Proof. By Lemma 2.3,

$$\hat{\mathbb{E}}[\Delta X_n \mid \mathcal{F}_{n-1}] = \frac{\mathbb{E}[L_N \Delta X_n \mid \mathcal{F}_{n-1}]}{\mathbb{E}[L_N \mid \mathcal{F}_{n-1}]} = \sum_{j=0}^{d} v_j \hat{P}_{n,j} = \mathbf{v}\hat{P}_n = A_n$$

for all n. On the other hand, from (2.7), we have

$$Y = Y_N = Y_n + \sum_{i=n+1}^{N} \big(-g_i(Z_i) + Z_i^\top \Delta X_i\big) = Y_n + \sum_{i=n+1}^{N} \big(-B_i + Z_i^\top(\Delta X_i - A_i)\big).$$

Taking the conditional expectation under $\hat{\mathbb{P}}$, we get the conclusion. $\square$

Example 2.3. Let $N=1$, $d=1$, $\Omega=\{+,-\}$, $v_0=-1, v_1=1$, and $\Delta X_1(\pm)=\pm 1$. Then $L=\mathbb{Z}$, $\mathbf{v}=(-1,1)$, $\Theta=[-1,1]$ and $\mathbf{v}\mathbf{v}^\top=2$. Following the proof of Theorem 2.1, the solution of the linear BSΔE $\Delta Y_1=-AZ_1+Z_1\Delta X_1$ can be constructed as

$$Y_0=\frac{Y_1(+)+Y_1(-)}{2}+A\frac{Y_1(+)-Y_1(-)}{2}=\frac{1+A}{2}Y_1(+)+\frac{1-A}{2}Y_1(-)$$

for any $A\in\mathbb{R}$. The expression given by Proposition 2.2 is $Y_0=\hat{E}[Y_1]$, which can be directly seen with $\hat{P}_1=((1-A)/2,(1+A)/2)^\top$ for $A\in\Theta=[-1,1]$. For $A\notin\Theta$, we observe that Y_0 is not increasing in either $Y_1(+)$ or $Y_1(-)$. In particular, Y_0 cannot be represented as an expectation in this case.

The set Θ plays a key role also for a comparison theorem. Let $\mathcal{B}$ denote the set of the sequence $g=\{g_n\}_{n=1}^N\in\mathcal{A}$ with

$$g_n(z_2)-g_n(z_1)\geq\min_{\theta\in\Theta}\theta^\top(z_2-z_1) \tag{2.10}$$

for all $z_1,z_2\in\mathbb{R}^d$.

Proposition 2.3. (Comparison theorem.) *For $i=1,2$, let $Y^{(i)}$ be $\mathscr{F}_N$-measurable random variables with $Y^{(1)}\geq Y^{(2)}$, and $g^{(i)}=\{g_n^{(i)}\}\in\mathcal{A}$ with $g_n^{(1)}\geq g_n^{(2)}$. Let $\mathcal{E}_n^{(i)}(Y^{(i)})$ denote $\mathcal{E}_n^g(Y^{(i)})$ for $g=g^{(i)}$, $i=1,2$ respectively. Assume also $g^{(i)}\in\mathcal{B}$ for either $i=1$ or $i=2$. Then*

$$\mathcal{E}_n^{(1)}(Y^{(1)})\geq\mathcal{E}_n^{(2)}(Y^{(2)}),\qquad n=0,1,\ldots,N.$$

Proof. Note first that

$$\min_{\omega\in\Omega}z^\top\Delta X_n(\omega)=\min_{p\in\Delta_d}z^\top\mathbf{v}p=\min_{\theta\in\Theta}z^\top\theta \tag{2.11}$$

for any $z\in\mathbb{R}^d$ and n. Therefore, under (2.10) for $g=g^{(i)}$, $g^{(i)}$ is balanced in the terminology of [15], so the result follows from Theorem 3.2 of [15]. Here we repeat essentially the same proof for the readers' convenience. Let $\left\{\left(Y_n^{(i)},Z_n^{(i)}\right)\right\}$ be the solution of (2.7) with $g=g^{(i)}$ and $Y_N=Y^{(i)}$. We have $Y_N^{(1)}\geq Y_N^{(2)}$ by assumption. Suppose $Y_k^{(1)}\geq Y_k^{(2)}$ for some k. Then

$$0\leq Y_k^{(1)}-Y_k^{(2)}=Y_{k-1}^{(1)}-Y_{k-1}^{(2)}-g_k^{(1)}\big(Z_k^{(1)}\big)+g_k^{(2)}\big(Z_k^{(2)}\big)+\big(Z_k^{(1)}-Z_k^{(2)}\big)^\top\Delta X_k$$

and so

$$\big(Z_k^{(1)}-Z_k^{(2)}\big)^\top\Delta X_k\geq -Y_{k-1}^{(1)}+Y_{k-1}^{(2)}+g_k^{(1)}\big(Z_k^{(1)}\big)-g_k^{(2)}\big(Z_k^{(2)}\big).$$

Since the right-hand side is $\mathscr{F}_{k-1}$-measurable, this implies further

$$\min_{\theta\in\Theta}\theta^\top\big(Z_k^{(1)}-Z_k^{(2)}\big)\geq -Y_{k-1}^{(1)}+Y_{k-1}^{(2)}+g_k^{(1)}\big(Z_k^{(1)}\big)-g_k^{(2)}\big(Z_k^{(2)}\big)$$

by (2.11). Therefore

$$\begin{aligned}Y_{k-1}^{(1)}-Y_{k-1}^{(2)}&\geq g_k^{(1)}\big(Z_k^{(1)}\big)-g_k^{(2)}\big(Z_k^{(2)}\big)-\min_{\theta\in\Theta}\theta^\top\big(Z_k^{(1)}-Z_k^{(2)}\big)\\&=g_k^{(1)}\big(Z_k^{(2)}\big)-g_k^{(2)}\big(Z_k^{(2)}\big)+g_k^{(1)}\big(Z_k^{(1)}\big)-g_k^{(1)}\big(Z_k^{(2)}\big)-\min_{\theta\in\Theta}\theta^\top\big(Z_k^{(1)}-Z_k^{(2)}\big)\\&\geq 0\end{aligned}$$

under (2.10) for $g_n = g_n^{(1)}$, and also

$$\begin{aligned} Y_{k-1}^{(1)} - Y_{k-1}^{(2)} &\geq g_k^{(1)}(Z_k^{(1)}) - g_k^{(2)}(Z_k^{(2)}) - \min_{\theta\in\Theta} \theta^\top (Z_k^{(1)} - Z_k^{(2)}) \\ &= g_k^{(1)}(Z_k^{(1)}) - g_k^{(2)}(Z_k^{(1)}) + g_k^{(2)}(Z_k^{(1)}) - g_k^{(2)}(Z_k^{(2)}) - \min_{\theta\in\Theta} \theta^\top (Z_k^{(1)} - Z_k^{(2)}) \\ &\geq 0 \end{aligned}$$

under (2.10) for $g_n = g_n^{(2)}$. The result then follows by induction. □

Remark 2.3. A sufficient condition for g_n to meet (2.10) is that $g_n(z)$ is continuously differentiable in z with $\nabla g_n(z)$ taking values in Θ. Indeed, by Taylor's theorem,

$$g_n(z_1) - g_n(z_2) = A_n^\top (z_1 - z_2), \quad A_n = \int_0^1 \nabla g_n(z_2 + t(z_1 - z_2))\, \mathrm{d}t,$$

and then notice that A_n is Θ-valued because Θ is a convex set.

Example 2.4. (*Locally entropic monetary utility.*) Let $\{(\hat{P}_{n,0}, \ldots, \hat{P}_{n,d})^\top\}$ be a Δ_d-valued predictable process, let $\{B_n\}$ and $\{\Gamma_n\}$ be positive predictable processes, and

$$g_n(z) = -\frac{1}{\Gamma_n} \log\left(\sum_{j=0}^{d} \mathrm{e}^{-\Gamma_n z^\top v_j} \hat{P}_{n,j}\right) - \frac{1}{\Gamma_n} \log B_n. \tag{2.12}$$

Using a similar calculation to Example 2.1, we deduce the relation

$$Y_{n-1} = -\frac{1}{\Gamma_n}\{\log \hat{\mathbb{E}}[\mathrm{e}^{-\Gamma_n Y_n} \mid \mathcal{F}_{n-1}] + \log B_n\}, \quad n = N, \ldots, 1.$$

In particular, when $B_n = 1$, $\mathcal{E}_{n-1}^g$ is locally the minus of the entropic risk measure with risk-aversion parameter Γ_n extending (2.9). In contrast to the dynamic entropic risk measure studied in [1], we have the time-consistency property $\mathcal{E}_m^g(\mathcal{E}_n^g(Y)) = \mathcal{E}_m^g(Y)$ for any $m \leq n$ when $B_n = 1$ for all n even if the process $\{\Gamma_n\}$ is not constant. We allow $B_n \neq 1$ in order to include an example in Section 3. We call $\mathcal{E}_n^g$ a locally entropic monetary utility. A brief numerical study for this utility is provided in Appendix B. We have

$$\nabla g_n(z) = \frac{\hat{\mathbb{E}}[\Delta X_n\, \mathrm{e}^{-\Gamma_n z^\top \Delta X_n} \mid \mathscr{F}_{n-1}]}{\hat{\mathbb{E}}[\mathrm{e}^{-\Gamma_n z^\top \Delta X_n} \mid \mathscr{F}_{n-1}]} = \mathbf{v}\hat{P}_n(z),$$

where $\hat{P}_n(z) = (\hat{P}_{n,0}(z), \ldots, \hat{P}_{n,d}(z))^\top$ and

$$\hat{P}_{n,j}(z) = \frac{\mathrm{e}^{-\Gamma_n z^\top v_j} \hat{P}_{n,j}}{\sum_{k=0}^{d} \mathrm{e}^{-\Gamma_n z^\top v_k} \hat{P}_{n,k}}.$$

Since $\hat{P}_n(z)$ is continuous in z and Δ_d-valued for all n, by Remark 2.3, the assumptions of Proposition 2.3 on $g_n^{(i)}$ are satisfied.

Next, we seek a robust representation of $\mathcal{E}^g$ when g is concave. Let $\mathcal{C}$ denote the set of $g = \{g_n\} \in \mathcal{B}$ with $g_n(z)$ being concave in z for all n.

Lemma 2.5. *Let $g = \{g_n\} \in \mathcal{A}$. Then $g \in \mathcal{C}$ if and only if*

$$g_n(z) = \min_{\theta\in\Theta}\{z^\top \theta + b_n(\theta)\}, \tag{2.13}$$

where

$$b_n(\theta) = \sup_{z\in\mathbb{R}^d} \{g_n(z) - z^\top \theta\}.$$

Proof. If g_n is concave, then it is continuous on the interior of its domain that is $\mathbb{R}^d$. Therefore by a well-known fact on the Legendre transform, we have

$$g_n(z) = \inf_{x\in\mathbb{R}^d} \{z^\top x + b_n(x)\}.$$

Let $x \notin \Theta$. Since Θ is a closed convex set of $\mathbb{R}^d$, by the Hahn–Banach theorem (or the separating hyperplane theorem), there exists $z_0 \in \mathbb{R}^d$ such that

$$\min_{\theta\in\Theta} z_0^\top \theta > z_0^\top x.$$

Using (2.10), for $z = \alpha z_0$, $\alpha > 0$,

$$g_n(z) - z^\top x \geq g_n(0) + \min_{\theta\in\Theta} \theta^\top z - z^\top x = g_n(0) + \alpha \min_{\theta\in\Theta} z_0^\top (\theta - x).$$

Since the last term is positive, letting $\alpha \to \infty$, we conclude $b_n(x) = \infty$. This implies

$$g_n(z) = \inf_{\theta\in\Theta} \{z^\top \theta + b_n(\theta)\} = \inf_{(\theta,b)\in A_n} \{z^\top \theta + b\},$$

where $A_n = \{(\theta, b) \in \Theta \times \mathbb{R}\,; g_n(w) \leq w^\top \theta + b$ for all $w \in \mathbb{R}^d\}$. Fix n and z and then take a sequence $\{(\theta_k, b_k)\} \subset A_n$ such that $z^\top \theta_k + b_k \to g_n(z)$. Since Θ is compact, there exists a converging subsequence $\{\theta_{k_j}\}$ with limit $\theta_* \in \Theta$. We have $b_{k_j} = z^\top \theta_{k_j} + b_{k_j} - z^\top \theta_{k_j} \to g_n(z) - z^\top \theta_* =: b_*$. Also, $w^\top \theta_{k_j} + b_{k_j} \geq g_n(w)$ for all w implies $w^\top \theta_* + b_* \geq g_n(w)$ for all w, hence $(\theta_*, b_*) \in A_n$. Thus we obtain (2.13). Conversely, if (2.13) is true, then $g_n(z)$ is concave, being the minimum of concave (affine) functions. Since

$$z_1^\top \theta + b_n(\theta) = z_2^\top \theta + b_n(\theta) + \theta^\top (z_1 - z_2) \geq z_2^\top \theta + b_n(\theta) + \min_{\theta\in\Theta} \theta^\top (z_1 - z_2),$$

we derive (2.10) from (2.13). □

Let $\mathcal{P}_N$ denote the set of the probability measures on $(\Omega, \mathscr{F}_N)$ absolutely continuous with respect to $\mathbb{P}$. For $\hat{\mathbb{P}} \in \mathcal{P}_N$, there corresponds a Δ_d-valued predictable process $\{\hat{P}_n\}$ by (2.4). The measure $\hat{\mathbb{P}}$ is recovered from $\{\hat{P}_n\}$ by (2.5). Let $\hat{\mathbb{E}}$ denote the expectation under $\hat{\mathbb{P}}$ and define

$$c_n^g(\hat{\mathbb{P}}) = \hat{\mathbb{E}}\left[\sum_{i=n+1}^{N} b_i(\mathbf{v}\hat{P}_i) \middle| \mathscr{F}_n \right],$$

where b_i is associated with $g = \{g_n\} \in \mathcal{C}$ via (2.13). The following theorem shows the nature of the g-expectation with the gradient constraint as a nonlinear expectation, taking care of Knightian uncertainty, refining a general convex duality result in [18] for an explicit representation of a penalty function.

Theorem 2.2. *Let $g \in \mathcal{C}$. Then, for any $\mathscr{F}_N$-measurable random variable Y,*

$$\mathcal{E}_n^g(Y) = \min_{\hat{\mathbb{P}}\in\mathcal{P}_N} \left\{ \hat{\mathbb{E}}[Y \mid \mathscr{F}_n] + c_n^g(\hat{\mathbb{P}}) \right\}, \quad n = 0, 1, \ldots, N. \tag{2.14}$$

Proof. By (2.13), we have

$$g_n(z) \le z^\top \mathbf{v}\hat{P}_n + b_n(\mathbf{v}\hat{P}_n)$$

for any $\hat{\mathbb{P}} \in \mathcal{P}_N$. Therefore, by Propositions 2.2 and 2.3, we have

$$\mathcal{E}_n^g(Y) \le \hat{\mathbb{E}}[Y \mid \mathscr{F}_n] + c_n^g(\hat{\mathbb{P}})$$

for any $\hat{\mathbb{P}} \in \mathcal{P}_N$. On the other hand, for any $Y \in \mathscr{F}_N$, there exists the solution $\{(Y_n, Z_n)\}$ of (2.7) with $Y_N = Y$. By (2.13), there exists $\hat{P}_n$ such that

$$g_n(Z_n) = Z_n^\top \mathbf{v}\hat{P}_n + b_n(\mathbf{v}\hat{P}_n)$$

for each n. Since $\{g_n\}$ and $\{Z_n\}$ are predictable, $\{\hat{P}_n\}$ is a Δ_d-valued predictable process. Let $\hat{\mathbb{P}} \in \mathcal{P}_N$ be associated with $\{\hat{P}_n\}$. Then $\{(Y_n, Z_n)\}$ solves the BSDE with $\hat{g}_n(z) = z^\top \mathbf{v}\hat{P}_n + b_n(\mathbf{v}\hat{P}_n)$ as well, and so by Proposition 2.2,

$$\mathcal{E}_n^g(Y) = Y_n = \hat{\mathbb{E}}[Y \mid \mathscr{F}_n] + c_n^g(\hat{\mathbb{P}}),$$

which implies (2.14). □

Corollary 2.1. *Let $g \in \mathcal{C}$. Let Y and Y' be $\mathscr{F}_N$-measurable random variables.*

(i) *If $Y \ge Y'$, then $\mathcal{E}_n^g(Y) \ge \mathcal{E}_n^g(Y')$, $n = 0, 1, \dots, N$.*

(ii) *For any $\mathscr{F}_n$-measurable* [0,1]*-valued random variable λ,*

$$\mathcal{E}_n^g(\lambda Y + (1-\lambda)Y') \ge \lambda \mathcal{E}_n^g(Y) + (1-\lambda)\mathcal{E}_n^g(Y'), \quad n = 0, 1, \dots, N.$$

Example 2.5. Let $\Theta_n \subset \Theta$ and

$$g_n(z) = \inf_{\theta \in \Theta_n} z^\top \theta, \quad n = 1, \dots, N.$$

Here, the set Θ_n can be random in such a way that g_n is $\mathscr{F}_{n-1} \otimes \mathscr{B}(\mathbb{R}^d)$-measurable. Then we have (2.13) with b_n such that $b_n(\theta) = 0$ if $\theta \in \bar{\Theta}_n$ while $b_n(\theta) = \infty$ otherwise, where $\bar{\Theta}_n$ is the closure of Θ_n. In particular when $\Theta_n = \Theta$, by Theorem 2.2,

$$\mathcal{E}_n^g(Y) = \min_{\hat{\mathbb{P}} \in \mathcal{P}_N} \hat{\mathbb{E}}[Y \mid \mathscr{F}_n], \quad n = 1, \dots, N,$$

for any $\mathscr{F}_N$-measurable random variable Y. Note also that

$$\mathcal{E}_0^g(Y) = \min_{\hat{\mathbb{P}} \in \mathcal{P}_N} \hat{\mathbb{E}}[Y] = \min_{\omega \in \Omega} Y(\omega).$$

When $\Theta_n = \{\mathbf{v}P_n^{(1)}, \dots, \mathbf{v}P_n^{(m)}\}$ for a Δ_d-valued predictable process $\{P_n^{(i)}\}$, letting $\mathbb{E}^{(x)}$ denote the expectation under the measure determined by $\{P_n^{(x_n)}\}$ for $x = (x_1, \dots, x_N) \in \{1, \dots, m\}^N$, by Theorem 2.2, we have

$$\mathcal{E}_n^g(Y) = \min_x \mathbb{E}^{(x)}[Y \mid \mathscr{F}_n], \quad n = 1, \dots, N.$$

2.5. Filtration consistent nonlinear expectations

Inspired by Coquet et al. [16], we call $\mathcal{E}\colon L^0(\Omega, \mathscr{F}_N, \mathbb{P}) \to \mathbb{R}$ a filtration consistent nonlinear expectation if:

(i) $Y \geq Y' \Rightarrow \mathcal{E}(Y) \geq \mathcal{E}(Y')$,

(ii) $Y \geq Y'$ and $\mathcal{E}(Y) = \mathcal{E}(Y') \Rightarrow Y = Y'$,

(iii) $\mathcal{E}(c) = c$ for any constant $c \in \mathbb{R}$, and

(iv) for any $n = 1, \ldots, N$ and Y, there exists an $\mathscr{F}_n$-measurable η such that $\mathcal{E}(Y1_A) = \mathcal{E}(\eta 1_A)$ for any $A \in \mathscr{F}_n$.

Further, η is uniquely determined as shown in [16]. Let it be denoted by $\mathcal{E}_n(Y)$. It follows that

$$\mathcal{E}_n(\mathcal{E}_m(Y)) = \mathcal{E}_n(Y) \tag{2.15}$$

for any $m \geq n$, and

$$\mathcal{E}_n(Y)1_A = \mathcal{E}_n(Y1_A) \tag{2.16}$$

for any $A \in \mathscr{F}_n$.

Proposition 2.4. *Let $g = \{g_n\} \in \mathcal{A}$, and assume that for $n = 1, \ldots, N$,*

(i) *$g_n(0) = 0$ and*

(ii) *for any $z_1, z_2 \in \mathbb{R}^d$*

$$g_n(z_1) - g_n(z_2) \geq \min_{\theta \in \Theta} \theta^\top (z_1 - z_2),$$

with equality holding only if $z_1 = z_2$.

Then $\mathcal{E}_0^g$ is a filtration consistent nonlinear expectation with $\mathcal{E}_n = \mathcal{E}_n^g$ and a translation invariance property,

$$\mathcal{E}_n(Y + \eta) = \mathcal{E}_n(Y) + \eta, \tag{2.17}$$

for any $\mathscr{F}_N$-measurable random variable Y, $\mathscr{F}_n$-measurable random variable η, and $n = 1, \ldots, N$.

Proof. From Proposition 2.3 and its proof, we observe the first two properties of filtration consistent nonlinear expectation. By $g_n(0) = 0$ we derive (2.15) and (2.16), from which the other properties follow. □

The following theorem is a discrete analog of Theorem 7.1 of [16].

Proposition 2.5. *Let $\mathcal{E}$ be a filtration consistent nonlinear expectation with the translation invariance property (2.17). Let $g_n(z) = \mathcal{E}_{n-1}(z^\top \Delta X_n)$. Then $\mathcal{E}_n = \mathcal{E}_n^g$.*

Proof. We have $\mathcal{E}_N(Y) = Y$ for any Y, so the claim is true for $n = N$. Assume $\mathcal{E}_k(Y) = \mathcal{E}_k^g(Y)$ for $k \geq n$. Then, by (2.2), there exists $\mathscr{F}_{n-1}$-measurable A_n and Z_n such that

$$\mathcal{E}_n(Y) = A_n + Z_n^\top \Delta X_n.$$

Then, by (2.15) and (2.17),

$$\mathcal{E}_{n-1}(Y) = A_n + \mathcal{E}_{n-1}(Z_n^\top \Delta X_n).$$

The last term is $g_n(Z_n)$ by (2.16). Therefore

$$\mathcal{E}_n(Y) - \mathcal{E}_{n-1}(Y) = -g_n(Z_n) + Z_n^\top \Delta X_n,$$

which implies that $\mathcal{E}_{n-1}(Y) = \mathcal{E}^g_{n-1}(Y)$. The result follows by induction. □

3. Market equilibrium analysis

3.1. Monetary utility maximization

Now we consider $\{X_n\}$ to be a d-dimensional asset price process. The lattice L is then understood as the price grid. For any $\{\mathscr{F}_n\}$-predictable $\mathbb{R}^d$-valued process $Z = \{Z_n\}$ and $w \in \mathbb{R}$,

$$W_n(w, Z) := w + \sum_{i=1}^{n} Z_i^\top \Delta X_i$$

represents the wealth process associated with the portfolio strategy Z and the initial wealth w. Applying Theorem 2.1 with $g_n = 0$, $n = 1, \ldots, N$, we observe that the market is complete, that is, for any $\mathscr{F}_N$-measurable random variable Y, there exists a predictable process Z and $w \in \mathbb{R}$ such that $Y = W_N(w, Z)$.

Consider an agent whose utility functional is $\mathcal{E}^g_0$ for $g \in \mathcal{C}$. This utility is monetary in the sense that for all $n = 0, \ldots, N$, $\mathcal{E}^g_n(Y + A) = \mathcal{E}^g_n(Y) + A$ for any $\mathscr{F}_N$-measurable random variable Y and $\mathscr{F}_n$-measurable random variable A. When assuming $g_n(0) = 0$ for all n in addition, the utility is normalized in the sense that $\mathcal{E}^g_n(0) = 0$, $n = 0, \ldots, N$, and it is time-consistent in the sense that $\mathcal{E}^g_m(\mathcal{E}^g_n(Y)) = \mathcal{E}^g_m(Y)$ for any $m \leq n$ and for any $\mathscr{F}_N$-measurable random variable Y. The simplest example is $\mathcal{E}^g_n(Y) = \mathbb{E}[Y \mid \mathscr{F}_n]$ corresponding to $g_n(z) = z^\top \mathbb{E}[\Delta X_n \mid \mathscr{F}_{n-1}]$. More generally, $\mathcal{E}^g_n(Y)$ is a conditional expectation with respect to a probability measure when g_n are linear for all n by Proposition 2.2. The driver $\{g_n\}$ should reflect the agent's belief in the distribution of the price process $\{X_n\}$. For example, if $g_n = 0$ for all n, then $\mathcal{E}^g_n(Y) = \mathbb{E}_\mathbb{Q}[Y \mid \mathscr{F}_n]$ irrespective of $\mathbb{P}$. The choice of nonlinear $\{g_n\}$ accommodates a nonlinear evaluation of risk, extending the exponential utility (2.9). In light of Theorem 2.2, our problem can be interpreted as a robust utility maximization.

The agent's objective is to maximize $\mathcal{E}^g_0(H + W_N(w, \pi))$ among the predictable process $\pi = \{\pi_n\}$, where H is a given $\mathscr{F}_N$-measurable random variable representing an initial endowment of the agent, i.e. an initially endowed asset or a scheduled random cashflow. Since the utility is monetary, it suffices to treat the case $w = 0$. Let

$$Y^\pi_n = \mathcal{E}^g_n(H + W_N(0, \pi) - W_n(0, \pi)) = \mathcal{E}^g_n(H + W_N(0, \pi)) - W_n(0, \pi). \tag{3.1}$$

Then the problem is equivalent to maximizing Y^π_0 among π. The following theorem characterizes the maximizer.

Theorem 3.1. *Assume that there exists a predictable process* $\{Z^\dagger_n\}$ *such that*

$$g_n(Z^\dagger_n) = \sup_{z \in \mathbb{R}^d} g_n(z), \qquad n = 1, \ldots, N. \tag{3.2}$$

Then

$$\max_\pi Y^\pi_0 = \max_\pi \mathcal{E}^g_0(H + W_N(0, \pi)) = \mathcal{E}^g_0(H + W_N(0, \pi^*)),$$

where $\pi_n^* = Z_n^\dagger - Z_n^*$, $Z_n^* = Z_n^H + Z_n^g$,

$$\begin{aligned} Z_n^H &= (d+1)(\mathbf{v}\mathbf{v}^\top)^{-1}\mathbb{E}_{\mathbb{Q}}[H\Delta X_n \mid \mathscr{F}_{n-1}], \\ Z_n^g &= (d+1)(\mathbf{v}\mathbf{v}^\top)^{-1}\mathbb{E}_{\mathbb{Q}}[G_n\Delta X_n \mid \mathscr{F}_{n-1}], \end{aligned} \tag{3.3}$$

and

$$G_n = \sum_{i=n+1}^{N} \mathbb{E}_{\mathbb{Q}}[g_i(Z_i^\dagger) \mid \mathscr{F}_n]$$

for $n = 1, \dots, N$. *Moreover,*

$$Y_n^{\pi^*} = \mathbb{E}_{\mathbb{Q}}[H \mid \mathscr{F}_n] + G_n \tag{3.4}$$

for $n = 0, 1, \dots, N$. *If in addition* $Z_n^\dagger$ *is unique, then* π_n^* *is the unique maximizer.*

Proof. Let Y_n^* denote the right-hand side of (3.4). Then $\{(Y_n^*, Z_n^*)\}$ is the solution of the BSΔE

$$\Delta Y_n^* = -g_n(Z_n^\dagger) + (Z_n^*)^\top \Delta X_n, \quad Y_N^* = H, \tag{3.5}$$

by Theorem 2.1. Note that

$$\Delta Y_n^\pi = -g_n(Z_n) + Z_n^\top \Delta X_n - \pi_n^\top \Delta X_n, \quad Y_N^\pi = H,$$

for a predictable process $\{Z_n\}$ by (3.1). This means $\{(Y_n^\pi, Z_n^\pi)\}$, $Z_n^\pi = Z_n - \pi_n$ solves the BSΔE

$$\Delta Y_n^\pi = -h_n^\pi(Z_n^\pi) + (Z_n^\pi)^\top \Delta X_n, \quad Y_N^\pi = H,$$

where $h_n^\pi(z) = g_n(z + \pi_n)$. For any predictable process π, we have $h_n^\pi \le g_n(Z_n^\dagger)$. Therefore $Y_n^\pi \le Y_n^*$ by Proposition 2.3. Notice also that by choosing $\pi_n^* = Z_n^\dagger - Z_n^*$ we have $g_n(Z^\dagger) = h_n^{\pi^*}(Z_n^*)$, so that $\{(Y_n^*, Z_n^*)\}$ satisfies the same BSΔE as $\{(Y_n^{\pi^*}, Z_n^{\pi^*})\}$. Hence $Y_n^* = Y_n^{\pi^*}$. □

Remark 3.1. Applying Theorem 2.1 to $g_n = 0$ and $Y_N = H$, we have

$$H = \mathbb{E}_{\mathbb{Q}}[H] + \sum_{n=1}^{N} Z_n^H \Delta X_n$$

with $\{Z_n^H\}$ defined by (3.3). Therefore the optimal strategy $\pi^* = Z_n^\dagger - Z_n^* = -Z_n^H + Z_n^\dagger - Z_n^g$ of Theorem 3.1 is decomposed into the hedging part $-Z_n^H$ and the optimal investment part $Z_n^\dagger - Z_n^g$.

Example 3.1. (*Locally entropic monetary utility.*) Consider (2.12). Let Δ_d° denote the interior of Δ_d and assume $\{(\hat{P}_{n,1}, \dots, \hat{P}_{n,d})^\top\}$ to be Δ_d°-valued. Then the map $z \mapsto g_n(z)$ is strictly concave and its unique maximizer is given by $Z_n^\dagger = b(y)$ of (2.3) for

$$y = \frac{1}{\Gamma_n} \log \hat{P}_n := \frac{1}{\Gamma_n}\big(\log \hat{P}_{n,1}, \dots, \log \hat{P}_{n,d}\big)^\top,$$

or equivalently

$$Z_n^\dagger = \frac{1}{\Gamma_n}(\mathbf{v}\mathbf{v}^\top)^{-1}\mathbf{v} \log \hat{P}_n. \tag{3.6}$$

Indeed, $v_j^\top Z_n^\dagger = \Gamma_n^{-1} \log \hat{P}_{n,j} - a(y)$ implies

$$\sum_{j=0}^{d} v_j \, \mathrm{e}^{-\Gamma_n v_j^\top Z_n^\dagger} \hat{P}_{n,j} = 0,$$

and thus $\nabla g_n(Z_n^\dagger) = 0$ for all n. We also have

$$\begin{aligned}
\frac{1}{\Gamma_n} \log B_n + g_n(Z_n^\dagger) &= -\frac{1}{\Gamma_n} \log\left(\sum_{j=0}^{d} \mathrm{e}^{-\Gamma_n v_j^\top Z_n^\dagger} \hat{P}_{n,j} \right) \\
&= -\frac{1}{\Gamma_n} \log\left((d+1) \, \mathrm{e}^{\frac{1}{d+1} \mathbf{1}^\top \log \hat{P}_n} \right) \\
&= \frac{1}{\Gamma_n} \frac{1}{d+1} \sum_{j=0}^{d} \log \frac{1}{(d+1)\hat{P}_{n,j}} \\
&= \frac{1}{\Gamma_n} D_{\mathrm{KL}}(Q_n || \hat{P}_n) \\
&\geq 0, \qquad (3.7)
\end{aligned}$$

where D_{KL} denotes the Kullback–Leibler divergence on Δ_d and $Q_n = \mathbf{1}/(d+1)$. By (3.3) and (3.6),

$$\pi_n^* = Z_n^\dagger - Z_n^g - Z_n^H = -Z_n^H + (\mathbf{v}\mathbf{v}^\top)^{-1} \mathbf{v} \left(\frac{\log \hat{P}_n}{\Gamma_n} - \hat{Y}_n \right),$$

where $\hat{Y}_n = (\hat{Y}_{n,0}, \ldots, \hat{Y}_{n,d})^\top$ and $\hat{Y}_{n,j} = \mathbb{E}_{\mathbb{Q}}[G_n \mid \mathscr{F}_{n-1}, \Delta X_n = v_j]$. The first term $-Z_n^H$ is the hedging term as noted in Remark 3.1. The term

$$Z_n^\dagger = (\mathbf{v}\mathbf{v}^\top)^{-1} \mathbf{v} \frac{\log \hat{P}_n}{\Gamma_n} \approx (\mathbf{v}\mathbf{v}^\top)^{-1} \mathbf{v} \frac{\hat{P}_n - \mathbf{1}}{\Gamma_n} = \frac{1}{\Gamma_n} (\mathbf{v}\mathbf{v}^\top)^{-1} \hat{\mathbb{E}}[\Delta X_n \mid \mathscr{F}_{n-1}]$$

can be interpreted as the discrete counterpart of the Merton portfolio (see e.g. Remark 8.9 of [27]). The term $\hat{Y}_n$ adjusts the expected return depending on the stochastic dynamics of $D_{\mathrm{KL}}(Q_i || \hat{P}_i)$ for $i \geq n+1$. Indeed, when $B_n = 1$ and $\Gamma_n = \gamma$ for all n and a constant $\gamma > 0$ as in (2.8), we have

$$\begin{aligned}
G_n &= \sum_{i=n+1}^{n} \mathbb{E}_{\mathbb{Q}}[g_i(Z_i^\dagger) \mid \mathscr{F}_n] \\
&= \frac{1}{\gamma} \sum_{i=n+1}^{n} \mathbb{E}_{\mathbb{Q}}[D_{\mathrm{KL}}(Q_i || \hat{P}_i) | \mathscr{F}_n] \\
&= \frac{1}{\gamma} \mathbb{E}_{\mathbb{Q}}\left[\log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\hat{\mathbb{P}}} - \log \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\hat{\mathbb{P}}} \,\middle|\, \mathscr{F}_n \right] \,\middle|\, \mathscr{F}_n \right]
\end{aligned}$$

by (3.7), where $\hat{\mathbb{P}}$ is associated with $\{\hat{P}_n\}$ by (2.5). Note also that $Z_n^g = 0$ if G_n is deterministic, which is the case when $\{\hat{P}_n\}$, $\{\Gamma_n\}$, and $\{B_n\}$ are deterministic.

Example 3.2. Let $d = 2$ and

$$\mathbf{v} = \begin{pmatrix} 0 & 1 & -1 \\ -2c & c & c \end{pmatrix},$$

where $c \in \mathbb{R}$. Then we have

$$X_{n,2} = c\left(3\sum_{k=1}^{n} |\Delta X_{k,1}|^2 - 2n\right).$$

Indeed, if $X_n = (n-a-b)v_0 + av_1 + bv_2$ for $(a, b) \in \mathbb{N}^2$, then

$$a + b = \sum_{k=1}^{n} |\Delta X_{k,1}|^2, \quad X_{n,2} = -2c(n-a-b) + c(a+b) = c(3(a+b) - 2n).$$

Regarding $\{X_{n,1}\}$ as a price process of an asset, the above identity allows us to interpret $X_{N,2}$ as an affine transform of the variance swap payoff of the asset. Further, regarding $\mathbb{Q}$ as the pricing measure, $\{X_{n,2}\}$ corresponds to the price process of the variance swap payoff. Note that $\{X_{n,1}\}$ describes a trinomial model for the asset, which is not complete. The variance swap trading makes the two-dimensional market $\{X_n\}$ complete.

3.2. General equilibrium

Consider m agents who maximize respective utilities $\mathcal{E}_0^{(i)}(H^{(i)} + W_N(0, \pi^{(i)}))$, $i = 1, \ldots, m$ through trading strategies $\pi^{(i)}$ of the d-dimensional asset $\{X_n\}$, where $\mathcal{E}^{(i)}$ is the solution map of the BS∆E (2.7) with $g = g^{(i)} \in \mathcal{C}$, and $H^{(i)}$ is an $\mathscr{F}_N$-measurable random variable representing an endowment for the agent i, $i = 1, \ldots, m$. Let H_n denote the total supply vector of the asset vector X_n and assume $\{H_n\}$ to be an $\mathbb{R}^d$-valued predictable process. We say the market is in general equilibrium if there exist predictable processes $\pi^{(i)} = \{\pi_n^{(i)}\}$, $i = 1, \ldots, m$ such that

(i) $\mathcal{E}_0^{(i)}(H^{(i)} + W_N(0, \pi^{(i)})) = \max_\pi \mathcal{E}_0^{(i)}(H^{(i)} + W_N(0, \pi))$ for all $i = 1, \ldots, m$, and

(ii) $\sum_{i=1}^m \pi_n^{(i)} = H_n$ for all $n = 1, \ldots, N$,

where the maximum is among all predictable processes π.

Proposition 3.1. *Let $H^{(i)}$ and $\tilde{H}^{(i)}$, $i = 1, \ldots, m$ be $\mathscr{F}_N$-measurable random variables, and let $\{H_n\}$ and $\{\tilde{H}_n\}$ be $\mathbb{R}^d$-valued predictable processes satisfying*

$$\sum_{i=1}^{m} H^{(i)} + \sum_{n=1}^{N} H_n^\top \Delta X_n = \sum_{i=1}^{m} \tilde{H}^{(i)} + \sum_{n=1}^{N} \tilde{H}_n^\top \Delta X_n.$$

Then the market with the endowments $H^{(i)}$ and the total supply $\{H_n\}$ is in general equilibrium if and only if the market with the endowments $\tilde{H}^{(i)}$ and total supply $\{\tilde{H}_n\}$ is in general equilibrium.

Proof. Let $Z^{\dagger(i)}$, $Z^{H(i)}$, and $Z^{g(i)}$, respectively, denote $Z^\dagger$, Z^H, and Z^g in Theorem 3.1 with $g = g^{(i)}$ and $H = H^{(i)}$. Then, by Theorem 3.1,

$$-H_n + \sum_{i=1}^{m} \pi_n^{(i)} = -H_n - \sum_{i=1}^{m} Z^{H(i)} + \sum_{i=1}^{m} \left(Z^{\dagger(i)} - Z^{g(i)}\right) = -Z^H + \sum_{i=1}^{m} \left(Z^{\dagger(i)} - Z^{g(i)}\right),$$

where Z^H is defined by (3.3) with

$$H = \sum_{i=1}^{m} H^{(i)} + \sum_{n=1}^{N} H_n^\top \Delta X_n.$$

Therefore, whether or not $\sum_{i=1}^m \pi_n^{(i)} = H_n$ depends on $H^{(i)}$ and $\{H_n\}$ only through H. □

We are interested in conditions on $g^{(i)}$ for the market to be in general equilibrium. In light of Proposition 3.1, we assume hereafter $H^{(i)} = 0$ ($i \geq 2$) and $H_n = 0$ ($n \geq 1$), without loss of generality. Let H denote $H^{(1)}$.

For functions $f^{(1)}$ and $f^{(2)}$ on $\mathbb{R}^d$, define the sup-convolution $f^{(1)} \Box f^{(2)}$ by

$$f^{(1)} \Box f^{(2)}(z) = \sup_{x \in \mathbb{R}^d} \{f^{(1)}(x) + f^{(2)}(z - x)\}.$$

For the drivers $g^{(i)} = \{g_n^{(i)}\}$, $i = 1, \ldots, m$ of the m agents' utilities, let

$$g_n(z) = g_n^{(1)} \Box \cdots \Box g_n^{(m)}(z). \tag{3.8}$$

Lemma 3.1. *For all n,*

$$\sup_{z \in \mathbb{R}^d} g_n(z) = \sum_{i=1}^{m} \sup_{z \in \mathbb{R}^d} g_n^{(i)}(z). \tag{3.9}$$

Proof. It is trivial that $g_n(z)$ is upper-bounded by the right-hand side sum for any z, and hence its supremum is upper-bounded. Conversely, let $\left\{z_k^{(i)}\right\}$ be a sequence for which $\lim_{k\to\infty} g_n^{(i)}\left(z_k^{(i)}\right) = \sup_{z\in\mathbb{R}^d} g_n^{(i)}(z)$ for each i. Then

$$\sum_{i=1}^{m} g_n^{(i)}\left(z_k^{(i)}\right) \leq g_n\left(\sum_{i=1}^{m} z_k^{(i)}\right) \leq \sup_{z \in \mathbb{R}^d} g_n(z)$$

for any k, and the limit is similarly bounded. $\Box$

A single agent whose utility is $\mathcal{E}_0^g$ with $g = \{g_n\}$ defined by (3.8) and whose endowment is H is called the representative agent of the market. The following theorem reduces the general equilibrium problem for the multi-agent market to the one for a single agent market. This extends the idea of the well-known Gorman aggregation theorem.

Proposition 3.2. *Assume that there exists a unique predictable process $\{Z_n^{\dagger(i)}\}$ for each $i = 1, \ldots, m$ such that*

$$g_n^{(i)}(Z_n^{\dagger(i)}) = \sup_{z \in \mathbb{R}^d} g_n^{(i)}(z)$$

for all $n = 1, \ldots, N$. Assume further that the maximizer of the map $z \mapsto g_n(z)(\omega)$ is unique for each n and $\omega \in \Omega$. The market for the m agents is in general equilibrium if and only if the market for the representative agent is in general equilibrium.

Proof. Let $\{(Y_n^{*(i)}, Z_n^{*(i)})\}$ be the solution of

$$\Delta Y_n^{*(i)} = -g_n^{(i)}(Z_n^{\dagger(i)}) + (Z_n^{*(i)})^\top \Delta X_n,$$

with $Y_N^{*(1)} = H$ and $Y_N^{*(i)} = 0$ for $i \geq 2$. By Theorem 3.1, the unique optimal strategy $\pi^{(i)}$ for the agent i is given by $\pi_n^{(i)} = Z_n^{\dagger(i)} - Z_n^{*(i)}$. By (3.9), we have

$$\sum_{i=1}^{m} g_n^{(i)}(Z_n^{\dagger(i)}) = \sup_{z \in \mathbb{R}^d} g_n(z) = g_n(Z_n^{\dagger}), \quad Z_n^{\dagger} := \sum_{i=1}^{m} Z_n^{\dagger(i)}.$$

Therefore

$$Y_n^* := \sum_{i=1}^{m} Y_n^{*(i)}, \quad Z_n^* := \sum_{i=1}^{m} Z_n^{*(i)}$$

solves

$$\Delta Y_n^* = -g_n(Z_n^\dagger) + (Z_n^*)^\top \Delta X_n, \qquad Y_N^* = H.$$

Since $Z_n^\dagger$ is the unique maximizer of g_n, the unique optimal strategy for the representative agent is $\pi_n^* = Z_n^\dagger - Z_n^*$, again by Theorem 3.1. Hence

$$\sum_{i=1}^m \pi_n^i = \sum_{i=1}^m Z_n^{*(i)} - \sum_{i=1}^m Z_n^{\dagger(i)} = Z_n^* - Z_n^\dagger = \pi_n^*.$$

Therefore $\sum_{i=1}^m \pi_n^i = 0$ if and only if $\pi_n^* = 0$. □

3.3. Equilibrium in a single agent market

By Proposition 3.2, it suffices to consider the case $m = 1$ in order to characterize the general equilibrium. Let $m = 1$, and put $g_n = g_n^{(1)}$ and $\pi_n = \pi_n^{(1)}$. The market is in general equilibrium if and only if $\mathcal{E}_0^g(H) = \max_\pi \mathcal{E}_0^g(H + W_N(0, \pi))$, that is, $\pi_n^* \equiv 0$ is the maximizer. The following theorems characterize the general equilibrium by a backward recurrence relation for the BSΔE driver $\{g_n\}$.

Theorem 3.2. *The market is in general equilibrium if* (3.2) *holds with*

$$Z_n^\dagger = (d+1)(\mathbf{v}\mathbf{v}^\top)^{-1}\mathbb{E}_\mathbb{Q}[(H + G_n)\Delta X_n \mid \mathscr{F}_{n-1}], \qquad n = 1, \ldots, N, \tag{3.10}$$

where

$$G_n = \mathbb{E}_\mathbb{Q}\left[\sum_{i=n+1}^N \sup_{z\in\mathbb{R}^d} g_i(z) \middle| \mathscr{F}_n\right].$$

Conversely, if the market is in general equilibrium and there exist sequences of maximizers $\{Z_n^\dagger\}$ of $\{g_n\}$, then (3.10) *is one of them.*

Proof. Notice that the right-hand side of (3.10) coincides with Z_n^* defined in Theorem 3.1. If (3.2) holds with (3.10), then we conclude that $\pi_n^* = 0$ is optimal by Theorem 3.1, which means that the market is in general equilibrium. Conversely, if the market is in general equilibrium, then by Theorem 3.1, Z_n^* should coincide with a maximizer of g_n if any. □

Proposition 3.3. *Let $f_n : \Omega \times \mathbb{R}^d \times \Delta_d \to \mathbb{R}$ be $\mathscr{F}_{n-1} \otimes \mathscr{B}(\mathbb{R}^d \times \Delta_d)$-measurable, concave on $\mathbb{R}^d$ and continuously differentiable on $\mathbb{R}^d$ with ∇f_n taking values in Θ, and $0 \in \nabla f_n(z, \Delta_d)$ for all $z \in \mathbb{R}^d$, $n = 1, \ldots, N$. Then there exists a Δ_d-valued predictable process $\{\hat{P}_n\}$ such that the market with $g = \{g_n\}$, $g_n(z) = f_n(z, \hat{P}_n)$, is in general equilibrium.*

Proof. Note first that $g \in \mathcal{C}$ by Remark 2.3. We construct $\{\hat{P}_n\}$ inductively. By the assumption, there exists $\hat{P}_N$ such that $\nabla f_N(Z_N^\dagger, \hat{P}_N) = 0$ for $Z_N^\dagger$ defined by (3.10) for $n = N$. Given $\hat{P}_k$ for $k \geq n + 1$, let $Z_n^\dagger$ be defined by (3.10) with $g_k(z) = f_k\big(z, \hat{P}_k\big)$. By the assumption, there exists $\hat{P}_n$ such that $\nabla f_n(Z_n^\dagger, \hat{P}_n) = 0$. Since $z \mapsto f_n(z, \hat{P}_n)$ is concave, $Z_n^\dagger$ is a maximizer of $g_n(z) = f_n(z, \hat{P}_n)$. The result then follows from Theorem 3.2. □

Example 3.3. (*Locally entropic monetary utility.*) We consider (2.12) again with $\{\hat{P}_n\}$, $\hat{P}_n = (\hat{P}_{n,1}, \ldots, \hat{P}_{n,d})^\top$, being Δ_d°-valued. The driver g_n is of the form $g_n(z) = (f(\Gamma_n z, \hat{P}_n) - \log B_n)/\Gamma_n$, where

$$f(z, p) = -\log \sum_{j=0}^d \mathrm{e}^{-z^\top v_j} p_j, \qquad z \in \mathbb{R}^d, \qquad p = (p_0, \ldots, p_d)^\top \in \Delta_d. \tag{3.11}$$

By (3.6), we have $\nabla f_n(Z_n, \hat{P}_n) = 0$ if and only if

$$Z_n = \frac{1}{\Gamma_n}(\mathbf{v}\mathbf{v}^\top)^{-1}\mathbf{v}\log \hat{P}_n.$$

By Theorem 3.2, the market is in general equilibrium if and only if

$$\mathbf{v}\log \hat{P}_n = (d+1)\Gamma_n \mathbb{E}_{\mathbb{Q}}[(H+G_n)\Delta X_n \mid \mathscr{F}_{n-1}], \quad n = 1, \ldots, N.$$

This is a backward recurrence equation for $\{\hat{P}_n\}$ because

$$G_n = \mathbb{E}_{\mathbb{Q}}\left[\sum_{i=n+1}^{N} \frac{1}{\Gamma_i}(D_{\mathrm{KL}}(Q_i||\hat{P}_i) - \log B_i) \,\middle|\, \mathscr{F}_n\right]$$

by (3.7). Since $\mathbf{v}$ has rank d and $\mathbf{v}\mathbf{1} = 0$, an equivalent condition is that

$$\hat{P}_{n,j} = \frac{\mathrm{e}^{\Gamma_n \bar{Y}_{n,j}}}{\sum_{k=0}^{d} \mathrm{e}^{\Gamma_n \bar{Y}_{n,k}}}, \quad \bar{Y}_{n,j} = \mathbb{E}_{\mathbb{Q}}[H + G_n \mid \mathscr{F}_{n-1}, \Delta X_n = v_j], \quad j = 0, \ldots, d, \tag{3.12}$$

for $n = 1, \ldots, N$. Let $Y_n^* = \mathbb{E}_{\mathbb{Q}}[H \mid \mathscr{F}_n] + G_n$. Then

$$Y_n = \mathbb{E}_{\mathbb{Q}}[Y_n \mid \mathscr{F}_{n-1}, \Delta X_n] = \sum_{j=0}^{d} \bar{Y}_{n,j} 1_{\{\Delta X_n = v_j\}},$$

which implies

$$\hat{P}_{n,j} 1_{\{\Delta X_n = v_j\}} = \frac{\mathrm{e}^{\Gamma_n Y_n^*}}{(d+1)\mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{\Gamma_n Y_n^*} \mid \mathscr{F}_{n-1}]} 1_{\{\Delta X_n = v_j\}}, \quad j = 0, \ldots, d,$$

under (3.12). Therefore the market is in general equilibrium if and only if

$$\frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{Q}} = \prod_{n=1}^{N} \frac{\mathrm{e}^{\Gamma_n Y_n^*}}{\mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{\Gamma_n Y_n^*} \mid \mathscr{F}_{n-1}]},$$

where $\hat{\mathbb{P}}$ is defined by (2.5). Further, by (3.5), we have

$$\begin{aligned}
\log \hat{\mathbb{E}}[\mathrm{e}^{-\Gamma_n Y_n^*} \mid \mathscr{F}_{n-1}] &= -\Gamma_n (Y_{n-1}^* - g_n(Z_n^\dagger)) + \log \hat{\mathbb{E}}[\mathrm{e}^{-\Gamma_n (\Delta X_n)^\top Z_n^\dagger} \mid \mathscr{F}_{n-1}] \\
&= -\Gamma_n (Y_{n-1}^* - g_n(Z_n^\dagger)) - f_n(\Gamma_n Z_n^\dagger, \hat{P}_n) \\
&= -\Gamma_n Y_{n-1}^* - \log B_n.
\end{aligned}$$

This implies

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\hat{\mathbb{P}}} = \prod_{n=1}^{N} \frac{\mathrm{e}^{-\Gamma_n Y_n^*}}{\hat{\mathbb{E}}[\mathrm{e}^{-\Gamma_n Y_n^*} \mid \mathscr{F}_{n-1}]} = \exp\left\{\sum_{n=1}^{N} -\Gamma_n \Delta Y_n^*\right\} \prod_{n=1}^{N} B_n. \tag{3.13}$$

In particular, when $B_n = 1$ and $\Gamma_n = \gamma$ for all n and a constant $\gamma > 0$ as in (2.8), we have

$$\frac{d\mathbb{Q}}{d\hat{\mathbb{P}}} = \frac{e^{-\gamma H}}{\hat{\mathbb{E}}[e^{-\gamma H}]}, \tag{3.14}$$

which characterizes the equilibrium probability measure $\hat{\mathbb{P}}$ and is consistent with the well-known computation under exponential utility.

Remark 3.2. In addition to the conditions of Proposition 3.3, if f_n is of the form $f_n(z, p) = f(\Gamma_n z, p)/\Gamma_n$ for a smooth function $f: \mathbb{R}^d \times \Delta_d \to \mathbb{R}$ with $\nabla f(0, p) = p$ and a positive predictable process $\{\Gamma_n\}$ as in Example 3.3 with $B_n = 1$, then we have

$$g_n(z) = f_n(z, \hat{P}_n) \approx z^\top \hat{P}_n$$

considering Γ_n to be small. This approximation implies in turn

$$\mathcal{E}_n^g(Y) \approx \hat{\mathbb{E}}[Y \mid \mathscr{F}_n]$$

in light of Proposition 2.2, where $\hat{\mathbb{E}}$ is the expectation under $\hat{\mathbb{P}}$ defined by (2.5). Therefore, in this case, extending Example 3.3, we can interpret Γ_n as a risk-aversion parameter and $\hat{\mathbb{P}}$ as the belief of the agent. If the market is in general equilibrium,

$$\hat{\mathbb{E}}[\Delta X_n \mid \mathscr{F}_{n-1}] = \mathbf{v}\hat{P}_n$$

is interpreted as an equilibrium return.

Example 3.4. If $H = h(X_N)$ and $g_n(z) = f_n(X_{n-1}, z)$ for deterministic functions h and f_n, as in Proposition 2.1, and if $f(x,z)$ is strictly concave and continuously differentiable in z, then the condition (3.10) follows from the deterministic identity

$$\nabla_z f_n(x, (\mathbf{v}\mathbf{v}^\top)^{-1}\mathbf{v}\mathcal{N}u_n(x)) = 0, \quad x \in L, \ n = 1, \ldots, N,$$

where $\mathcal{N}u_n$ is as in Proposition 2.1. For example, for the market with no random endowments ($H^{(i)} = 0$) and unit total supply ($H_n = \mathbf{1}$), we have $H = \mathbf{1}^\top X_N$.

Example 3.5. Consider (2.12) with $B_n = 1$, $\Gamma_n = \gamma_n(X_{n-1})$ and $\hat{P}_n = p_n(X_{n-1})$ for deterministic functions $\gamma_n: L \to (0, \infty)$ and $p_n: L \to \Delta_d^\circ$. Assume also $H = h(X_N)$ as in Example 3.4. Then, from Examples 3.3 and 3.4, the market is in general equilibrium if

$$\mathbf{v}\log p_n(x) = \gamma_n(x)\mathbf{v}\mathcal{N}u_n(x), \quad x \in L, \ n = 1, \ldots, N.$$

The function $u_n(x)$ is computed backward inductively without using $p_n(x)$. For a given function $\gamma_n(x)$, there exists a unique $p_n(x) \in \Delta_d^\circ$ satisfying this equation for each $x \in L$. For the sequence of such functions $p_n(x)$ obtained in the backward manner, the Δ_d-valued sequence $\hat{P}_n = p_n(X_{n-1})$ defines a unique equilibrium probability $\hat{\mathbb{P}}$ by (2.5) associated with the sequence $\Gamma_n = \gamma_n(X_{n-1})$. The equilibrium return is approximated as

$$\hat{\mathbb{E}}[\Delta X_n \mid \mathscr{F}_{n-1}] = \mathbf{v}\hat{P}_n \approx \mathbf{v}\log\hat{P}_n = \gamma_n(X_{n-1})\mathbf{v}\mathcal{N}u_n(X_{n-1})$$

using $\mathbf{v}\mathbf{1} = 0$.

3.4. Equilibrium under heterogeneous beliefs

Here we assume $m > 1$ again and give more explicit computations of the sup-convolution in special cases. First, we consider a homogeneous case, i.e. the case where all of the drivers $g^{(i)}$ have the same functional form determined by a common Δ_d-valued predictable process $\{\hat{P}_n\}$ as

$$g_n^{(i)}(z) = \frac{1}{\Gamma_n^{(i)}} f_n(\Gamma_n^{(i)} z, \hat{P}_n), \tag{3.15}$$

where $f_n \colon \Omega \times \mathbb{R}^d \times \Delta_d \to \mathbb{R}$ is as in Proposition 3.3 and $\{\Gamma_n^{(i)}\}$, $i = 1, \ldots, m$ are positive predictable processes quantifying each agent's risk preference; see Remark 3.2. By induction, we can show that

$$g_n(z) := g_n^{(1)} \square \cdots \square g_n^{(m)}(z) = \frac{1}{\Gamma_n} f_n(\Gamma_n z, \hat{P}_n),$$

with

$$\frac{1}{\Gamma_n} = \sum_{i=1}^m \frac{1}{\Gamma_n^{(i)}}. \tag{3.16}$$

Proposition 3.4. *Under (3.15), there exists a Δ_d-valued predictable process $\{\hat{P}_n\}$ such that the market is in general equilibrium.*

Proof. The proof follows from Propositions 3.3 and 3.2. □

Example 3.6. (*Locally entropic monetary utility.*) Let $f_n(z, p) = f(z, p)$ defined by (3.11). The predictable processes $\{\Gamma_n^{(i)}\}$ and $\{\Gamma_n\}$ are then interpreted as the local risk-aversion parameter for the agent i and for the representative agent respectively. Then (3.12) defines the unique sequence $\{\hat{P}_n\}$ such that the market is in general equilibrium. The equilibrium probability measure $\hat{\mathbb{P}}$ satisfies (3.13) with $B_n = 1$. When $\Gamma_n = \gamma$ for all n for a constant $\gamma > 0$, then the representative agent has the exponential utility (2.9) and the equilibrium probability measure $\hat{\mathbb{P}}$ is characterized by (3.14).

Now we consider a heterogeneous case. We assume locally entropic monetary utilities $g_n^{(i)}(z) = f(\Gamma_n^{(i)} z, \hat{P}_n^{(i)}) / \Gamma_n^{(i)}$, where f is defined by (3.11), and $\{\Gamma_n^{(i)}\}$ and

$$\{\hat{P}_n^{(i)}\} = \left\{ \left(\hat{P}_{n,0}^{(i)}, \ldots, \hat{P}_{n,d}^{(i)} \right)^\top \right\},$$

respectively, are $(0, \infty)$-valued and Δ_d°-valued predictable processes for each $i = 1, \ldots, m$. Each sequence $\{\hat{P}_n^{(i)}\}$ determines a probability measure $\hat{P}^{(i)}$ on $\mathscr{F}_N$ by (2.5), which is interpreted as the agent i's belief in the law of $\{X_n\}$.

Proposition 3.5. *Define $\{\Gamma_n\}$ by (3.16). Then*

$$g_n^{(1)} \square \cdots \square g_n^{(m)}(z) = \frac{1}{\Gamma_n} f(\Gamma_n z, \tilde{P}_n) - \frac{1}{\Gamma_n} \log B_n,$$

where $\tilde{P}_n = (\tilde{P}_{n,0}, \ldots, \tilde{P}_{n,d})^\top$ and

$$\tilde{P}_{n,j} = \frac{1}{B_n} \prod_{i=1}^m \left(\hat{P}_{n,j}^{(i)} \right)^{\Gamma_n / \Gamma_n^{(i)}}, \quad B_n = \sum_{j=0}^d \prod_{i=1}^m \left(\hat{P}_{n,j}^{(i)} \right)^{\Gamma_n / \Gamma_n^{(i)}}.$$

Proof. The case $m = 2$ follows by solving the equation

$$\nabla f\big(\Gamma_n^{(1)} x, \hat{P}_n^{(1)}\big) = \nabla f\big(\Gamma_n^{(2)}(z - x), \hat{P}_n^{(2)}\big)$$

in $x \in \mathbb{R}^d$; see Lemma A.1. The general case then follows by induction. □

Remark 3.3. By Lemma A.2 we have $B_n \le 1$, with equality holding if and only if $\hat{P}_n^{(i)} = \hat{P}_n^{(1)}$ for all i.

By Proposition 3.5, the representative agent's market falls into Example 3.3. In particular, when $\Gamma_n = \gamma > 0$ (a constant), the market is in general equilibrium if and only if $\tilde{P}_n = \hat{P}_n$, where

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\hat{\mathbb{P}}} = \frac{\mathrm{e}^{-\gamma H} \prod_{n=1}^N B_n}{\hat{\mathbb{E}}[\mathrm{e}^{-\gamma H} \prod_{n=1}^N B_n]}.$$

To highlight the outcome of heterogeneous beliefs, let us further assume there are only two agents ($m = 2$) with constant risk aversion $\Gamma_n^{(i)} = \gamma_i > 0$ and with no endowment ($H = 0$). In this case, $\nabla g^{(i)}(0, \mathbb{Q}) = 0$, and so, if the two agents have a common belief $\hat{\mathbb{P}}$, we need $\hat{\mathbb{P}} = \mathbb{Q}$ for the market to be in general equilibrium. The optimal strategies are simply $\pi_n^{(1)} = \pi_n^{(2)} = 0$. On the other hand, for any Δ_d°-valued deterministic sequence $\{\hat{P}_n^{(1)}\}$, by choosing $\hat{P}_n^{(2)}$ as

$$\hat{P}_{n,j}^{(2)} = \frac{(\hat{P}_{n,j}^{(1)})^{-\gamma_2/\gamma_1}}{\sum_{k=0}^d (\hat{P}_{n,k}^{(1)})^{-\gamma_2/\gamma_1}},$$

we have $\tilde{P}_{n,j} = 1/(d+1)$ for all n and j, which makes this market with heterogeneous beliefs be in general equilibrium. The individual optimal strategies $\pi_n^{(1)} = -\pi_n^{(2)}$ are non-zero; the agents bet on their beliefs.

Another observation is that the equilibrium return is mostly affected by the belief of the least risk-averse agent. Indeed, if $\Gamma_n^{(1)} \ll \Gamma_n^{(i)}$ for $i \ge 2$, we have $\Gamma_n / \Gamma_n^{(1)} \approx 1$, while $\Gamma_n / \Gamma_n^{(i)} \approx 0$ for $i \ge 2$. Therefore $\tilde{P}_n \approx \hat{P}_n^{(1)}$.

Remark 3.4. The product $\prod_n B_n$ corresponds to the consensus characteristic introduced in a continuous-time framework [26] of heterogeneous beliefs. It can be interpreted as a discounting factor, and was further investigated in [21].

Appendix A. Computation of sup-convolution

Lemma A.1. *Let $\alpha > 0$, $\beta > 0$, $p_j > 0$, $q_j > 0$, $j = 0, \dots, d$, and $z \in \mathbb{R}^d$. Then*

$$\sup_{x \in \mathbb{R}^d} \left\{ -\frac{1}{\alpha} \log \sum_{j=0}^d \mathrm{e}^{-\alpha x^\top v_j} p_j - \frac{1}{\beta} \log \sum_{j=0}^d \mathrm{e}^{-\beta (z-x)^\top v_j} q_j \right\}$$

$$= -\frac{1}{\gamma} \log \sum_{j=0}^d \mathrm{e}^{-\gamma z^\top v_j} p_j^{\gamma/\alpha} q_j^{\gamma/\beta},$$

where $\gamma = 1/(1/\alpha + 1/\beta)$.

Proof. The first-order condition is

$$\sum_{j=0}^{d} v_j\left(\frac{\mathrm{e}^{-\alpha x^\top v_j}p_j}{S(x,\,\alpha,\,\{p_k\})}-\frac{\mathrm{e}^{-\beta(z-x)^\top v_j}q_j}{S(z-x,\,\beta,\,\{q_k\})}\right)=0,$$

where

$$S(x,\,\alpha,\,\{p_k\})=\sum_{j=0}^{d}\mathrm{e}^{-\alpha x^\top v_j}p_j.$$

Since $\mathbf{v1}=0$ and rank $\mathbf{v}=d$, the first-order condition is met if and only if

$$-\alpha x^\top v_j+\log p_j=-\beta(z-x)^\top v_j+\log q_j+c(x),\quad j=0,\ldots,d,$$

for a function c. Substituting

$$x^\top v_j=\frac{\beta}{\alpha+\beta}z^\top v_j+\frac{\log p_j-\log q_j-c(x)}{\alpha+\beta},$$

we obtain

$$-\frac{1}{\alpha}\log\sum_{j=0}^{d}\mathrm{e}^{-\alpha x^\top v_j}p_j=-\frac{1}{\alpha}\log\sum_{j=0}^{d}\mathrm{e}^{-\gamma z^\top v_j}p_j^{\gamma/\alpha}q_j^{\gamma/\beta}-\frac{c(x)}{\alpha+\beta},$$
$$-\frac{1}{\beta}\log\sum_{j=0}^{d}\mathrm{e}^{-\beta(z-x)^\top v_j}q_j=-\frac{1}{\beta}\log\sum_{j=0}^{d}\mathrm{e}^{-\gamma z^\top v_j}p_j^{\gamma/\alpha}q_j^{\gamma/\beta}+\frac{c(x)}{\alpha+\beta},$$

hence the result. □

Lemma A.2. *Let* $(p_{i,0},\ldots,p_{i,d})^\top$, $i=1,\ldots,m$ *be* m *points in* Δ_d°. *Let* $\gamma_i>0$ *for* $i=1,\ldots,m$ *and*

$$\gamma=\left(\sum_{i=1}^{m}\frac{1}{\gamma_i}\right)^{-1}.$$

Then

$$\sum_{j=0}^{d}\prod_{i=1}^{m}p_{i,j}^{\gamma/\gamma_i}\le 1.$$

Proof. The case $m=1$ is trivial. Let

$$\hat{\gamma}_k=\left(\sum_{i=1}^{k}\frac{1}{\gamma_i}\right)^{-1},\quad k=1,\ldots,m.$$

If the inequality is true when $m=k$, then

$$\sum_{j=0}^{d}\prod_{i=1}^{k+1}p_{i,j}^{\hat{\gamma}_{k+1}/\gamma_i}=\sum_{j=0}^{d}p_{k+1,j}\left(\frac{\prod_{i=1}^{k}p_{i,j}^{\hat{\gamma}_k/\gamma_i}}{p_{k+1,j}}\right)^{\hat{\gamma}_{k+1}/\hat{\gamma}_k}\le\left(\sum_{j=0}^{d}\prod_{i=1}^{k}p_{i,j}^{\hat{\gamma}_k/\gamma_i}\right)^{\hat{\gamma}_{k+1}/\hat{\gamma}_k}\le 1$$

by Jensen's inequality. We obtain the result by induction. □

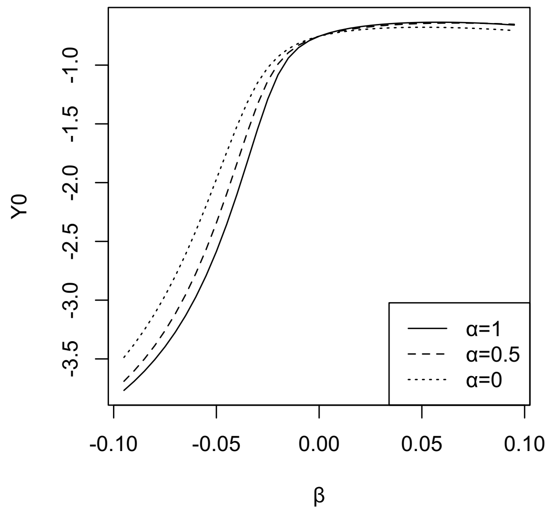

FIGURE 1. $Y_0 = \mathcal{E}_0^g(Y_N)$ as a function of $\beta \in (-0.1, 0.1)$ for $\alpha = 0, 0.5$ and 1.

Appendix B. Numerical experiment

Here we give a brief numerical experiment on the locally entropic monetary utility defined by (2.12). We focus on the case $B_n = 1$ and $\hat{P}_{n,j} = 1/(d+1)$ for all n and j. The purpose here is to examine numerically the effect of the local risk-aversion process $\{\Gamma_n\}$. More specifically, we consider an auto-regressive structure

$$\log \frac{\Gamma_{n+1}}{\gamma} = \alpha \log \frac{\Gamma_n}{\gamma} + \frac{\beta}{\sqrt{N}} \mathbf{1}^\top \Delta X_n$$

with constants $\gamma > 0$, $\alpha \in [0, 1]$ and $\beta \in \mathbb{R}$, and compute the g-expectation $\mathcal{E}_0^g(Y_N)$ for $Y_N = (Nd)^{-1/2}\mathbf{1}^\top X_N$. When β is negative, the negative values of $\mathbf{1}^\top \Delta X_n$ push Γ_n up, that is, the utility becomes more risk-averse when $\mathbf{1}^\top \Delta X_n$ is negative. We are interested in how such dynamics affects the initial utility value $Y_0 = \mathcal{E}_0^g(Y_N)$.

For example, for $d = 100$, $N = 3$, $\gamma = 1$, and $\Sigma/(d+1)$ being the identity matrix, Figure 1 shows the shapes of $Y_0 = \mathcal{E}_0^g(Y_N)$ as a function of $\beta \in (-0.1, 0.1)$ for $\alpha = 0, 0.5$, and 1. For the negative region of β, we have monotone shapes, which means that the larger the variance of the process $\{\Gamma_n\}$, the less the initial utility. This monotonicity is lost in the positive region of β.

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