J. Appl. Prob. **49,** 1015–1035 (2012)
Printed in England

ON A NEW CLASS OF TEMPERED STABLE DISTRIBUTIONS: MOMENTS AND REGULAR VARIATION

MICHAEL GRABCHAK,* *University of North Carolina at Charlotte*

Abstract

We extend the class of tempered stable distributions, which were first introduced in Rosiński (2007). Our new class allows for more structure and more variety of the tail behaviors. We discuss various subclasses and the relations between them. To characterize the possible tails, we give detailed results about finiteness of various moments. We also give necessary and sufficient conditions for the tails to be regularly varying. This last part allows us to characterize the domain of attraction to which a particular tempered stable distribution belongs.

Keywords: Infinite divisibility; tempered stable distribution; regular variation; tail behavior

2010 Mathematics Subject Classification: Primary 60E07
Secondary 60G51

1. Introduction

Tempered stable distributions were defined in Rosiński [20] as a class of models obtained by modifying the Lévy measures of stable distributions by multiplying their densities by completely monotone functions. This allows for models that are similar to stable distributions in some central region, but possess lighter (i.e. tempered) tails. It has been observed that these models provide a good fit to data in a variety of applications. These include mathematical finance [9], [15], biostatistics [2], [19], computer science [13], and physics [8], [18]. An explanation for why such models might appear in applications is given in [11].

The purpose of this paper is twofold. First, we provide necessary and sufficient conditions for tempered stable distributions to have regularly varying tails. This is important both from a theoretical perspective, since it will allow us to classify which domain of attraction a tempered stable distribution belongs to, and from an applied point of view, since such models are often used in practice.

Our second purpose is to introduce the class of p-tempered α-stable distributions, where $p > 0$ and $\alpha < 2$. The parameter p controls the amount of tempering, while α is the index of stability of the corresponding stable distribution. Clearly, the case where $\alpha \leq 0$ no longer has any meaning in terms of tempering stable distributions; however, it allows the class to be more flexible. In fact, within certain subclasses, the case where $\alpha \leq 0$ has been shown to provide a good fit to data; see, e.g. [2] or [9].

This class combines a number of important subclasses that have been studied separately in the literature. In particular, when $p = 1$ and $\alpha \in (0, 2)$, it coincides with Rosiński's [20] tempered stable distributions. When $p = 2$ and $\alpha \in [0, 2)$, it coincides with the class of

Received 30 August 2011.
* Postal address: University of North Carolina at Charlotte, 376 Fretwell Hall, 9201 University City Blvd, Charlotte, NC 28223, USA. Email address: mgrabcha@uncc.edu

tempered infinitely divisible distributions defined in [6]. If we allow the distributions to have a Gaussian part then we would have the class $J_{\alpha,p}$ defined in [16]. This, in turn, contains important subclasses including the Thorin class (when $p = 1$ and $\alpha = 0$), the Goldie–Steutel–Bondesson class (when $p = 1$ and $\alpha = -1$), the class of type-M distributions (when $p = 2$ and $\alpha = 0$), and the class of type-G distributions (when $p = 2$ and $\alpha = -1$). For more information on these classes, see [3], [4], and the references therein.

This paper is structured as follows. In Section 2 we define p-tempered α-stable distributions and state some basic results. We show that, as with tempered stable distributions, for a fixed α and p, all elements of this class are uniquely determined by a Rosiński measure R and a shift b. The remaining two sections are concerned with relating the tails of the Rosiński measure to the tails of the distribution. In Section 3 we give necessary and sufficient conditions for the existence of moments and exponential moments. We also give explicit formulae for the cumulants. Finally, in Section 4 we give necessary and sufficient conditions for the tails to be regularly varying. Specifically, we show that the tails of a p-tempered α-stable distribution are regularly varying if and only if the tails of the corresponding Rosiński measure are regularly varying.

Before proceeding, recall that the characteristic function of an infinitely divisible distribution μ on $\mathbb{R}^d$ can be written as $\hat{\mu}(z) = \exp\{C_\mu(z)\}$, where

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + \mathrm{i}\langle b, z\rangle + \int_{\mathbb{R}^d}\left(\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\frac{\langle z, x\rangle}{1+|x|^2}\right)M(\mathrm{d}x), \tag{1}$$

A is a symmetric nonnegative-definite $d \times d$ matrix, $b \in \mathbb{R}^d$, and M satisfies

$$M(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d}(|x|^2 \wedge 1)M(\mathrm{d}x) < \infty.$$

The measure μ is uniquely identified by the Lévy triplet (A, M, b) and we write $\mu = \mathrm{ID}(A, M, b)$.

2. p-tempered α-stable distributions

Recall that, for $\alpha \in (0, 2)$, the Lévy measure of an α-stable distribution with spectral measure σ is given by

$$L(B) = \int_{\mathbb{S}^{d-1}}\int_0^\infty \mathbf{1}_B(ru)r^{-\alpha-1}\mathrm{d}r\sigma(\mathrm{d}u), \qquad B \in \mathfrak{B}(\mathbb{R}^d).$$

By analogy, we define the following.

Definition 1. Fix $\alpha < 2$ and $p > 0$. An infinitely divisible probability measure μ is called a *p-tempered α-stable distribution* if it has no Gaussian part and its Lévy measure is given by

$$M(B) = \int_{\mathbb{S}^{d-1}}\int_0^\infty \mathbf{1}_B(ru)q(r^p, u)r^{-\alpha-1}\mathrm{d}r\sigma(\mathrm{d}u), \qquad B \in \mathfrak{B}(\mathbb{R}^d), \tag{2}$$

where σ is a finite Borel measure on $\mathbb{S}^{d-1}$ and $q\colon (0,\infty)\times\mathbb{S}^{d-1} \mapsto (0,\infty)$ is a Borel function such that, for all $u \in \mathbb{S}^{d-1}$, $q(\cdot, u)$ is completely monotone and

$$\lim_{r\to\infty} q(r, u) = 0. \tag{3}$$

We denote the class of p-tempered α-stable distributions by TS^p_α. If, in addition,

$$\lim_{r\downarrow 0} q(r,u) = 1 \quad \text{for every } u \in \mathbb{S}^{d-1}$$

then μ is called a *proper p-tempered α-stable distribution*.

Remark 1. The complete monotonicity of $q(\cdot,u)$ implies that, for each $u \in \mathbb{S}^{d-1}$, the function $q(r,u)$ is differentiable and monotonically decreasing in r. Moreover, by Bernstein's theorem (see, e.g. Theorem 1a of [10, Section XIII.4]),

$$q(r^p,u) = \int_{(0,\infty)} \mathrm{e}^{-r^p s} Q_u(\mathrm{d}s) \tag{4}$$

for some measurable family $\{Q_u\}_{u\in\mathbb{S}^{d-1}}$ of Borel measures on $(0,\infty)$. For a guarantee that we can take the family to be measurable, see Remark 3.2 of [4]. Note that the condition $\lim_{r\downarrow 0} q(r,u) = 1$ for every $u \in \mathbb{S}^{d-1}$ is equivalent to the condition that $\{Q_u\}_{u\in\mathbb{S}^{d-1}}$ is a family of probability measures.

Remark 2. From (4), it follows that, as p increases, the tails of M (as given in (2)) go to zero quicker. In this sense p controls the extent to which the tails of the Lévy measure are tempered.

Remark 3. For $\alpha \in (0,2)$ and $p > 0$, all proper p-tempered α-stable distributions belong to the class of generalized tempered stable distributions defined in [21]. Many important results about their Lévy processes are given there. These include short time behavior, conditions for absolute continuity with respect to the underlying stable process, and a series representation; see Theorems 3.1, 4.1, and 5.5 of [21] for details.

Remark 4. From Theorem 15.10 of [25], it follows that p-tempered α-stable distributions are self-decomposable if and only if $q(r^p,u)r^{-\alpha}$ is a decreasing function of r for every $u \in \mathbb{S}^{d-1}$. By Remark 1, this always holds when $\alpha \in [0,2)$. Thus, when $\alpha \in [0,2)$, p-tempered α-stable distributions inherit properties of self-decomposable distributions. In particular, if they are nondegenerate then they are absolutely continuous with respect to the Lebesgue measure in d-dimensions and when $d=1$, they are unimodal.

Following [20], we will reparametrize the Lévy measure M into a form that is often easier to work with. Let Q be a Borel measure on $\mathbb{R}^d$ given by

$$Q(A) = \int_{\mathbb{S}^{d-1}} \int_{(0,\infty)} \mathbf{1}_A(ru) Q_u(\mathrm{d}r)\sigma(\mathrm{d}u), \qquad A \in \mathfrak{B}(\mathbb{R}^d).$$

Note that $Q(\{0\}) = 0$. Define a Borel measure R on $\mathbb{R}^d$ by

$$R(A) = \int_{\mathbb{R}^d} \mathbf{1}_A\bigg(\frac{x}{|x|^{1+1/p}}\bigg)|x|^{\alpha/p} Q(\mathrm{d}x), \qquad A \in \mathfrak{B}(\mathbb{R}^d), \tag{5}$$

and again note that $R(\{0\}) = 0$. To get the inverse transformation, we have

$$Q(A) = \int_{\mathbb{R}^d} \mathbf{1}_A\bigg(\frac{x}{|x|^{p+1}}\bigg)|x|^{\alpha} R(\mathrm{d}x), \qquad A \in \mathfrak{B}(\mathbb{R}^d).$$

The following result extends Theorem 2.3 of [20].

Theorem 1. *Fix $p > 0$. Let M be given by (2), and let R be given by (5).*

1. *We can write*

$$M(A) = \int_{\mathbb{R}^d} \int_0^\infty \mathbf{1}_A(tx) t^{-1-\alpha} \mathrm{e}^{-t^p} \mathrm{d}t\, R(\mathrm{d}x), \qquad A \in \mathfrak{B}(\mathbb{R}^d), \tag{6}$$

or, equivalently,

$$M(A) = p^{-1} \int_{\mathbb{R}^d} \int_0^\infty \mathbf{1}_A(t^{1/p} x) t^{-1-\alpha/p} \mathrm{e}^{-t} \mathrm{d}t\, R(\mathrm{d}x), \qquad A \in \mathfrak{B}(\mathbb{R}^d).$$

2. *Equation (6) defines a Lévy measure if and only if either $R = 0$ or the following hold:*

$$\alpha < 2, \qquad R(\{0\}) = 0, \tag{7}$$

and

$$\int_{\mathbb{R}^d} (|x|^2 \wedge |x|^\alpha) R(\mathrm{d}x) < \infty \quad \text{if } \alpha \in (0, 2), \tag{8a}$$

$$\int_{\mathbb{R}^d} (|x|^2 \wedge [1 + \log^+ |x|]) R(\mathrm{d}x) < \infty \quad \text{if } \alpha = 0, \tag{8b}$$

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1) R(\mathrm{d}x) < \infty \quad \text{if } \alpha < 0. \tag{8c}$$

Moreover, when R satisfies these conditions, M is the Lévy measure of a p-tempered α-stable distribution and it uniquely determines R.

3. *A p-tempered α-stable distribution is proper if and only if in addition to (7) and (8) R satisfies*

$$\int_{\mathbb{R}^d} |x|^\alpha R(\mathrm{d}x) < \infty. \tag{9}$$

4. *If R satisfies (9) then in (2) the measure σ is given by*

$$\sigma(B) = \int_{\mathbb{R}^d} \mathbf{1}_B\left(\frac{x}{|x|}\right) |x|^\alpha R(\mathrm{d}x), \qquad B \in \mathfrak{B}(\mathbb{S}^{d-1}).$$

Note that, for all $\alpha < 2$, the conditions in (8) imply the necessity of

$$\int_{\mathbb{R}^d} (|x|^2 \wedge |x|^\alpha) R(\mathrm{d}x) < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} (|x|^2 \wedge 1) R(\mathrm{d}x) < \infty.$$

Before proving Theorem 1, we will translate the integrability conditions on R into integrability conditions on $\{Q_u\}_{u \in \mathbb{S}^{d-1}}$ and σ.

Corollary 1. *Fix $p > 0$, let M be given by (2), and let $\{Q_u\}$ be as in (4). Then M is a Lévy measure if and only if either*

$$Q_u(\mathbb{R}_+) = 0 \quad \sigma\text{-almost everywhere}$$

or $\alpha < 2$ and

$$\int_{\mathbb{S}^{d-1}} \int_0^\infty (t^{-(2-\alpha)/p} \wedge 1) Q_u(\mathrm{d}t) \sigma(\mathrm{d}u) < \infty, \qquad \alpha \in (0, 2),$$

$$\int_{\mathbb{S}^{d-1}} \int_0^\infty (t^{-2/p} \wedge [1 + \log^+(t^{-1/p})]) Q_u(\mathrm{d}t) \sigma(\mathrm{d}u) < \infty, \qquad \alpha = 0,$$

$$\int_{\mathbb{S}^{d-1}} \int_0^\infty (t^{-(2-\alpha)/p} \wedge t^{\alpha/p}) Q_u(\mathrm{d}t) \sigma(\mathrm{d}u) < \infty, \qquad \alpha < 0.$$

https://doi.org/10.1239/jap/1354716655 Published online by Cambridge University Press

Note that these conditions guarantee that, for any $p > 0$ and σ-almost everywhere u, (3) holds and $\int_{\mathbb{R}^d} \mathrm{e}^{-r^p s} Q_u(\mathrm{d}s) < \infty$.

Proof of Theorem 1. We omit most parts of the proof because they are similar to the case when $p = 1$ and $\alpha \in (0, 2)$, which is given in [20]. We only show that, when M is given by (6), it is a Lévy measure if and only if (7) and (8) hold. Assume that $R \neq 0$, since the other case is trivial. We have

$$M(\{0\}) = \int_{\mathbb{R}^d} \int_0^\infty \mathbf{1}_{\{0\}}(tx) t^{-\alpha-1} \mathrm{e}^{-t^p} \mathrm{d}t\, R(\mathrm{d}x) = \int_{\{0\}} \int_0^\infty t^{-1-\alpha} \mathrm{e}^{-t^p} \mathrm{d}t\, R(\mathrm{d}x),$$

which equals 0 if and only if $R(\{0\}) = 0$.

Now assume that $\int (|x|^2 \wedge 1) M(\mathrm{d}x) < \infty$. For any $\varepsilon > 0$,

$$\begin{aligned}
\infty &> \int_{|x|\le 1} |x|^2 M(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d} |x|^2 \int_0^{|x|^{-1}} t^{1-\alpha} \mathrm{e}^{-t^p} \mathrm{d}t\, R(\mathrm{d}x) \\
&\ge \int_{|x|\le 1/\varepsilon} |x|^2 \int_0^{\varepsilon} t^{1-\alpha} \mathrm{e}^{-t^p} \mathrm{d}t\, R(\mathrm{d}x) \\
&\ge \mathrm{e}^{-\varepsilon^p} \int_{|x|\le 1/\varepsilon} |x|^2 \int_0^{\varepsilon} t^{1-\alpha} \mathrm{d}t\, R(\mathrm{d}x).
\end{aligned}$$

Since $R \neq 0$, for this to be finite for all $\varepsilon > 0$, it is necessary that $\alpha < 2$. Taking $\varepsilon = 1$ gives the necessity of $\int_{|x|\le 1} |x|^2 R(\mathrm{d}x) < \infty$. Observe that

$$\begin{aligned}
\infty &> \int_{|x|\ge 1} M(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d} \int_{|x|^{-1}}^\infty t^{-1-\alpha} \mathrm{e}^{-t^p} \mathrm{d}t\, R(\mathrm{d}x) \\
&\ge \int_{|x|\ge 1} \int_{|x|^{-1}}^\infty t^{-1-\alpha} \mathrm{e}^{-t^p} \mathrm{d}t\, R(\mathrm{d}x) \\
&\ge \int_1^\infty t^{-1-\alpha} \mathrm{e}^{-t^p} \mathrm{d}t \int_{|x|\ge 1} R(\mathrm{d}x) + \mathrm{e}^{-1} \int_{|x|\ge 1} \int_{|x|^{-1}}^1 t^{-1-\alpha} \mathrm{d}t\, R(\mathrm{d}x).
\end{aligned}$$

This implies the necessity of $\int_{|x|\ge 1} R(\mathrm{d}x) < \infty$ and $\int_{|x|\ge 1} \int_{|x|^{-1}}^1 t^{-1-\alpha} \mathrm{d}t\, R(\mathrm{d}x) < \infty$. When $\alpha < 0$, we are done. When $\alpha = 0$, this implies the finiteness of $\int_{|x|\ge 1} \log |x| R(\mathrm{d}x)$, and when $\alpha \in (0, 2)$, it implies the finiteness of $\int_{|x|\ge 1} |x|^\alpha R(\mathrm{d}x)$. Thus, (7) and (8) hold.

Now assume that (7) and (8) hold. We have

$$\begin{aligned}
\int_{|x|\le 1} |x|^2 M(\mathrm{d}x) &= \int_{\mathbb{R}^d} |x|^2 \int_0^{|x|^{-1}} t^{1-\alpha} \mathrm{e}^{-t^p} \mathrm{d}t\, R(\mathrm{d}x) \\
&\le \int_{|x|\le 1} |x|^2 R(\mathrm{d}x) \int_0^\infty t^{1-\alpha} \mathrm{e}^{-t^p} \mathrm{d}t + \int_{|x|>1} |x|^2 \int_0^{|x|^{-1}} t^{1-\alpha} \mathrm{d}t\, R(\mathrm{d}x) \\
&= p^{-1} \Gamma\Big(\frac{2-\alpha}{p}\Big) \int_{|x|\le 1} |x|^2 R(\mathrm{d}x) + (2-\alpha)^{-1} \int_{|x|>1} |x|^\alpha R(\mathrm{d}x) \\
&< \infty.
\end{aligned}$$

Let $D = \sup_{t\ge 1} t^{2-\alpha}\mathrm{e}^{-t^p}$. We have

$$\int_{|x|\ge 1} M(\mathrm{d}x) = \int_{\mathbb{R}^d}\int_{|x|^{-1}}^{\infty} t^{-1-\alpha}\mathrm{e}^{-t^p}\mathrm{d}t\,R(\mathrm{d}x)$$
$$\le D\int_{|x|\le 1}\int_{|x|^{-1}}^{\infty} t^{-3}\mathrm{d}t\,R(\mathrm{d}x) + \int_{|x|>1}\int_{|x|^{-1}}^{\infty} t^{-1-\alpha}\mathrm{e}^{-t^p}\mathrm{d}t\,R(\mathrm{d}x).$$

The first integral in the above equals $0.5D\int_{|x|\le 1}|x|^2R(\mathrm{d}x)$, which is assumed finite. The second integral can be written as

$$\int_{|x|>1}\int_{|x|^{-1}}^{1} t^{-1-\alpha}\mathrm{e}^{-t^p}\mathrm{d}t\,R(\mathrm{d}x) + \int_{1}^{\infty} t^{-1-\alpha}\mathrm{e}^{-t^p}\mathrm{d}t\int_{|x|>1}R(\mathrm{d}x).$$

Of these, the second integral is finite since $\int_{|x|>1}R(\mathrm{d}x) < \infty$. The first is bounded by $\int_{|x|>1}(|x|^\alpha - 1)R(\mathrm{d}x)/\alpha$ when $\alpha \ne 0$ and by $\int_{|x|>1}\log|x|R(\mathrm{d}x)$ when $\alpha = 0$. The fact that both of these are assumed to be finite gives the result.

Definition 2. The unique measure in (5) is called the *Rosiński measure* of the corresponding p-tempered α-stable distribution.

Remark 5. For $\alpha \in (0, 2)$ and $p = 1$, the Rosiński measure was called the spectral measure in [20]. For $\alpha \in [0, 2)$ and $p = 2$, the Rosiński measure was introduced in a slightly different parametrization in [6].

Remark 6. Fix $\alpha < 2$ and $p > 0$, and let $\mu \in \mathrm{TS}^p_\alpha$ with Rosiński measure R. Then $\mu = \mathrm{ID}(0, M, b)$ for some $b \in \mathbb{R}^d$ and M uniquely determined by R. We write $\mathrm{TS}^p_\alpha(R, b)$ to denote this distribution.

Theorem 1 shows that, for a fixed $p > 0$ and $\alpha < 2$, the Rosiński measure is uniquely determined by the Lévy measure. This leaves the question of whether all of the parameters are jointly identifiable. Unfortunately, this is not the case. As we will show below, even for a fixed $p > 0$ the parameters α and R are not jointly identifiable. However, using ideas similar to those in [20], we will show that, for a fixed $p > 0$, in the subclass of proper tempered stable distributions, they are jointly identifiable. On the other hand, for a fixed $\alpha < 2$, even in the subclass of proper tempered stable distributions, the parameters p and R are not jointly identifiable. We begin with the following lemma.

Lemma 1. *Fix $\alpha < 2$ and $p > 0$, and let M be the Lévy measure of a p-tempered α-stable distribution with Rosiński measure $R \ne 0$.*

1. *The map $s \mapsto s^\alpha M(|x| > s)$ is decreasing and $\lim_{s\to\infty} s^\alpha M(|x| > s) = 0$.*
2. *If $\alpha \in (0, 2)$ then*
$$\lim_{s\downarrow 0} s^\alpha M(|x| > s) = \frac{1}{\alpha}\int_{\mathbb{R}^d}|x|^\alpha R(\mathrm{d}x)$$
and if $\alpha \le 0$ then
$$\lim_{s\downarrow 0} s^\alpha M(|x| > s) = \infty.$$
3. *If $\alpha < 0$ then*
$$\lim_{s\downarrow 0} s^\alpha M(|x| < s) = \frac{1}{|\alpha|}\int_{\mathbb{R}^d}|x|^\alpha R(\mathrm{d}x)$$

https://doi.org/10.1239/jap/1354716655 Published online by Cambridge University Press

and if $\alpha \in [0, 2)$ *then, for all* $s > 0$,

$$M(|x| < s) = \infty.$$

Lemma 1 extends Corollary 2.5 of [20]. Note that it implies that, in the subclass of proper tempered stable distributions, both $\lim_{s\downarrow 0} s^\alpha M(|x| > s) = \infty$ and $M(|x| < s) = \infty$ if and only if $\alpha = 0$.

Proof of Lemma 1. We begin with the first part. Since

$$\begin{aligned} s^\alpha M(|x| > s) &= s^\alpha \int_{\mathbb{R}^d} \int_{s|x|^{-1}}^\infty t^{-1-\alpha} \mathrm{e}^{-t^p} \mathrm{d}t\, R(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \int_{|x|^{-1}}^\infty t^{-1-\alpha} \mathrm{e}^{-(st)^p} \mathrm{d}t\, R(\mathrm{d}x), \end{aligned} \tag{10}$$

the map $s \mapsto s^\alpha M(|x| > s)$ is decreasing. For large enough s, the integrand in (10) is bounded by $t^{-1-\alpha}\mathrm{e}^{-t^p}$, which is integrable. Thus, by dominated convergence,

$$\lim_{s\to\infty} s^\alpha M(|x| > s) = 0.$$

For the second part, by (10) and the monotone convergence theorem,

$$\lim_{s\downarrow 0} s^\alpha M(|x| > s) = \int_{\mathbb{R}^d} \int_{|x|^{-1}}^\infty t^{-1-\alpha} \mathrm{d}t\, R(\mathrm{d}x).$$

Thus, if $\alpha \in (0, 2)$ then

$$\lim_{s\downarrow 0} s^\alpha M(|x| > s) = \frac{1}{\alpha} \int_{\mathbb{R}^d} |x|^\alpha R(\mathrm{d}x),$$

and if $\alpha \le 0$ then

$$\lim_{s\downarrow 0} s^\alpha M(|x| > s) = \infty.$$

Now for the third part. If $\alpha \in [0, 2)$ then, for all $s > 0$,

$$\begin{aligned} M(|x| < s) &= \int_{\mathbb{R}^d} \int_0^{s|x|^{-1}} t^{-1-\alpha} \mathrm{e}^{-t^p} \mathrm{d}t\, R(\mathrm{d}x) \\ &\ge \int_{\mathbb{R}^d} \mathrm{e}^{-(s/|x|)^p} \int_0^{s|x|^{-1}} t^{-1-\alpha} \mathrm{d}t\, R(\mathrm{d}x) \\ &= \infty. \end{aligned}$$

If $\alpha < 0$ then

$$\begin{aligned} \lim_{s\downarrow 0} s^\alpha M(|x| < s) &= \lim_{s\downarrow 0} s^\alpha \int_{\mathbb{R}^d} \int_0^{s|x|^{-1}} t^{-1-\alpha} \mathrm{e}^{-t^p} \mathrm{d}t\, R(\mathrm{d}x) \\ &= \lim_{s\downarrow 0} \int_{\mathbb{R}^d} \int_0^{|x|^{-1}} t^{-1-\alpha} \mathrm{e}^{-(st)^p} \mathrm{d}t\, R(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \int_0^{|x|^{-1}} t^{-1-\alpha} \mathrm{d}t\, R(\mathrm{d}x) \\ &= \frac{1}{|\alpha|} \int_{\mathbb{R}^d} |x|^\alpha R(\mathrm{d}x), \end{aligned}$$

where the third line follows by the monotone convergence theorem.

Combining Lemma 1 with (9) gives the following.

Proposition 1. *In the subclass of proper tempered stable distributions with parameter $p > 0$ fixed, the parameters R and α are jointly identifiable.*

However, in general, the parameters α and p are not identifiable. This will become apparent from the following results.

Proposition 2. *Fix $\alpha < 2$ and $\beta \in (\alpha, 2)$, and let $K = \int_0^\infty s^{\beta-\alpha-1}\mathrm{e}^{-s^p}\mathrm{d}s$. If $\mu = \mathrm{TS}_\beta^p(R, b)$ and*

$$R'(A) = K^{-1}\int_{\mathbb{R}^d}\int_0^1 \mathbf{1}_A(ux)u^{-\beta-1}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u\,R(\mathrm{d}x),$$

then R' is the Rosiński measure of a p-tempered α-stable distribution and $\mu = \mathrm{TS}_\alpha^p(R', b)$.

Proof. First we will show that R' is the Rosiński measure of some p-tempered α-stable distribution. Let $C = \max_{u\in[0,0.5]}(1-u^p)^{(\beta-\alpha)/p-1}$. We have

$$\begin{aligned}
K\int_{|x|\le 1}|x|^2R'(\mathrm{d}x) &= \int_{\mathbb{R}^d}|x|^2\int_0^{1\wedge|x|^{-1}}u^{1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u\,R(\mathrm{d}x)\\
&\le \int_{|x|\le 2}|x|^2R(\mathrm{d}x)\int_0^1 u^{1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u\\
&\quad + C\int_{|x|>2}|x|^2\int_0^{|x|^{-1}}u^{1-\beta}\mathrm{d}u\,R(\mathrm{d}x)\\
&= \int_{|x|\le 2}|x|^2R(\mathrm{d}x)\int_0^1 u^{1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u\\
&\quad + \frac{C}{2-\beta}\int_{|x|\ge 2}|x|^\beta R(\mathrm{d}x)\\
&< \infty.
\end{aligned}$$

If $\alpha \in (0, 2)$ then

$$\begin{aligned}
K\int_{|x|>2}|x|^\alpha R'(\mathrm{d}x) &= \int_{|x|\ge 2}|x|^\alpha\int_{|x|^{-1}}^{1/2}u^{\alpha-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u\,R(\mathrm{d}x)\\
&\quad + \int_{|x|\ge 2}|x|^\alpha\int_{1/2}^1 u^{\alpha-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u\,R(\mathrm{d}x)\\
&\le C\int_{|x|\ge 2}|x|^\alpha\int_{|x|^{-1}}^\infty u^{\alpha-1-\beta}\mathrm{d}u\,R(\mathrm{d}x)\\
&\quad + \int_{|x|>2}|x|^\beta R(\mathrm{d}x)\int_{1/2}^1 u^{\alpha-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u.
\end{aligned}$$

Here the first integral equals $(C/(\beta-\alpha))\int_{|x|\ge 2}|x|^\beta R(\mathrm{d}x) < \infty$ and the second is also finite. Now assume that $\alpha = 0$ and fix $\varepsilon \in (0, \beta)$. By Equation 4.1.37 of [1], there exists a $C_\varepsilon > 0$ such that, for all $u > 0$, $\log u \le C_\varepsilon u^\varepsilon$. Thus,

$$K\int_{|x|>2}\log|x|R'(\mathrm{d}x) \le KC_\varepsilon\int_{|x|>2}|x|^\varepsilon R'(\mathrm{d}x),$$

which is finite by arguments similar to the previous case. When $\alpha < 0$,

$$
\begin{aligned}
K\int_{|x|>2} R'(\mathrm{d}x) &= \int_{|x|>2}\int_{|x|^{-1}}^{1} u^{-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u\, R(\mathrm{d}x)\\
&\le C\int_{|x|>2}\int_{|x|^{-1}}^{1} u^{-1-\beta}\mathrm{d}u\, R(\mathrm{d}x)\\
&\quad + \int_{|x|>2} R(\mathrm{d}x)\int_{1/2}^{1} u^{-1-\beta}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u.
\end{aligned}
$$

Here the second integral is finite. For $\beta \neq 0$, the first equals $(C/\beta)\int_{|x|>2}(|x|^\beta - 1)R(\mathrm{d}x)$ which is finite, and, for $\beta = 0$, it equals $\int_{|x|>2}\log|x|R(\mathrm{d}x) < \infty$.

Now, let M' be the Lévy measure of $\mathrm{TS}^p_\alpha(R', b)$. By (6),

$$
\begin{aligned}
M'(A) &= K^{-1}\int_{\mathbb{R}^d}\int_0^\infty\int_0^1 \mathbf{1}_A(utx)t^{-1-\alpha}\mathrm{e}^{-t^p}u^{-\beta-1}(1-u^p)^{(\beta-\alpha)/p-1}\mathrm{d}u\,\mathrm{d}t\,R(\mathrm{d}x)\\
&= K^{-1}\int_{\mathbb{R}^d}\int_0^\infty\int_0^t \mathbf{1}_A(vx)t^{\beta-\alpha-1}\mathrm{e}^{-t^p}v^{-\beta-1}\left(1-\frac{v^p}{t^p}\right)^{(\beta-\alpha)/p-1}\mathrm{d}v\,\mathrm{d}t\,R(\mathrm{d}x)\\
&= K^{-1}\int_{\mathbb{R}^d}\int_0^\infty\int_v^\infty \mathbf{1}_A(vx)t^{p-1}\mathrm{e}^{-t^p}v^{-\beta-1}(t^p-v^p)^{(\beta-\alpha)/p-1}\mathrm{d}t\,\mathrm{d}v\,R(\mathrm{d}x)\\
&= K^{-1}\int_{\mathbb{R}^d}\int_0^\infty \mathbf{1}_A(vx)\mathrm{e}^{-v^p}v^{-\beta-1}\mathrm{d}v\,R(\mathrm{d}x)\int_0^\infty \mathrm{e}^{-s^p}s^{\beta-\alpha-1}\mathrm{d}s\\
&= \int_{\mathbb{R}^d}\int_0^\infty \mathbf{1}_A(vx)\mathrm{e}^{-v^p}v^{-\beta-1}\mathrm{d}v\,R(\mathrm{d}x),
\end{aligned}
$$

where the second line follows by the substitution $v = ut$ and the fourth by the substitution $s^p = t^p - v^p$.

To show a similar result for the parameter p, we need some additional notation. For $r \in (0,1)$, let f_r be the density of the r-stable distribution with

$$\int_0^\infty \mathrm{e}^{-tx}f_r(x)\,\mathrm{d}x = \mathrm{e}^{-t^r}.$$

Such a density exists by Proposition 1.2.12 of [23]. However, the only case where an explicit formula is known is

$$f_{0.5}(s) = (2\sqrt{\pi})^{-1}\mathrm{e}^{-1/(4s)}s^{-3/2}\,\mathbf{1}_{[s>0]}$$

(see Examples 2.13 and 8.11 of [25]). From Theorem 5.4.1 of [27], it follows that if $X \sim f_r$ and $\beta \ge 0$, then

$$\mathrm{E}\,|X|^{-\beta} < \infty.$$

Proposition 3. *Fix $\alpha < 2$ and $0 < p < q$. If $\mu = \mathrm{TS}^p_\alpha(R, b)$ and*

$$R'(A) = \int_{\mathbb{R}^d}\int_0^\infty \mathbf{1}_A(s^{-1/q}x)s^{\alpha/q}f_{p/q}(s)\,\mathrm{d}s\,R(\mathrm{d}x),$$

then R' is the Rosiński measure of a q-tempered α-stable distribution and $\mu = \mathrm{TS}^q_\alpha(R', b)$. Moreover, μ is a proper p-tempered α-stable distribution if and only if it is a proper q-tempered α-stable distribution.

https://doi.org/10.1239/jap/1354716655 Published online by Cambridge University Press

This implies that, for a fixed α, the parameters p and R are not jointly identifiable even within the subclass of proper tempered stable distributions.

Proof of Proposition 3. First we show that R' is, in fact, the Rosiński measure of a q-tempered α-stable distribution. We have

$$\begin{aligned}
\int_{|x|\le 1} |x|^2 R'(\mathrm{d}x) &= \int_{\mathbb{R}^d} |x|^2 \int_{|x|^q}^{\infty} s^{-(2-\alpha)/q} f_{p/q}(s)\,\mathrm{d}s\, R(\mathrm{d}x) \\
&\le \int_{|x|\le 1} |x|^2 \int_0^{\infty} s^{-(2-\alpha)/q} f_{p/q}(s)\,\mathrm{d}s\, R(\mathrm{d}x) \\
&\quad + \int_{|x|>1} |x|^{\alpha} R(\mathrm{d}x) \int_0^{\infty} f_{p/q}(s)\,\mathrm{d}s \\
&< \infty.
\end{aligned}$$

If $\alpha \neq 0$ and $\beta = \alpha \vee 0$, then

$$\begin{aligned}
\int_{|x|>1} |x|^{\beta} R'(\mathrm{d}x) &= \int_{\mathbb{R}^d} |x|^{\beta} \int_0^{|x|^q} s^{-(\beta-\alpha)/q} f_{p/q}(s)\,\mathrm{d}s\, R(\mathrm{d}x) \\
&\le \int_{|x|\le 1} |x|^2 \int_0^{\infty} s^{-(2-\alpha)/q} f_{p/q}(s)\,\mathrm{d}s\, R(\mathrm{d}x) \\
&\quad + \int_{|x|>1} |x|^{\beta} \int_0^{\infty} s^{-(\beta-\alpha)/q} f_{p/q}(s)\,\mathrm{d}s\, R(\mathrm{d}x) \\
&< \infty.
\end{aligned}$$

If $\alpha = 0$ then

$$\begin{aligned}
\int_{|x|>1} \log|x| R'(\mathrm{d}x) &= \int_{\mathbb{R}^d} \int_0^{|x|^q} \log|x s^{-1/q}| f_{p/q}(s)\,\mathrm{d}s\, R(\mathrm{d}x) \\
&\le \int_{|x|\le 1} |x|^2 R(\mathrm{d}x) \int_0^{\infty} s^{-2/q} f_{p/q}(s)\,\mathrm{d}s \\
&\quad + \int_{|x|>1} \log|x| R(\mathrm{d}x) \int_0^{\infty} f_{p/q}(s)\,\mathrm{d}s \\
&\quad + \int_{|x|>1} R(\mathrm{d}x) \int_0^{\infty} s^{-1/q} f_{p/q}(s)\,\mathrm{d}s \\
&< \infty,
\end{aligned}$$

where the inequality uses the fact that $\log|x| \le |x|$ (see Equation 4.1.36 of [1]).

If M' is the Lévy measure of $\mathrm{TS}^q_\alpha(R', b)$ then, by (6),

$$\begin{aligned}
M'(A) &= \int_{\mathbb{R}^d} \int_0^{\infty} \int_0^{\infty} \mathbf{1}_A(s^{-1/q} t x) t^{-1-\alpha} \mathrm{e}^{-t^q}\,\mathrm{d}t\, s^{\alpha/q} f_{p/q}(s)\,\mathrm{d}s\, R(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d} \int_0^{\infty} \mathbf{1}_A(vx) v^{-1-\alpha} \int_0^{\infty} \mathrm{e}^{-v^q s} f_{p/q}(s)\,\mathrm{d}s\,\mathrm{d}v\, R(\mathrm{d}x) \\
&= \int_{\mathbb{R}^d} \int_0^{\infty} \mathbf{1}_A(vx) v^{-1-\alpha} \mathrm{e}^{-v^p}\,\mathrm{d}v\, R(\mathrm{d}x).
\end{aligned}$$

The last part follows from (9) and the fact that

$$\int_{\mathbb{R}^d} |x|^\alpha R'(\mathrm{d}x) = \int_{\mathbb{R}^d} |x|^\alpha R(\mathrm{d}x) \int_0^\infty s^{-\alpha/q} s^{\alpha/q} f_{p/q}(s)\,\mathrm{d}s = \int_{\mathbb{R}^d} |x|^\alpha R(\mathrm{d}x).$$

Propositions 2 and 3 give a constructive proof of the following result, a version of which was shown in [16].

Corollary 2. *Fix* $\alpha < 2$ *and* $p > 0$*, and let* $\mu \in \mathrm{TS}^p_\alpha$.

1. *For any* $q \geq p$, $\mu \in \mathrm{TS}^q_\alpha$.
2. *For any* $\beta \leq \alpha$, $\mu \in \mathrm{TS}^p_\beta$.

We end this section by characterizing when a p-tempered α-stable distribution is β-stable for some $\beta \in (0, 2)$.

Proposition 4. *Fix* $\alpha < 2$, $p > 0$, *and* $\beta \in (0, 2)$. *Let* μ *be a* β*-stable distribution with spectral measure* $\sigma \neq 0$. *If* $\beta \leq \alpha$ *then* $\mu \notin \mathrm{TS}^p_\alpha$. *If* $\beta \in (0 \vee \alpha, 2)$ *then* $\mu = \mathrm{TS}^p_\alpha(R, b)$ *and*

$$R(A) = K^{-1} \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}_A(r\xi) r^{-1-\beta} \mathrm{d}r \sigma(\mathrm{d}\xi), \qquad A \in \mathfrak{B}(\mathbb{R}^d), \tag{11}$$

where $K = \int_0^\infty t^{\beta-\alpha-1} \mathrm{e}^{-t^p} \mathrm{d}t$.

Note that

$$\int_{\mathbb{R}^d} |x|^\alpha R(\mathrm{d}x) = K^{-1} \sigma(\mathbb{S}^{d-1}) \int_0^\infty r^{-(\beta-\alpha)-1} \mathrm{d}r = \infty.$$

Thus, by part 3 of Theorem 1, no stable distributions are proper p-tempered α-stable.

Proof of Proposition 4. If $\mu \in \mathrm{TS}^p_\alpha$ then its Lévy measure can be written as (2). By uniqueness of the polar decomposition of Lévy measures (see Lemma 2.1 of [4]), the function $q(r, u) = r^{(\alpha-\beta)/p}$. This is not completely monotone when $\beta < \alpha$, and it does not satisfy (3) when $\beta = \alpha$.

Now assume that $\beta > \alpha$ and let R be as in (11). In this case $R(\{0\}) = 0$ and, for any $\gamma \in [0, \beta)$,

$$\int_{\mathbb{R}^d} (|x|^2 \wedge |x|^\gamma) R(\mathrm{d}x) = K^{-1} \sigma(\mathbb{S}^{d-1}) \int_0^\infty (r^{1-\beta} \wedge r^{\gamma-\beta-1})\,\mathrm{d}r < \infty.$$

Thus, by Theorem 1, R is the Rosiński measure of a p-tempered α-stable distribution. If M is the Lévy measure of $\mathrm{TS}^p_\alpha(R, b)$ then

$$\begin{aligned}
M(A) &= K^{-1} \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^\infty \mathbf{1}_A(rt\xi) t^{-1-\alpha} \mathrm{e}^{-t^p} \mathrm{d}t r^{-1-\beta} \mathrm{d}r \sigma(\mathrm{d}\xi) \\
&= K^{-1} \int_0^\infty t^{\beta-\alpha-1} \mathrm{e}^{-t^p} \mathrm{d}t \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}_A(r\xi) r^{-1-\beta} \mathrm{d}r \sigma(\mathrm{d}\xi) \\
&= \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}_A(r\xi) r^{-1-\beta} \mathrm{d}r \sigma(\mathrm{d}\xi),
\end{aligned}$$

which is the Lévy measure of μ.

3. Moments

In this section we give necessary and sufficient conditions for the finiteness of moments and exponential moments. We also give explicit formulae for the cumulants when they exist. This is useful, for instance, in parameter estimation by the method of moments. First we introduce some notation. For any $x \in \mathbb{R}^d$, let x_i be the ith component. For simplicity, throughout this section, we will use M to denote the Lévy measure of a p-tempered α-stable distribution.

Let k be a d-dimensional vector of nonnegative integers. Let C_μ be as in (1). Recall that we define the cumulant

$$c_k = (-i)^{\sum k_i} \frac{\partial^{\sum k_i}}{\partial z_d^{k_d} \cdots \partial z_1^{k_1}} C_\mu(z) \bigg|_{z=0},$$

when the derivative exists and is continuous in a neighborhood of 0. Cumulants can be uniquely expressed in terms of moments. Let $X \sim \mu$. When $k_i = 1$ and $k_j = 0$ for all $j \neq i$, then $c_k = \mathrm{E}\, X_i$; when $k_i = 2$ and $k_j = 0$ for all $j \neq i$, then $c_k = \mathrm{var}(X_i)$; and when, for some $i \neq j$, $k_i = k_j = 1$ and $k_\ell = 0$ for all $\ell \neq i, j$, then $c_k = \mathrm{cov}(X_i, X_j)$. In the statement of the following theorem, we adopt the convention that $0^0 = 1$.

Theorem 2. *Fix $\alpha < 2$ and $p > 0$, and let $\mu = \mathrm{TS}_\alpha^p(R, b)$.*

1. *If $\alpha \in (0, 2)$ and $q_1, \dots, q_d \geq 0$ with $q := \sum_{j=1}^d q_j < \alpha$, then*

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^d |x_j|^{q_j} \right) \mu(\mathrm{d}x) \leq \int_{\mathbb{R}^d} |x|^q \mu(\mathrm{d}x) < \infty. \tag{12}$$

2. *If $\alpha \in (0, 2)$ then*

$$\int_{\mathbb{R}^d} |x|^\alpha \mu(\mathrm{d}x) < \infty \quad \Longleftrightarrow \quad \int_{|x|>1} |x|^\alpha \log |x| R(\mathrm{d}x) < \infty. \tag{13}$$

Additionally, if $q_1, \dots, q_d \geq 0$ with $\sum_{j=1}^d q_j = \alpha$ then

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^d |x_j|^{q_j} \right) \mu(\mathrm{d}x) < \infty$$

if and only if

$$\int_{|x|>1} \left(\prod_{j=1}^d |x_j|^{q_j} \right) \log |x| R(\mathrm{d}x) < \infty. \tag{14}$$

3. *If $q > (\alpha \vee 0)$ then*

$$\int_{\mathbb{R}^d} |x|^q \mu(\mathrm{d}x) < \infty \quad \Longleftrightarrow \quad \int_{|x|>1} |x|^q R(\mathrm{d}x) < \infty.$$

Additionally, if $q_1, \dots, q_d \geq 0$ with $\sum_{j=1}^d q_j > (\alpha \vee 0)$ then

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^d |x_j|^{r_j} \right) \mu(\mathrm{d}x) < \infty \quad \text{for all } r_k \in [0, q_k],\ k = 1, \dots, d,$$

if and only if

$$\int_{|x|>1}\left(\prod_{j=1}^d |x_j|^{r_j}\right) R(\mathrm{d}x) < \infty \quad \textit{for all } r_k \in [0, q_k],\ k = 1, \ldots, d. \tag{15}$$

4. *Let $q_1, \ldots, q_d$ be nonnegative integers, and let $q = \sum_{i=1}^d q_i$. Furthermore, if $q = \alpha$, assume that (14) holds and, if $q > \alpha$, assume that (15) holds. If $q_i = q = 1$ for some i then*

$$c_{(q_1,\ldots,q_d)} = b_i + \int_{\mathbb{R}^d}\int_0^\infty x_i \frac{|x|^2}{1+|x|^2t^2} t^{2-\alpha}\mathrm{e}^{-t^p}\,\mathrm{d}t\,R(\mathrm{d}x).$$

If $q \geq 2$ then

$$c_{(q_1,\ldots,q_d)} = p^{-1}\Gamma\left(\frac{n-\alpha}{p}\right)\int_{\mathbb{R}^d}\left(\prod_{j=1}^d x_j^{q_j}\right)R(\mathrm{d}x).$$

For proper 1-tempered α-stable distributions with $\alpha \in (0, 2)$, a somewhat weaker version of part 4 above was given in [26].

Proof of Theorem 2. By Corollary 25.8 of [25], the condition $\int_{\mathbb{R}^d}|x|^q\mu(\mathrm{d}x) < \infty$ is equivalent to the condition $\int_{|x|>1}|x|^q M(\mathrm{d}x) < \infty$. Similarly, by Theorem 1 of [24], the condition $\int_{\mathbb{R}^d}(\prod_{j=1}^d |x_j|^{r_j})\mu(\mathrm{d}x) < \infty$ for all $r_k \in [0, q_k],\ k = 1, \ldots, d$, is equivalent to the condition $\int_{|x|>1}(\prod_{j=1}^d |x_j|^{r_j})M(\mathrm{d}x) < \infty$ for all $r_k \in [0, q_k],\ k = 1, \ldots, d$.

We will now transfer the integrability conditions from M to R. Let $f_q(x)$ be either $|x|^q$ or $\prod_{j=1}^d |x_j|^{r_j}$, where $\sum_{j=1}^d r_j = q$. By (6),

$$\int_{|x|>1} f_q(x)M(\mathrm{d}x) = \int_{\mathbb{R}^d}\int_{|x|^{-1}}^\infty f_q(x)t^{q-1-\alpha}\mathrm{e}^{-t^p}\,\mathrm{d}t\,R(\mathrm{d}x).$$

From here, (13) and parts 1 and 3 follow by arguments similar to those in Proposition 2.7 of [20]. The second half of part 2 follows, essentially, from arguments similar to those in Proposition 2.7 of [20] as well, but to guarantee that the integral remains finite for all $r_k \in [0, q_k)$, we use (12).

For general infinitely divisible distributions, the form of the cumulants in terms of the Lévy measure is given in Theorems 5.1 and 5.2 of [12]. From this, part 4 follows by using (6) and simplifying.

In the rest of this section we will give conditions for the finiteness of certain exponential moments.

Theorem 3. *Fix $\alpha < 2$, $p \in (0, 1]$, and $\theta > 0$. Let $\mu = \mathrm{TS}^p_\alpha(R, b)$.*

1. *If $\alpha \in (0, 2)$ then*

$$\int_{\mathbb{R}^d}\mathrm{e}^{\theta|x|^p}\mu(\mathrm{d}x) < \infty \iff R(\{|x| > \theta^{-1/p}\}) = 0.$$

2. *If $\alpha < 0$ then $\int_{\mathbb{R}^d}\mathrm{e}^{\theta|x|^p}\mu(\mathrm{d}x) < \infty$ if and only if*

$$R(\{|x| \geq \theta^{-1/p}\}) = 0 \quad \textit{and} \quad \int_{0<|x|^{-p}-\theta<1}(|x|^{-p}-\theta)^{\alpha/p}R(\mathrm{d}x) < \infty.$$

https://doi.org/10.1239/jap/1354716655 Published online by Cambridge University Press

3. *If* $\alpha = 0$ *then* $\int_{\mathbb{R}^d} \mathrm{e}^{\theta|x|^p}\mu(\mathrm{d}x) < \infty$ *if and only if*

$$R(\{|x| \geq \theta^{-1/p}\}) = 0 \quad \textit{and} \quad \int_{0<|x|^{-p}-\theta<1} |\log(|x|^{-p}-\theta)| R(\mathrm{d}x) < \infty.$$

This implies that, unless $R = 0$, it is impossible to have $\int_{\mathbb{R}^d} \mathrm{e}^{\theta|x|^p}\mu(\mathrm{d}x) < \infty$ for all $\theta > 0$. Note that in parts 2 and 3 of Theorem 3 we have the condition $R(\{|x| \geq \theta^{-1/p}\}) = 0$, whereas in part 1 we have a similar condition, but with strict inequality. Note also that the set $\{0 < |x|^{-p} - \theta < 1\} = \{(1+\theta)^{-1/p} < |x| < \theta^{-1/p}\}$. The latter form may be somewhat more appealing, but it loses emphasis on why the integrals may diverge.

Proof of Theorem 3. The proof of part 1 is similar to the proof of Proposition 2.7 of [20]. Now fix $\alpha \leq 0$. By Corollary 25.8 of [25], the finiteness of $\int_{\mathbb{R}^d} \mathrm{e}^{\theta|x|^p}\mu(\mathrm{d}x)$ is equivalent to the finiteness of $\int_{|x|>1} \mathrm{e}^{\theta|x|^p} M(\mathrm{d}x)$. We have

$$\begin{aligned}\int_{|x|>1} \mathrm{e}^{\theta|x|^p} M(\mathrm{d}x) &= \int_{\mathbb{R}^d}\int_{|x|^{-1}}^{\infty} \mathrm{e}^{(\theta|x|^p-1)t^p} t^{-\alpha-1}\mathrm{d}t\, R(\mathrm{d}x)\\ &\geq \int_{|x|^p\geq\theta^{-1}}\int_{\theta^{1/p}}^{\infty} t^{-\alpha-1}\mathrm{d}t\, R(\mathrm{d}x).\end{aligned}$$

This shows the necessity of $R(\{|x| \geq \theta^{-1/p}\}) = 0$ in both parts 2 and 3. We will henceforth assume that this property holds both when showing necessity and sufficiency. We have

$$\begin{aligned}\int_{|x|>1} \mathrm{e}^{\theta|x|^p} M(\mathrm{d}x) &= \int_{|x|<\theta^{-1/p}}\int_{|x|^{-1}}^{\infty} \mathrm{e}^{(\theta|x|^p-1)t^p} t^{-1-\alpha}\mathrm{d}t\, R(\mathrm{d}x)\\ &= p^{-1}\int_{0<|x|^{-p}-\theta} (1-\theta|x|^p)^{\alpha/p}\int_{|x|^{-p}-\theta}^{\infty} \mathrm{e}^{-u}u^{-1-\alpha/p}\mathrm{d}u\, R(\mathrm{d}x).\end{aligned}$$

This can be divided into two parts:

$$\begin{aligned}&p^{-1}\int_{1\leq|x|^{-p}-\theta} (|x|^{-p}-\theta)^{-|\alpha|/p}|x|^{-|\alpha|}\int_{|x|^{-p}-\theta}^{\infty} \mathrm{e}^{-u}u^{-1+|\alpha|/p}\mathrm{d}u\, R(\mathrm{d}x)\\ &+ p^{-1}\int_{0<|x|^{-p}-\theta<1} (|x|^{-p}-\theta)^{-|\alpha|/p}|x|^{-|\alpha|}\int_{|x|^{-p}-\theta}^{\infty} \mathrm{e}^{-u}u^{-1+|\alpha|/p}\mathrm{d}u\, R(\mathrm{d}x)\\ &=: p^{-1}(I_1+I_2).\end{aligned}$$

Let $C_\theta := \sup_{u>1} \mathrm{e}^{-u}u^{-1+|\alpha|/p}(u+\theta)^{(2+|\alpha|)/p+1}$. We have

$$\begin{aligned}I_1 &\leq \int_{1\leq|x|^{-p}-\theta} |x|^{-|\alpha|}\int_{|x|^{-p}-\theta}^{\infty} \mathrm{e}^{-u}u^{-1+|\alpha|/p}\mathrm{d}u\, R(\mathrm{d}x)\\ &\leq C_\theta\int_{1\leq|x|^{-p}-\theta} |x|^{-|\alpha|}\int_{|x|^{-p}-\theta}^{\infty} (u+\theta)^{-(2+|\alpha|)/p-1}\mathrm{d}u\, R(\mathrm{d}x)\\ &= C_\theta\frac{p}{2-\alpha}\int_{|x|\leq(1+\theta)^{-1/p}} |x|^2 R(\mathrm{d}x)\\ &< \infty.\end{aligned}$$

Thus, finiteness is determined by I_2.

https://doi.org/10.1239/jap/1354716655 Published online by Cambridge University Press

If $\alpha < 0$ and $0 < |x|^{-p} - \theta < 1$, we have

$$\int_1^\infty \mathrm{e}^{-u} u^{-1+|\alpha|/p} \mathrm{d}u \le \int_{|x|^{-p}-\theta}^\infty \mathrm{e}^{-u} u^{-1+|\alpha|/p} \mathrm{d}u \le \Gamma\biggl(\frac{|\alpha|}{p}\biggr)$$

and

$$\theta^{|\alpha|/p} \le |x|^{-|\alpha|} \le (1+\theta)^{|\alpha|/p}.$$

Thus, when $\alpha < 0$, I_2 is finite if and only if

$$\int_{0<|x|^{-p}-\theta<1} (|x|^{-p} - \theta)^{-|\alpha|/p} R(\mathrm{d}x) < \infty.$$

If $\alpha = 0$ then, for $0 < |x|^{-p} - \theta < 1$, we have

$$\int_{|x|^{-p}-\theta}^\infty \mathrm{e}^{-u} u^{-1} \mathrm{d}u = \int_1^\infty \mathrm{e}^{-u} u^{-1} \mathrm{d}u + \int_{|x|^{-p}-\theta}^1 \mathrm{e}^{-u} u^{-1} \mathrm{d}u,$$

where the first integral is finite. For the second, we have

$$\int_{|x|^{-p}-\theta}^1 \mathrm{e}^{-u} u^{-1} \mathrm{d}u \le \int_{|x|^{-p}-\theta}^1 u^{-1} \mathrm{d}u = -\log(|x|^{-p} - \theta)$$

and

$$\int_{|x|^{-p}-\theta}^1 \mathrm{e}^{-u} u^{-1} \mathrm{d}u \ge \mathrm{e}^{-1} \int_{|x|^{-p}-\theta}^1 u^{-1} \mathrm{d}u = -\mathrm{e}^{-1} \log(|x|^{-p} - \theta).$$

Thus, when $\alpha = 0$, the finiteness of I_2 is equivalent to the finiteness of

$$-\int_{0<|x|^{-p}-\theta<1} \log(|x|^{-p} - \theta) R(\mathrm{d}x) = \int_{0<|x|^{-p}-\theta<1} |\log(|x|^{-p} - \theta)| R(\mathrm{d}x).$$

This completes the proof.

Theorem 4. *Fix $\alpha < 2$ and $p > 0$, and let $\mu = \mathrm{TS}^p_\alpha(R, b)$.*

1. *If $q \in (0, 1]$ with $q < p$ then, for any $\theta > 0$,*

$$\int_{\mathbb{R}^d} \mathrm{e}^{\theta|x|^q} \mu(\mathrm{d}x) < \infty \tag{16}$$

whenever

$$\int_{|x|>1} \exp\{A_{p,q}(\theta|x|^q)^{p/(p-q)}\} |x|^{-\alpha q/(p-q)} R(\mathrm{d}x) < \infty,$$

where $A_{p,q} = (q/p)^{q/(p-q)}(1 - q/p)$.

2. *If $R \ne 0$ then $\int_{\mathbb{R}^d} \mathrm{e}^{\theta|x|\log|x|} \mu(\mathrm{d}x) = \infty$ for every $\theta > 0$.*

For the case where $\alpha \in (0, 2)$, $p = 2$, and $q = 1$, a necessary and sufficient condition for (16) is given in [6]. Their method of proof is easily extended to the case when $\alpha < 2$ and $p = 2q$. In this case, the necessary and sufficient condition for (16) is

$$\int_{|x|>1} \mathrm{e}^{\theta^2|x|^{2q}/4} |x|^{-q-\alpha} R(\mathrm{d}x) < \infty.$$

https://doi.org/10.1239/jap/1354716655 Published online by Cambridge University Press

Proof of Theorem 4. We begin with part 1. Fix $c = (2p/(\theta q))^{1/p}$. By Corollary 25.8 of [25], the problem is equivalent to the finiteness of

$$\begin{aligned}\int_{|x|>1} e^{\theta|x|^q} M(dx) &= \int_{|x|\le c}\int_{|x|^{-1}}^{\infty} e^{\theta|x|^q t^q - t^p} t^{-1-\alpha} dt\, R(dx)\\ &\quad + \int_{|x|>c}\int_{|x|^{-1}}^{(\theta|x|^q)^{1/(p-q)}} e^{\theta|x|^q t^q - t^p} t^{-1-\alpha} dt\, R(dx)\\ &\quad + \int_{|x|>c}\int_{(\theta|x|^q)^{1/(p-q)}}^{\infty} e^{\theta|x|^q t^q - t^p} t^{-1-\alpha} dt\, R(dx)\\ &=: I_1 + I_2 + I_3.\end{aligned}$$

For the first integral, we have

$$\begin{aligned}I_1 &= \int_{|x|\le c}\int_{|x|^{-1}}^{\infty} e^{\theta|x|^q t^q - t^p} t^{1-\alpha} t^{-2} dt\, R(dx)\\ &\le \int_{|x|\le c} |x|^2 R(dx) \int_{c^{-1}}^{\infty} e^{\theta c^q t^q - t^p} t^{1-\alpha} dt\\ &< \infty.\end{aligned}$$

For the third integral, by the substitution $u = t^{p-q}/(\theta|x|^q)$, we have

$$\begin{aligned}I_3 &= \frac{1}{p-q}\int_{|x|>c} (\theta|x|^q)^{-\alpha/(p-q)} \int_1^{\infty} e^{-(1-1/u)(u\theta|x|^q)^{p/(p-q)}} u^{-1-\alpha/(p-q)} du\, R(dx)\\ &\le \frac{1}{p-q}\int_{|x|>c} (\theta|x|^q)^{-\alpha/(p-q)} R(dx) \int_1^{\infty} e^{-(1-1/u)(u\theta c^q)^{p/(p-q)}} u^{-1-\alpha/(p-q)} du.\end{aligned}$$

Clearly, this is finite for $\alpha \in [0, 2)$. We will show that it is, in fact, always finite when $I_2 < \infty$. To see this, observe that, after the substitution $u = t^{p-q}/(\theta|x|^q)$, we have

$$\begin{aligned}I_2 &= \frac{1}{p-q}\int_{|x|>c} (\theta|x|^q)^{-\alpha/(p-q)} \int_{|x|^{-p}/\theta}^{1} e^{(1/u-1)(u\theta|x|^q)^{p/(p-q)}} u^{-1-\alpha/(p-q)} du\, R(dx)\\ &\ge \frac{1}{p-q}\int_{|x|>c} (\theta|x|^q)^{-\alpha/(p-q)} R(dx) \int_{c^{-p}/\theta}^{1} u^{-1-\alpha/(p-q)} du. \qquad (17)\end{aligned}$$

Thus, everything is determined by I_2.

Note that, as a function of u, $(1/u-1)(u\theta|x|^q)^{p/(p-q)}$ is strictly increasing until $u = q/p$, where it attains a maximum and is then decreasing. Thus,

$$\begin{aligned}&\int_{|x|^{-p}/\theta}^{q/(2p)} e^{(1/u-1)(u\theta|x|^q)^{p/(p-q)}} u^{-1-\alpha/(p-q)} du\\ &\qquad \le e^{(2p/q-1)(\theta|x|^q q/(2p))^{p/(p-q)}} (|x|^p\theta)^{0\vee[1+\alpha/(p-q)]} \qquad (18)\end{aligned}$$

and, for some constant $C > 0$,

$$\int_{q/(2p)}^{1} e^{(1/u-1)(u\theta|x|^q)^{p/(p-q)}} u^{-1-\alpha/(p-q)} du \le C e^{(p/q-1)[qp^{-1}\theta|x|^q]^{p/(p-q)}}. \qquad (19)$$

https://doi.org/10.1239/jap/1354716655 Published online by Cambridge University Press

Note that $(p/q-1)(q/p)^{p/(p-q)} = A_{p,q}$. Observing that the right-hand side of (19) goes to ∞ faster than the right-hand side of (18), and combining this with (17) gives part 1.

Now to show part 2. For any $h > 0$, let $T_h = \{|x| > h\}$. Assume that $R \neq 0$. Since $R(\{0\}) = 0$, there exists a $\varepsilon > 0$ such that $R(T_\varepsilon) > 0$. Thus, for any $h > 0$,

$$\begin{aligned} M(T_h) &= \int_{\mathbb{R}^d} \int_{h|x|^{-1}}^\infty \mathrm{e}^{-t^p} t^{-1-\alpha} \mathrm{d}t\, R(\mathrm{d}x) \\ &\geq \int_{|x|>\varepsilon} \int_{h\varepsilon^{-1}}^\infty \mathrm{e}^{-t^p} t^{-1-\alpha} \mathrm{d}t\, R(\mathrm{d}x) \\ &= R(T_\varepsilon) \int_{h\varepsilon^{-1}}^\infty \mathrm{e}^{-t^p} t^{-1-\alpha} \mathrm{d}t \\ &> 0. \end{aligned}$$

From here, the result follows by Theorem 26.1 of [25].

4. Regular variation

In this section we give necessary and sufficient conditions for tempered stable distributions to have regularly varying tails. To simplify the notation, we adopt the following convention. For $c \in \mathbb{R}$ and real-valued functions f, g with g strictly positive in some neighborhood of ∞, we write $f(t) \sim cg(t)$ as $t \to \infty$ to mean

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = c.$$

We now recall what it means for a measure to have regularly varying tails.

Definition 3. Fix $\varrho \geq 0$. Let R be a Borel measure on $\mathbb{R}^d$ such that, for some $T > 0$,

$$R(\{x \in \mathbb{R}^d : |x| > T\}) < \infty$$

and, for all $s > 0$,

$$R(\{x \in \mathbb{R}^d : |x| > s\}) > 0.$$

We say that R has *regularly varying tails with index* ϱ if there exists a finite Borel measure $\sigma \neq 0$ on $\mathbb{S}^{d-1}$ such that, for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$,

$$\lim_{r\to\infty} \frac{R(|x| > rt : x/|x| \in D)}{R(|x| > r)} = t^{-\varrho} \frac{\sigma(D)}{\sigma(\mathbb{S}^{d-1})}.$$

When this holds, we write $R \in \mathrm{RV}_{-\varrho}(\sigma)$.

Clearly, a measure $R \in \mathrm{RV}_{-\varrho}(\sigma)$ if and only if there exists a slowly varying function ℓ such that, for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$,

$$R\left(|x| > t, \frac{x}{|x|} \in D\right) \sim \sigma(D) t^{-\varrho} \ell(t) \quad \text{as } t \to \infty. \tag{20}$$

It is well known (see, e.g. [5]) that if $R \in \mathrm{RV}_{-\varrho}(\sigma)$ then

$$\int_{|x|\geq T} |x|^\gamma R(\mathrm{d}x) \begin{cases} < \infty & \text{if } \gamma < \varrho, \\ = \infty & \text{if } \gamma > \varrho. \end{cases} \tag{21}$$

Let $\mu = \mathrm{TS}^p_\alpha(R, b)$. If $\alpha \in (0, 2)$ then Theorem 2 implies that $\int_{\mathbb{R}^d} |x|^\varrho \mu(\mathrm{d}x) < \infty$ for all $\varrho \in [0, \alpha)$, and, hence, by (21), μ cannot have regularly varying tails with index $\varrho < \alpha$. However, other tail indices are possible. We will now categorize when μ has regularly varying tails.

Theorem 5. *Fix $\alpha < 2$ and $p > 0$. Let $\mu = \mathrm{TS}^p_\alpha(R, b)$, and let M be the Lévy measure of μ. If $\varrho > \alpha \vee 0$ then*

$$\mu \in \mathrm{RV}_{-\varrho}(\sigma) \iff M \in \mathrm{RV}_{-\varrho}(\sigma) \iff R \in \mathrm{RV}_{-\varrho}(\sigma).$$

Moreover, if $M \in \mathrm{RV}_{-\varrho}(\sigma)$ then, for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$ and $\sigma(D) > 0$,

$$\lim_{r\to\infty} \frac{R(|x| > r,\ x/|x| \in D)}{M(|x| > r,\ x/|x| \in D)} = \frac{p}{\Gamma((\varrho - \alpha)/p)}.$$

Before proving the theorem let us state a useful corollary. Recall that, for $\gamma \in (0, 2)$, a probability measure μ is in the domain of attraction of a γ-stable distribution with spectral measure $\sigma \neq 0$ if and only if $\mu \in \mathrm{RV}_{-\gamma}(\sigma)$. See, e.g. [17] or [22], although they make the additional assumption that the limiting stable distribution is full.

Corollary 3. *Fix $\alpha < 2$ and $p > 0$, and let $\mu = \mathrm{TS}^p_\alpha(R, b)$. If $\sigma \neq 0$ is a finite Borel measure on $\mathbb{S}^{d-1}$ and $\gamma \in (0 \vee \alpha, 2)$, then μ is in the domain of attraction of a γ-stable distribution with spectral measure σ if and only if $R \in \mathrm{RV}_{-\gamma}(\sigma)$.*

In Theorem 5, the relationship between the regular variation of μ and M is well known; see, for example, [14]. A proof of the fact that $R \in \mathrm{RV}_{-\varrho}(\sigma)$ implies that $M \in \mathrm{RV}_{-\varrho}(\sigma)$ can be accomplished using standard tools. However, the other direction requires more complex arguments. For brevity, we use the same approach for both directions.

Let $k\colon (0, \infty) \mapsto \mathbb{R}$ be a Borel function. The *Mellin transform* of k is defined by

$$\hat{k}(z) = \int_0^\infty u^{z-1} k\left(\frac{1}{u}\right) \mathrm{d}u$$

for all $z \in \mathbb{C}$ for which the integral converges. We will need the following result, which combines Theorems 4.4.2 and 4.9.1 of [7].

Lemma 2. *Let $-\infty < \gamma < \rho < \tau < \infty$, $c \in \mathbb{R}$, and let ℓ be a slowly varying function. Assume that k is a continuous and nonnegative function on $(0, \infty)$ such that*

$$\sum_{-\infty<n<\infty} \max\{\mathrm{e}^{-\gamma n}, \mathrm{e}^{-\tau n}\} \sup_{\mathrm{e}^n \le x \le \mathrm{e}^{n+1}} k(x) < \infty$$

and

$$\hat{k}(z) \neq 0 \quad \textit{when } \operatorname{Re} z = \rho.$$

Let U be a monotone, right continuous function on $(0, \infty)$ with

$$\limsup_{r\downarrow 0} \frac{|U(r)|}{r^\gamma} < \infty.$$

Then

$$\int_{(0,\infty)} k\left(\frac{x}{t}\right) \mathrm{d}U(t) \sim c\rho\hat{k}(\rho) x^\rho \ell(x) \quad \textit{as } x \to \infty$$

if and only if

$$U(x) \sim c x^\rho \ell(x) \quad \textit{as } x \to \infty.$$

Let $\mu = \mathrm{TS}^p_\alpha(R, b)$, let M be the Lévy measure of μ, and assume that $\sigma \neq 0$ is a finite Borel measure on $\mathbb{S}^{d-1}$. For all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$, define, for $r > 0$,

$$M_D(r) = M\bigg(|x| > r,\ \frac{x}{|x|} \in D\bigg)$$

and

$$R_D(r) = R\bigg(|x| > r,\ \frac{x}{|x|} \in D\bigg).$$

Note that, for any integrable function $f : \mathbb{R} \to \mathbb{R}$,

$$\int_{x/|x| \in D} f(|x|) R(\mathrm{d}x) = -\int_{(0,\infty)} f(x)\,\mathrm{d}R_D(x). \tag{22}$$

Lemma 3. *If* $\varrho > \alpha \vee 0$ *and* $\ell \in \mathrm{RV}^\infty_0$, *then*

$$M_D(r) \sim \sigma(D) p^{-1} \Gamma\bigg(\frac{\varrho - \alpha}{p}\bigg) r^{-\varrho} \ell(r) \quad \textit{as } r \to \infty$$

if and only if

$$R_D(r) \sim \sigma(D) r^{-\varrho} \ell(r) \quad \textit{as } r \to \infty.$$

Proof. For simplicity, let $\beta = \alpha \vee 0$. Note that, by (6) and (22),

$$\begin{aligned} M_D(r) &= \int_{x/|x| \in D} \int_{r|x|^{-1}}^{\infty} t^{-1-\alpha} \mathrm{e}^{-t^p}\,\mathrm{d}t\, R(\mathrm{d}x) \\ &= -\int_{(0,\infty)} \int_{r/x}^{\infty} t^{-1-\alpha} \mathrm{e}^{-t^p}\,\mathrm{d}t\,\mathrm{d}R_D(x) \\ &= -\int_{(0,\infty)} k\bigg(\frac{r}{x}\bigg) R_D(\mathrm{d}x), \end{aligned}$$

where

$$k(s) = \int_s^\infty t^{-\alpha-1} \mathrm{e}^{-t^p}\,\mathrm{d}t = p^{-1} \int_{s^p}^\infty t^{-\alpha/p-1} \mathrm{e}^{-t}\,\mathrm{d}t.$$

For $\operatorname{Re} z < -\beta$,

$$\begin{aligned} \hat{k}(z) &= \int_0^\infty u^{z-1} k\bigg(\frac{1}{u}\bigg)\,\mathrm{d}u \\ &= \int_0^\infty u^{z-1} \int_{1/u}^\infty t^{-1-\alpha} \mathrm{e}^{-t^p}\,\mathrm{d}t\,\mathrm{d}u \\ &= \int_0^\infty u^{z+\alpha-1} \int_1^\infty t^{-1-\alpha} \mathrm{e}^{-(t/u)^p}\,\mathrm{d}t\,\mathrm{d}u \\ &= \int_0^\infty u^{-z-\alpha-1} \mathrm{e}^{-u^p} \mathrm{d}u \int_1^\infty t^{z-1}\,\mathrm{d}t \\ &= -\frac{1}{pz} \Gamma\bigg(\frac{-z-\alpha}{p}\bigg). \end{aligned}$$

Thus, since $-\varrho < -\beta$,

$$k(-\varrho) = \frac{1}{p\varrho} \Gamma\bigg(\frac{\varrho - \alpha}{p}\bigg).$$

From here, the result will follow by Lemma 2. We just need to verify that the assumptions hold.

https://doi.org/10.1239/jap/1354716655 Published online by Cambridge University Press

It is easy to see that k is a continuous, nonnegative function on $(0, \infty)$ and that $\hat{k}(z)$ has no zeros. Fix $\tau \in (-\varrho, -\beta)$ and $\gamma < -(\varrho \vee 2)$, and let $C = \sup_{t\geq 1} t^{-\alpha/p-1}\mathrm{e}^{-t/2}$. Note that $\gamma < \tau < 0$. We have

$$\begin{aligned} p\sum_{n=0}^{\infty} \max\{\mathrm{e}^{-\gamma n}, \mathrm{e}^{-\tau n}\} \sup_{\mathrm{e}^n \leq x \leq \mathrm{e}^{n+1}} k(x) &= \sum_{n=0}^{\infty} \mathrm{e}^{|\gamma| n} \int_{\mathrm{e}^{np}}^{\infty} t^{-\alpha/p-1}\mathrm{e}^{-t/2}\mathrm{e}^{-t/2}\,\mathrm{d}t \\ &\leq C\sum_{n=0}^{\infty} \mathrm{e}^{|\gamma| n} \int_{\mathrm{e}^{np}}^{\infty} \mathrm{e}^{-t/2}\,\mathrm{d}t \\ &= 2C\sum_{n=0}^{\infty} \mathrm{e}^{|\gamma| n}\mathrm{e}^{-\mathrm{e}^{np}/2} \\ &< \infty \end{aligned}$$

and

$$p\sum_{-\infty<n\leq -1} \max\{\mathrm{e}^{-\gamma n}, \mathrm{e}^{-\tau n}\} \sup_{\mathrm{e}^n \leq x \leq \mathrm{e}^{n+1}} k(x) = \sum_{n=1}^{\infty} \mathrm{e}^{-|\tau| n} \int_{\mathrm{e}^{-np}}^{\infty} t^{-\alpha/p-1}\mathrm{e}^{-t}\,\mathrm{d}t.$$

When $\alpha < 0$, this is bounded by

$$\int_0^{\infty} t^{-\alpha/p-1}\mathrm{e}^{-t}\,\mathrm{d}t \sum_{n=1}^{\infty} \mathrm{e}^{-|\tau| n} < \infty.$$

When $\alpha = 0$, it is bounded by

$$\sum_{n=1}^{\infty} \mathrm{e}^{-|\tau| n}\bigg(\int_{\mathrm{e}^{-np}}^{1} t^{-1}\,\mathrm{d}t + \int_1^{\infty} \mathrm{e}^{-t}\,\mathrm{d}t\bigg) = \sum_{n=1}^{\infty} \mathrm{e}^{-|\tau| n}(np + \mathrm{e}^{-1}) < \infty.$$

When $\alpha \in (0, 2)$, it is bounded by

$$\sum_{n=1}^{\infty} \mathrm{e}^{-|\tau| n} \int_{\mathrm{e}^{-np}}^{\infty} t^{-\alpha/p-1}\,\mathrm{d}t = \frac{p}{\alpha}\sum_{n=1}^{\infty} \mathrm{e}^{-|\tau| n+\alpha n} = \frac{p}{\alpha}\sum_{n=1}^{\infty} \mathrm{e}^{-(|\tau|-\alpha) n} < \infty.$$

Recall that $\gamma < -2$, and note that $-R_D(r)$ is a right-continuous, monotonically increasing function on $(0, \infty)$ with

$$\begin{aligned} \limsup_{r\downarrow 0} \frac{|-R_D(r)|}{r^{\gamma}} &\leq \limsup_{r\downarrow 0} r^2 \int_{|x|>r} R(\mathrm{d}x) \\ &\leq \limsup_{r\downarrow 0} \int_{1>|x|>r} |x|^2 R(\mathrm{d}r) + \limsup_{r\downarrow 0} r^2 R(|x| \geq 1) \\ &\leq \int_{|x|<1} |x|^2 R(\mathrm{d}r) \\ &< \infty. \end{aligned}$$

This completes the proof of Lemma 3.

The proof of Theorem 5 follows immediately from Lemma 3 and (20).

Acknowledgements

Most of the research for this paper was done while the author was a PhD student working with Professor Gennady Samorodnitsky. Professor Samorodnitsky's comments and support are gratefully acknowledged. This work was supported, in part, by funds provided by the University of North Carolina at Charlotte.

References

[1] Abramowitz, M. and Stegun, I. A. (1972). *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, 9th edn. Dover Publications, New York.
[2] Allen, O. O. (1992). Modelling heterogeneity in survival analysis by the compound Poisson distribution. *Ann. Appl. Prob.* **2,** 951–972.
[3] Aoyama, T., Maejima, M. and Rosiński, J. (2008). A subclass of type G selfdecomposable distributions on $\mathbb{R}^d$. *J. Theoret. Prob.* **21,** 14–34.
[4] Barndorff-Nielsen, O. E., Maejima, M. and Sato, K.-I. (2006). Some classes of multivariate infinitely divisible distributions admitting stochastic integral representations. *Bernoulli* **12,** 1–33.
[5] Basrak, B., Davis, R. A. and Mikosch, T. (2002). A characterization of multivariate regular variation. *Ann. Appl. Prob.* **12,** 908–920.
[6] Bianchi, M. L., Rachev, S. T., Kim, Y. S. and Fabozzi, F. J. (2011). Tempered infinitely divisible distributions and processes. *Theory Prob. Appl.* **55,** 2–26.
[7] Bingham, N. H., Goldie, C. M. and Teugels, J. L. (1987). *Regular Variation* (Encyclopedia Math. Appl. **27**). Cambridge University Press.
[8] Bruno, R., Sorriso-Valvo, L., Carbone, V. and Bavassano, B. (2004). A possible truncated-Lévy-flight statistics recovered from interplanetary solar-wind velocity and magnetic-field fluctuations. *Europhys. Lett.* **66,** 146–152.
[9] Carr, P., Geman, H., Madan, D. B. and Yor, M. (2002). The fine structure of asset returns: an empirical investigation. *J. Business* **75,** 305–332.
[10] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd edn. John Wiley, New York.
[11] Grabchak, M. and Samorodnitsky, G. (2010). Do financial returns have finite or infinite variance? A paradox and an explanation. *Quant. Finance* **10,** 883–893.
[12] Gupta, A. K., Shanbhag, D. N., Nguyen, T. T. and Chen, J. T. (2009). Cumulants of infinitely divisible distibutions. *Random Operators Stoch. Equat.* **17,** 103–124.
[13] Gyires, T. and Terdik, G. (2009). Does the Internet still demonstrate fractal nature? In *8th Internat. Conf. Networks*, IEEE Computer Society Press, Washington, DC, pp. 30–34.
[14] Hult, H. and Lindskog, F. (2006). On regular variation for infinitely divisible random vectors and additive processes. *Adv. Appl. Prob.* **38,** 134–148.
[15] Kim, Y. S., Rachev, S. T., Bianchi, M. L. and Fabozzi, F. J. (2010). Tempered stable and tempered infinitely divisible GARCH models. *J. Banking Finance* **34,** 2096–2109.
[16] Maejima, M. and Nakahara, G. (2009). A note on new classes of infinitely divisible distributions on $\mathbb{R}^d$. *Electron. Commun. Prob.* **14,** 358–371.
[17] Meerschaert, M. M. and Scheffler, H.-P. (2001). *Limit Distributions for Sums of Independent Random Vectors*. John Wiley, New York.
[18] Meerschaert, M. M., Zhang, Y. and Baeumer, B. (2008). Tempered anomalous diffusion in heterogeneous systems. *Geophys. Res. Lett.* **35**, 5pp.
[19] Palmer, K. J., Ridout, M. S. and Morgan, B. J. T. (2008). Modelling cell generation times by using the tempered stable distribution. *J. R. Statist. Soc. C* **57,** 379–397.
[20] Rosiński, J. (2007). Tempering stable processes. *Stoch. Process. Appl.* **117,** 677–707.
[21] Rosiński, J. and Sinclair, J. L. (2010). Generalized tempered stable processes. In *Stability in Probability* (Banach Center Publ. **90**), Polish Acad. Sci. Inst. Math. Warsaw, pp. 153–170.
[22] Rvačeva, E. L. (1962). On domains of attraction of multi-dimensional distributions. In *Selected Translations in Mathematical Statistics and Probability*, Vol. 2, American Mathematical Society, Providence, RI, pp. 183–205.
[23] Samorodnitsky, G. and Taqqu, M. S. (1994). *Stable Non-Gaussian Random Processes*. Chapman & Hall, New York.
[24] Sapatinas, T. and Shanbhag, D. N. (2010). Moment properties of multivariate infinitely divisible laws and criteria for multivariate self-decomposability. *J. Multivariate Anal.* **101,** 500–511.
[25] Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.
[26] Terdik, G. and Woyczyński, W. A. (2006). Rosiński measures for tempered stable and related Ornstein–Uhlenbeck processes. *Prob. Math. Statist.* **26,** 213–243.
[27] Uchaikin, V. V. and Zolotarev, V. M. (1999). *Chance and Stability*. VSP, Utrecht.