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KNOTS AND GRAVITY

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Abstract

In the loop representation theory of non-perturbative quantum gravity, gravitational states are described by functionals on the loop space of a 3-manifold. In the order to gain a deeper insight into the physical interpretation of loop states, a natural question arises: to wit, how are gravitations related to loops? Some light will be shed on this question by establishing a definite relationship between loops and 3-geometries of the 3-manifold.

1. Introduction

In the mid 80's, Ashtekar [1] formulated an alternative Hamiltonian approach to General Relativity. This led Rovelli *et al.* [4, 6] to formulate Quantum Gravity in terms of loops in a 3-manifold Σ. A *loop* in Σ is just a closed curve starting and ending at the same point. An *n-loop* is the set $\{\gamma^1, \dots, \gamma^n\}$ of n loops γ^i in Σ.

Very briefly, the loop representation of Quantum Gravity describes gravitational states via complex functionals Ψ on the space of multi-loops of Σ. The functionals describing the physical states of gravity satisfy

(1) Ψ is a constant on knot classes;
(2) Ψ has support on smooth multi-loops without intersections.[3]

The physical interpretation still remains an open question. However, Rovelli [5, p. 1661] sketched a heuristic argument revealing the emergence of a discrete structure to space-time at the Plank scale. It will be tersely shown here that certain choices of $\aleph_0$-loops relate to 3-geometries in a natural way. This in turn yield a deeper insight into the way loops and gravity are related.

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[3]Extensions to piecewise smooth loops have also been done. Refer to [2, 3] for more details.

2. Definitions and notations

All loops considered here will be piecewise smooth in some fixed 3-manifold Σ, where Σ is assumed to be smooth, closed, compact, orientable and Riemannian. By a *Riemannian* 3-metric q on Σ is meant a symmetric, covariant 2-tensor that is positive-definite at each point $x \in \Sigma$. The space of Riemannian 3-metrics on Σ will be denoted by Γ_2^+ and the space of (Riemannian) 3-geometries of Σ by $\mathscr{Q} \stackrel{\text{def}}{=} \Gamma_2^+/\mathrm{Diff}^+(\Sigma)$, where $\mathrm{Diff}^+(\Sigma)$ denotes the group of smooth, orientation-preserving diffeomorphisms on Σ and a *3-geometry* is defined by the equivalence class $[q] \stackrel{\text{def}}{=} \{f^*q \mid f \in \mathrm{Diff}^+(\Sigma)\}$ of metrics $q \in \Gamma_2^+$ related by coordinate transformations. The space Γ_2^+ is endowed with the compact C^∞-topology and $\mathscr{Q}$ is given the quotient topology.

Now, given curves $\gamma, \eta : I \to \Sigma$, $I \stackrel{\text{def}}{=} [0, 1]$, with $\gamma(0) = \gamma(1)$, define $\gamma * \eta$ by

$$\gamma * \eta(t) = \begin{cases} \gamma(2t) & \text{for } 0 \leqq t \leqq \frac{1}{2}, \\ \eta(2t-1) & \text{for } \frac{1}{2} \leqq t \leqq 1. \end{cases}$$

Call a curve γ a *q-geodesic* if γ is a (parametrized) geodesic in Σ with respect to $q \in \Gamma_2^+$.

DEFINITION 2.1. γ is a *piecewise geodesic loop* if $\exists\, q \in \Gamma_2^+$ and n smooth q-geodesics $\gamma_1, \dots, \gamma_n : I \to \Sigma$ such that $\gamma = \gamma_1 * \cdots * \gamma_n$.

Let $D_\Sigma \subset \Sigma$ denote a countably dense subset and let $\mathscr{M}_\infty\left[\Gamma_2^+\right]$, for each $q \in \Gamma_2^+$, be the set of $\aleph_0$-loops $\gamma = \{\gamma^i : i \in \mathbb{N}\}$ such that

(1) $\forall\, i$, γ^i is a piecewise, affinely parametrized, q-geodesic loop in Σ;
(2) γ is in bijective[4] correspondence with D_Σ under $\gamma^i \mapsto \gamma^i(0)$.

It is easy to see that conditions (1) and (2) together imply that each γ corresponds to a unique 3-geometry $[q_\gamma] \in \mathscr{Q}$. For suppose γ is both a q-geodesic loop as well as a q'-geodesic loop. Then, with respect to charts U_α,

$$\left(\ddot{\gamma}^i_\alpha\right)^l + \Gamma_\alpha(q)^l_{kj}\left(\dot{\gamma}^i_\alpha\right)^k\left(\dot{\gamma}^i_\alpha\right)^j \stackrel{\text{a.e.}}{=} 0 \quad \text{and} \quad \left(\ddot{\gamma}^i_\alpha\right)^l + \Gamma_\alpha(q')^l_{kj}\left(\dot{\gamma}^i_\alpha\right)^k\left(\dot{\gamma}^i_\alpha\right)^j \stackrel{\text{a.e.}}{=} 0$$

on $\gamma^i(I) \cap U_\alpha$ for each i (no summation over α), where *a.e.* means that the equality holds on $I - \{t_1, \dots, t_n\}$, with $0 \leqq n < \infty$ and $n = 0$ denoting the empty set. Hence, $\left(\Gamma_\alpha(q)^l_{kj} - \Gamma_\alpha(q')^l_{kj}\right)\left(\dot{\gamma}^i_\alpha\right)^k\left(\dot{\gamma}^i_\alpha\right)^j \stackrel{\text{a.e.}}{=} 0\ \forall\, \gamma^i \in \gamma$ and α. Thus by (2), $\Gamma(q)^l_{kj}(x) \equiv \Gamma(q')^l_{kj}(x)$ on a dense subset of Σ as $\overline{\bigcup\{\gamma^i(I) \mid \gamma^i \in \gamma\}} \equiv \Sigma$ by (2). So, invoking the continuity of $\Gamma(h)$ for $h = q, q'$, it follows at once that $\Gamma(q) \equiv \Gamma(q')$ on Σ. Now, with respect to local coordinate basis, $\Gamma(q)^l_{kj} = \frac{1}{2}q^{ih}(\partial_k q_{hj} + \partial_j q_{hk} - \partial_h q_{kj})$ (and likewise

[4]This condition may be relaxed to a surjection.

for q'); consequently, q and q' are related homothetically; that is, $\exists\, c > 0$ constant such that $q' = cq$.[5] More generally, q, q' are related to some smooth diffeomorphism.

As a converse remark, notice that if Σ were not separable or that $\gamma_q = \{\gamma_q^i \mid i \in \mathbb{N}\}$ were not chosen to satisfy (2), γ_q need not uniquely determine $[q] \in \mathcal{Q}$. Call $\mathcal{M}_\infty\left[\Gamma_2^+\right] \stackrel{\text{def}}{=} \bigcup_{q \in \Gamma_2^+} \mathcal{M}_\infty[q]$ the space of *piecewise geodesic* $\aleph_0$-*loops*. A suitable topology will be constructed on this space below.

Let $L_\Sigma\left[\Gamma_2^+\right]$ denote the set of all affinely parametrized, piecewise geodesic loops in Σ and let $L_\Sigma^\infty\left[\Gamma_2^+\right]$ denote the countably infinite (set-theoretic) product of $L_\Sigma\left[\Gamma_2^+\right]$. Define an equivalence relation $R_\infty \subset L_\Sigma^\infty\left[\Gamma_2^+\right] \times L_\Sigma^\infty\left[\Gamma_2^+\right]$ by $R_\infty = \{(\gamma, \gamma') : [\gamma] = [\gamma']\}$, where $[\eta] \stackrel{\text{def}}{=} \{\eta^i \in L_\Sigma\left[\Gamma_2^+\right] : \eta = (\eta^i)_{i=1}^\infty\}$ is just the set of components of the $\aleph_0$-loop η. Let $\pi_\Sigma : L_\Sigma^\infty\left[\Gamma_2^+\right] \to \mathcal{M}\left[\Gamma_2^+\right] \stackrel{\text{def}}{=} \mathcal{L}_\Sigma^\infty\left[\Gamma_2^+\right]/R_\infty$ be the natural map. If $\mathcal{M}_n\left[\Gamma_2^+\right]$ denotes the set of (affinely parametrized) piecewise geodesic n-loops, then $\mathcal{M}\left[\Gamma_2^+\right] \equiv \bigcup_{n=1}^\infty \mathcal{M}_n\left[\Gamma_2^+\right]$. Now, let $M_\infty \subset \mathcal{L}_\Sigma^\infty\left[\Gamma_2^+\right]$ be a subset satisfying

(a) for each $\gamma \stackrel{\text{def}}{=} (\gamma^i)_{i=1}^\infty \in M_\infty$, $\gamma^i \neq \gamma^j \;\; \forall i \neq j$,
(b) $\pi_\Sigma(M_\infty) = \mathcal{M}_\infty\left[\Gamma_2^+\right] \subset \mathcal{M}\left[\Gamma_2^+\right]$.

It is clear from the definition of M_∞ that there exists a family of subsets $M_\sigma \subset M_\infty$ satisfying

(i) $M_\infty = \bigcup_\sigma M_\sigma$,
(ii) $M_\sigma \cap M_{\sigma'} = \emptyset \;\; \forall \sigma \neq \sigma'$,
(iii) $\pi_\Sigma \mid M_\sigma : M_\sigma \to \mathcal{M}_\infty\left[\Gamma_2^+\right]$ is a bijection.

Let $h_\sigma \stackrel{\text{def}}{=} \pi_\Sigma \mid M_\sigma$ and for each $\gamma \in \mathcal{M}_\infty\left[\Gamma_2^+\right]$, set $\gamma_\sigma = h_\sigma^{-1}(\gamma) \in M_\sigma$.[6] The subsets M_σ can be endowed with a metric topology. A metric on M_σ will now be constructed. Firstly, fix a finite atlas $\mathfrak{U}$ on Σ. Secondly, note that if $\Omega_\Sigma = \{\gamma : I \to \Sigma \mid \gamma(0) = \gamma(1),\ \gamma \text{ continuous}\}$ denotes the loop space of Σ and if d_q is a (topological) metric on Σ (induced by a Riemannian 3-metric q) compatible with its manifold topology, then $d_\Omega(\gamma, \eta) \stackrel{\text{def}}{=} \sup_{t \in I} d_q(\gamma(t), \eta(t))$ defines a metric on Ω_Σ compatible with its compact-open topology.[7]

Now, given a pair of $\aleph_0$-loops $\gamma, \eta \in M_\sigma$, let

$$d'_\Omega\left(\gamma^i, \eta^i\right) \stackrel{\text{def}}{=} \text{ess sup}\left\{\left\|D^k\gamma^i(t) - D^k\eta^i(t)\right\| : t \in I,\ k \geqq 1\right\},$$

where sup runs over all relevant (finite) charts $(U, \varphi) \in \mathfrak{U}$, ess denoting that the expression $\left\|D^k\gamma^i(t) - D^k\eta^i(t)\right\|$ is defined on I *a.e.* — that is, it is *not* defined only on a *finite* (possibly zero) set of points in I wherein γ^i and η^i are not differentiable,

[5]Note trivially that as q, q' are positive-definite, $c < 0$ is not an admissible solution.
[6]The subscript σ on γ_σ will be omitted if no confusion should arise from the context.
[7]Observe trivially that the d_Ω-topology does not depend on the choice of the (admissible) 3-metric q since all (topological) metrics on Σ induced by (admissible) Riemannian 3-metrics q are equivalent.

and $D^k\gamma^i(t)$ denotes the kth differential of γ^i at t in abused notations. Finally, set $d_\sigma(\gamma,\eta) \overset{\text{def}}{=} \sup_i d_\Omega\left(\gamma^i,\eta^i\right) + \sup_i d'_\Omega\left(\gamma^i,\eta^i\right)$. It is routine to verify that d_σ is indeed a metric on M_σ.

REMARK 2.2. It can be shown that the d_σ-topology is compatible with the topology on M_σ generated by the subbasic sets $N_\varepsilon\left(\gamma;(U_{\alpha(i)},\varphi_{\alpha(i)})_{i=1}^{\infty},K\right)$ to be defined below, where $K \subset I$ is compact, $\gamma^i(K) \subset U_{\alpha(i)}$, and $(U_{\alpha(i)},\varphi_{\alpha(i)}) \in \overline{\mathfrak{U}}$ for each i, with $\overline{\mathfrak{U}}$ being the *maximal* atlas of Σ. Firstly, set $\alpha \overset{\text{def}}{=} \{\alpha(i) \mid 1 \leqq i \leqq \infty\}$ and denote $(U_{\alpha(i)},\varphi_{\alpha(i)})_i$ by $(U,\varphi)_\alpha$ for notational convenience. Next, let

$$d'_{\sigma\alpha K}\left(\gamma^i,\eta^i\right) \overset{\text{def}}{=} \operatorname{ess\,sup}\left\{\left\|D^k\varphi_{\alpha(i)}\circ\gamma^i(t) - D^k\varphi_{\alpha(i)}\circ\eta^i(t)\right\| : t \in K,\ k \geqq 1\right\}$$

whenever $\gamma^i(K),\eta^i(K) \subset U_{\alpha(i)}$ $\forall i$. Then, for a fixed $\gamma \in M_\sigma$ such that $\gamma^i(K) \subset U_{\alpha(i)}$ $\forall i$, let $N_\varepsilon(\gamma;(U,\varphi)_\alpha,K) \overset{\text{def}}{=} \{\eta \in M_\sigma \mid \bar{d}_{\sigma\alpha K}(\gamma,\eta) < \varepsilon, \eta^i(K) \subset U_{\alpha(i)}\ \forall i\}$, where

$$\bar{d}_{\sigma\alpha K}(\gamma,\eta) \overset{\text{def}}{=} \sup_i d_\Omega\left(\gamma^i,\eta^i\right) + \sup_i d'_{\sigma\alpha K}\left(\gamma^i,\eta^i\right).$$

In particular, the d_σ-topology does not depend on the particular choice of (admissible) finite atlas $\mathfrak{U}$ of Σ. Hence, in this sense, the d_σ-topology is well-defined.

It is easy to see from the construction that $h_{\sigma\sigma'} : M_\sigma \to M_{\sigma'}$ given by $\gamma_\sigma \mapsto \gamma_{\sigma'}$, where $h_\sigma(\gamma_\sigma) = \gamma = h_{\sigma'}(\gamma_{\sigma'})$, defines a homeomorphism. The existence of $h_{\sigma\sigma'}$ follows immediately from properties (a) and (iii) above. Hence, it is possible to endow $\mathscr{M}_\infty\left[\Gamma_2^+\right]$ with a topology so that each $h_\sigma : M_\sigma \to \mathscr{M}_\infty\left[\Gamma_2^+\right]$ defines a homeomorphism. In this paper, $\mathscr{M}_\infty\left[\Gamma_2^+\right]$ will be equipped with this topology. As an aside, if M_∞ were given the sum topology, $M_\infty \overset{\text{def}}{=} \bigoplus_\sigma M_\sigma$, then $h : M_\infty \to \mathscr{M}_\infty\left[\Gamma_2^+\right]$ defined by $h \mid M_\sigma = h_\sigma$ is a continuous open surjection.

3. Knots and 3-geometries

First of all, recall that a smooth *ambient isotopy* is a smooth deformation of one loop into another such that the surrounding manifold is smoothly transformed. More precisely, it is a smooth map $F : \Sigma \times I \to \Sigma \times I$ given by $(x,t) \mapsto (F_t(x),t)$ such that $F_0 = \mathrm{id}_\Sigma$ and $F_t \in \mathrm{Diff}(\Sigma)$ $\forall t \in I$. Let $\mathscr{G}_a^+ \subset C^\infty(\Sigma \times I, \Sigma \times I)$ be the set of (smooth) orientation-preserving, ambient isotopies on Σ.

If $\gamma,\eta \in \mathscr{L}_\Sigma$ are any pair of loops and γ is ambiently isotopic to η under some $F \in \mathscr{G}_a^+$, denote this by $F : \gamma \simeq \eta$. Now, given any pair of $\aleph_0$-loops $\gamma,\eta \in \mathscr{M}_\infty\left[\Gamma_2^+\right]$, define an equivalence relation R generated by $\simeq$ on $\mathscr{M}_\infty\left[\Gamma_2^+\right]$ as follows:

$$\gamma \simeq \eta \quad \Longleftrightarrow \quad \exists\, F \in \mathscr{G}_a^+ \text{ such that } F\cdot\gamma = \eta,$$

where $F \cdot \gamma \stackrel{\text{def}}{=} \{F_1 \circ \gamma^1, F_1 \circ \gamma^2, \dots\}$ and $F : \gamma^i \simeq \eta^i \ \forall i$. Then the space $\mathscr{K}[\Gamma_2^+]$ of equivalence classes of $\aleph_0$-loops in $\mathscr{M}_\infty[\Gamma_2^+]$ is defined to be the quotient space $\mathscr{M}_\infty[\Gamma_2^+]/R$. Henceforth, for simplicity, call an element $[\gamma] \stackrel{\text{def}}{=} \{\eta \in \mathscr{M}_\infty[\Gamma_2^+] : \eta \simeq \gamma\}$ of the quotient space $\mathscr{K}[\Gamma_2^+]$ a (*piecewise geodesic*) $\aleph_0$-*knot* and let $\kappa_\infty : \mathscr{M}_\infty[\Gamma_2^+] \to \mathscr{K}[\Gamma_2^+]$ denote the natural map. In the interest of simplicity, call $\gamma \in \mathscr{M}_\infty[\Gamma_2^+]$ a *piecewise* $(\aleph_0, q)$-*geodesic loop* whenever the 3-metric q is required to be specified.

LEMMA 3.1. *Let* $\gamma, \tilde{\gamma} \in \mathscr{M}_\infty[\Gamma_2^+]$ *be piecewise* $(\aleph_0, q)$- *and* $(\aleph_0, \tilde{q})$-*geodesic loops respectively. If* $\gamma \simeq \tilde{\gamma}$, *then* $\exists f \in \mathrm{Diff}^+(\Sigma)$ *such that* $q = f^*\tilde{q}$.

PROOF. Let $F \in \mathscr{G}_a^+$ be an ambient isotopy of γ and $\tilde{\gamma} : F \cdot \gamma = \tilde{\gamma}$. Then, evidently, $\tilde{\gamma}$ is a piecewise $(\aleph_0, (F_1^{-1})^* q)$-geodesic. However, $\tilde{\gamma}$ is also a piecewise $(\aleph_0, \tilde{q})$-geodesic; hence, by (2), $\exists f \in \mathrm{Diff}^+(\Sigma)$ such that $q = f^*\tilde{q}$, as required.

The main results of this paper will now be stated. In fact, the correspondence between loops and geometries can be easily sought simply by noting that each element in $\mathscr{M}_\infty[\Gamma_2^+]$ corresponds to a unique 3-geometry $[q]$ of Σ by construction.

THEOREM 3.2. *There exists a continuous, open surjection* $\hat{\chi} : \mathscr{M}_\infty[\Gamma_2^+] \to \mathscr{Q}$ *given by* $\gamma_q \mapsto [q]$, *where* γ_q *is a (piecewise)* $(\aleph_0, \tilde{q})$-*geodesic loop and* $q \in [q]$.

PROOF. The details can be found in [7, Theorem 4.1].

COROLLARY 3.3. *The map* $\hat{\chi}$ *induces a continuous, open surjection* $\chi : \mathscr{K}[\Gamma_2^+] \to \mathscr{Q}$ *given by* $[\gamma_q] \mapsto \hat{\chi}(\gamma_q)$, *where* $\gamma_q \in \kappa_\infty^{-1}([\gamma_q])$ *is any fixed representative.*

PROOF. The map χ is well-defined by Lemma 3.1. The result now follows immediately from Theorem 3.2, the openness of the projection map κ_∞ and from the commutativity of the following diagram:

$$\begin{array}{ccc} \mathscr{M}_\infty[\Gamma_2^+] & \xrightarrow{\hat{\chi}} & \mathscr{Q} \\ {\scriptstyle\kappa_\infty}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathscr{K}[\Gamma_2^+] & \xrightarrow{\chi} & \mathscr{Q}. \end{array}$$

4. Discussion

It is easy to observe from Theorem 3.2 that at the classical level, each $\aleph_0$-loop $\gamma \in \mathscr{M}_\infty[\Gamma_2^+]$ contains enough information to restrict the 3-manifold Σ together with

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its 3-geometry $[q]$. To see this, it is sufficient to note firstly that χ maps γ to a unique 3-geometry $[q]$. Then, by choosing any representative of $[q]$ and defining the closure of $\bigcup \{\gamma^i(I) \mid \gamma^i \in \gamma\}$ with respect to the metric induced by q yields the Riemannian manifold (Σ, q).

This in turn suggests that $\aleph_0$-loops are suitable candidates for the description of gravitational states. Heuristically, we may interpret a *knot state* $|[\gamma]\rangle$, $[\gamma] \in \mathscr{K}\left[\Gamma_2^+\right]$, as a state associated with a 3-manifold together with its Riemannian 3-geometry $(\Sigma, \chi([\gamma]))$. That is, each knot state $|[\gamma]\rangle$ corresponds to the global degrees of freedom of gravity. Secondly, functionals on $\mathscr{L}_\Sigma$ which describe gravitational states are constant on the $\mathscr{G}_a^+$-orbits of $\mathscr{L}_\Sigma$ — $\psi[\gamma] = \psi[\gamma'] \ \forall \gamma, \gamma' \in [\gamma]$, where $\psi : \mathscr{L}_\Sigma \to \mathbb{C}$ is a loop functional — due to the diffeomorphism constraint of general relativity (in the loop representation) [6, p. 132]. Surprisingly, this condition follows immediately from Corollary 3.3. For let $C(\mathscr{Q}, \mathbb{C})$ be the set of continuous functionals on $\mathscr{Q}$ and $C(\mathscr{K}\left[\Gamma_2^+\right], \mathbb{C})$ that of $\mathscr{K}\left[\Gamma_2^+\right]$. Then, $\forall\, \tilde{\Psi} \in C(\mathscr{Q}, \mathbb{C})$, $\tilde{\Psi} \circ \chi \in C(\mathscr{K}\left[\Gamma_2^+\right], \mathbb{C})$; that is, $\chi^* \left(C(\mathscr{Q}, \mathbb{C})\right) \subset C(\mathscr{K}\left[\Gamma_2^+\right], \mathbb{C})$, and the assertion thus follows.

This paper will conclude by outlining a prime motivation for studying the relationship between knots and geometries. It is possible to show, by relaxing the bijective condition of (2) imposed on γ — that is, $\gamma^i \mapsto \gamma^i(0)$ is a bijection — to a surjective one, and by imposing additional conditions on the $\aleph_0$-loops, that the resulting $\aleph_0$-loop space $\mathscr{M}_\infty$ admits a smooth manifold structure modelled on a locally convex topological vector space. This has the implication that $\mathscr{M}_\infty$ can be regarded as a configuration space for gravity in the sense of geometric quantization. Thus, in this sense, $\mathscr{M}_\infty$ has the interpretation of being the 'dynamical' space where 3-geometries evolve. This is of course rather speculative, and work in this area is currently in progress.

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