TRANSFORMATIONS OF WHICH THE PARTIAL DIFFERENTIAL EQUATION OF HEAT FLOW IS A DIFFERENTIAL INVARIANT

by D. H. PARSONS

(Received 28th December 1964)

The problem to be considered is that of finding transformations which leave unchanged the form of the equation of isotropic heat flow

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}-\frac{1}{K}\frac{\partial v}{\partial t}=0, \tag{1}$$

where K is a constant. From such a transformation, we can at once deduce, from any known integral of (1), a new integral which may depend upon one or more arbitrary constants.

This problem has been studied by P. Appell (**1**) for the one-dimensional form of the heat flow equation

$$\frac{\partial^2 v}{\partial x^2}-\frac{1}{K}\frac{\partial v}{\partial t}=0.$$

Appell showed that transformations having the required property are provided by any combinations of two basic types; firstly, the trivial transformations of the form

$$X = ax+b,\ T = a^2t+c,\ V = v\ (a,\ b,\ c\text{ being constants}),$$

and secondly, the now well-known transformation

$$X = x/t,\ T = -1/t,\ v = Vt^{-\frac{1}{2}}e^{-x^2/4Kt}.$$

Well-known analogues of these two types apply to the two- and three-dimensional forms of the heat flow equation.

2. We shall now show that there is, also, a non-trivial transformation of a different kind, which leaves the form of (1) unchanged. This is the transformation

$$\left.\begin{aligned} &X = x+2Kat,\ Y = y+2Kbt,\ Z = z+2Kct,\ T = t,\\ &v = Ve^{\{ax+by+cz+K(a^2+b^2+c^2)t\}}\end{aligned}\right\} \tag{2}$$

a, b, c being arbitrary constants.

https://doi.org/10.1017/S0013091500013274 Published online by Cambridge University Press

For, by direct calculation from (2), we have

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}-\frac{1}{K}\frac{\partial v}{\partial t} \equiv e^{\{ax+by+cz+K(a^2+b^2+c^2)t\}}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}+\frac{\partial^2 V}{\partial Z^2}-\frac{1}{K}\frac{\partial V}{\partial T}\right);$$

and therefore the form of (1) remains unaltered, x, y, z, t, v being replaced by X, Y, Z, T, V respectively.

From (2) and the above result, we see that if $v(x, y, z, t)$ be an integral (supposed known) of (1), so also is

$$e^{\{ax+by+cz+K(a^2+b^2+c^2)t\}}.\, v\{x+2Kat,\ y+2Kbt,\ z+2Kct,\ t\};$$

so that from one known integral we deduce at once an infinity of integrals, depending upon three arbitrary parameters, a, b, c.

The corresponding results for the two- and one-dimensional heat flow equations are obtained by writing $c = 0$ or $b = c = 0$, as the case may be, in the expressions above.

REFERENCE

(1) P. Appell, *J. Math. Pures et Appl.* (4) **8** (1892), 187.

The University
Reading