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GRAPHS WHOSE FULL AUTOMORPHISM GROUP IS A SYMMETRIC GROUP

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Abstract

We address the problem of describing all graphs Γ such that $\operatorname{Aut}\Gamma$ is a symmetric group, subject to certain restrictions on the sizes of the orbits of $\operatorname{Aut}\Gamma$ on vertices. As a corollary of our general results, we obtain a classification of all graphs Γ on v vertices with $\operatorname{Aut}\Gamma \cong S_n$, where

$$v < \min\left\{5n, \tfrac{1}{2}n(n-1)\right\}.$$

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Introduction

It has been known since Frucht's paper [1] of 1938 that, given any finite group G, there is a graph Γ such that the automorphism group of Γ is isomorphic to G. For certain groups G, such as S_n, this result is obvious, and it is more interesting to investigate the more general problem of describing all graphs Γ such that $\operatorname{Aut}\Gamma \cong G$. In this paper we address this problem for the symmetric groups S_n. This was considered for graphs with less than $2n$ vertices in [2] and [3]. Here we investigate the graphs Γ such that $\operatorname{Aut}\Gamma \cong S_n$, under the following far less restrictive hypothesis:

(*) all orbits of $\operatorname{Aut}\Gamma$ on the set $V\Gamma$ of vertices of Γ have size less than $\frac{1}{2}n(n-1)$.

It is an elementary consequence of (*) (see Proposition 1.2 below) that for $n > 6$, all the orbits of $\operatorname{Aut}\Gamma$ on $V\Gamma$ have size 1 or n. Let t be the number of orbits of size n. In Theorem 1.4 we show that Γ must satisfy various strong necessary conditions; and we conjecture (1.5) that these conditions on an arbitrary graph Δ are also sufficient to imply that $\operatorname{Aut}\Delta \cong S_n$. We prove Conjecture 1.5 for $1 \leqslant t \leqslant 4$ (see Theorem 2.7). In particular this gives a classification of all graphs Γ with $\operatorname{Aut}\Gamma \cong S_n$ $(n > 6)$ and $|V\Gamma| < \min\{5n, \frac{1}{2}n(n-1)\}$. This substantially improves the results of [2] and [3], and also solves various problems raised in [3, Section 4]. We include general descriptions of these graphs in an Appendix.

Finally, we remark that the methods of this paper will extend to the analysis of graphs with automorphism group S_n under weaker hypotheses than (*) (see Remark 3 after Theorem 2.7).

NOTATION. If G is a permutation group on a set Ω and $\Delta \subseteq \Omega$ then $G_{\{\Delta\}}$ denotes the setwise stabilizer of Δ in G; and if Ψ is a fixed set of G then G^{Ψ} denotes the action of G on Ψ. Also $\operatorname{Alt}(\Omega)$ and $\operatorname{Sym}(\Omega)$ denote, respectively, the alternating and symmetric groups on Ω.

1. A general result and a conjecture

We begin with an elementary proposition.

PROPOSITION 1.1. *Let $n > 6$ and let H be a proper subgroup of S_n with $|S_n : H| < \frac{1}{2}n(n-1)$. Then H is A_n, S_{n-1} or A_{n-1}.*

PROOF. If H is transitive and imprimitive on the n points with blocks of size a and $ab = n$ $(a \neq 1, b \neq 1)$, then $|H| \leqslant (a!)^b b!$, so

$$\tfrac{1}{2}n(n-1) > |S_n : H| \geqslant n!/\big((a!)^b b!\big)$$

which forces $n \leqslant 6$, a contradiction. If H is primitive on the n points and $H \neq A_n$ then a result of Bochert (Theorem 14.2 of [4]) gives $\frac{1}{2}n(n-1) > |S_n : H| \geqslant [\frac{1}{2}(n+1)]!$, forcing $n \leqslant 6$ or $n = 8$. The latter is impossible (an easy check) so this case cannot occur. Finally if H is intransitive with an orbit of size r then $\frac{1}{2}n(n-1) > |S_n : H| \geqslant \binom{n}{r}$, so that r is 1 or $n-1$ and H is S_{n-1} or A_{n-1}.

From Proposition 1.1 we see that if Γ is a graph with $\operatorname{Aut}\Gamma \cong S_n$ $(n > 6)$ and all orbits of $\operatorname{Aut}\Gamma$ on $V\Gamma$ have size less than $\frac{1}{2}n(n-1)$ then these orbit sizes all lie in $\{1, 2, n, 2n\}$. We shall easily show below (Proposition 1.2) that the orbit sizes 2 and $2n$ cannot occur, so for the remainder of this section we concentrate on the set $\mathscr{E}_n$ of graphs defined as follows:

DEFINITION. Let Γ be a graph and let $n \geqslant 2$. Then $\Gamma \in \mathscr{E}_n$ if and only if $\operatorname{Aut}\Gamma$ has a subgroup G isomorphic to S_n such that all orbits of G on $V\Gamma$ have size 1 or n.

Let $\Gamma \in \mathscr{E}_n$. Then we may take Γ to be a graph on $tn + r$ vertices $\{\alpha_{11},\ldots,\alpha_{1n},\ldots,\alpha_{t1},\ldots,\alpha_{tn},\phi_1,\ldots,\phi_r\}$ such that $\operatorname{Aut}\Gamma$ has a subgroup G isomorphic to S_n with r fixed points $\phi_1,\ldots,\phi_r$ and t orbits $\Delta_1,\ldots,\Delta_t$ of size n, where $\Delta_i = \{\alpha_{i1},\ldots,\alpha_{in}\}$ $(i = 1,\ldots,t)$. It is clear that each subgraph Δ_i is either the complete graph K_n or the null graph V_n and that for any i, j, ϕ_j is joined to all or no vertices in Δ_i. For any i, j, k define

$$\Gamma_j(\alpha_{ik}) = \{\alpha_{jl} \in \Delta_j \mid \alpha_{jl} \text{ is joined to } \alpha_{ik} \text{ in } \Gamma\}.$$

Then $\Gamma_j(\alpha_{ik})$ is a union of orbits of the stabiliser $G_{\alpha_{ik}}$ on Δ_j. Now if $n \neq 6$ then S_n has just one conjugacy class of subgroups of index n, so we may assume in this case that $G_{\alpha_{ik}} = G_{\alpha_{jk}}$ for all i, j, k; and S_6 has two conjugacy classes of subgroups of index 6, one class containing the stabilizer of one of the 6 points and the other containing a subgroup S_5 transitive on the 6 points. Hence for any $n \geqslant 2$ and any i, j, k we may assume that $G_{\alpha_{ik}}$ is either transitive on Δ_j or has orbits $\{\alpha_{jk}\}$ and $\Delta_j \setminus \{\alpha_{jk}\}$ on Δ_j. Consequently $\Gamma_j(\alpha_{ik})$ is one of the sets $\varnothing$, Δ_j, $\{\alpha_{jk}\}$ and $\Delta_j \setminus \{\alpha_{jk}\}$; and for any k, l, if $\Gamma_j(\alpha_{ik})$ is $\varnothing(\Delta_j, \{\alpha_{jk}\}, \Delta_j \setminus \{\alpha_{jk}\})$ then $\Gamma_j(\alpha_{il})$ is $\varnothing(\Delta_j, \{\alpha_{jl}\}, \Delta_j \setminus \{\alpha_{jl}\}$ respectively).

Note that the above analysis goes through if we replace S_n by the alternating group A_n. Using this analysis we now prove the result promised above.

PROPOSITION 1.2. *Let Γ be a graph with* $\operatorname{Aut}\Gamma \cong S_n$ $(n > 6)$ *and suppose that all orbits of* $\operatorname{Aut}\Gamma$ *on* $V\Gamma$ *have size less than* $\frac{1}{2}n(n-1)$. *Then all these orbits have size* 1 *or* n.

PROOF. Write $G = \operatorname{Aut}\Gamma$. By Proposition 1.1 all G-orbits on $V\Gamma$ have size 1, 2, n or $2n$. Let $H < G$ with $H \cong A_n$. Then all H-orbits on $V\Gamma$ have size 1 or n; let those of size n be $\Delta_1,\ldots,\Delta_t$ and let fix $H = \{\phi_1,\ldots,\phi_r\}$. By the above analysis, each Δ_i is K_n or V_n, each ϕ_j is joined to all or no vertices of Δ_i and, writing $\Delta_i = \{\alpha_{i1},\ldots,\alpha_{in}\}$, we may choose notation so that $\Gamma_j(\alpha_{ik})$ is one of $\varnothing$, Δ_j, $\{\alpha_{jk}\}$ and $\Delta_j \setminus \{\alpha_{jk}\}$ and if $\Gamma_j(\alpha_{ik})$ is $\varnothing$ $(\Delta_j$, $\{\alpha_{jk}\}$, $\Delta_j \setminus \{\alpha_{jk}\})$ then $\Gamma_j(\alpha_{il})$ is $\varnothing(\Delta_j$, $\{\alpha_{jl}\}$, $\Delta_j \setminus \{\alpha_{jl}\}$ respectively) (any i, j, k, l). It is clear from this that $\operatorname{Aut}\Gamma$ contains a subgroup K such that $K \cong S_n$, K has orbits $\Delta_1,\ldots,\Delta_t$ and fixes each ϕ_j. Hence $K = G$ and the orbits of G all have size 1 or n.

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We now resume our analysis of a graph $\Gamma \in \mathscr{E}_n$ on vertex set $\Delta_1 \cup \cdots \cup \Delta_t \cup \{\phi_1, \ldots, \phi_r\}$ as described above. From Γ we define a coloured graph Γ^* with vertex set $\{\delta_1, \ldots, \delta_t, \phi_1, \ldots, \phi_r\}$ having 3 vertex-colours (white, black and red) and 5 edge-colours ($0, 1, n-1, n$ and black) as follows:

(i) δ_i is coloured white if Δ_i is V_n, black if Δ_i is K_n $(i = 1, \ldots, t)$;

(ii) ϕ_j is coloured red $(j = 1, \ldots, r)$;

(iii) ϕ_j is joined to δ_i by a black edge if ϕ_j is joined in Γ to all vertices of Δ_i, and by no edge at all if not;

(iv) ϕ_i is joined to ϕ_j by a black edge if ϕ_i is joined to ϕ_j in Γ, and by no edge if not;

(v) the vertices δ_i, δ_j are joined by an edge coloured $0, 1, n-1$ or n as follows (if $n = 2$ then the labels 1 and $n-1$ should represent different colours):

$$\delta_i \times \overset{0}{\rule{2cm}{0.4pt}} \times \delta_j \qquad \text{if } \Gamma_j(\alpha_{ik}) = \varnothing,$$

$$\delta_i \times \overset{1}{\rule{2cm}{0.4pt}} \times \delta_j \qquad \text{if } \Gamma_j(\alpha_{ik}) = \{\alpha_{jk}\},$$

$$\delta_i \times \overset{n-1}{\rule{2cm}{0.4pt}} \times \delta_j \qquad \text{if } \Gamma_j(\alpha_{ik}) = \Delta_j \setminus \{\alpha_{jk}\},$$

$$\delta_i \times \overset{n}{\rule{2cm}{0.4pt}} \times \delta_j \qquad \text{if } \Gamma_j(\alpha_{ik}) = \Delta_j,$$

(here $\times$ represents a black or a white vertex). The automorphism group $\operatorname{Aut} \Gamma^*$ is the group of permutations of $V\Gamma^*$ preserving all vertex- and edge-colours. Clearly Γ can be reconstructed from Γ^*; so $\Gamma \leftrightarrow \Gamma^*$ is a 1-1 correspondence.

Now we define two further graphs from Γ^*: firstly, Γ_0^* is the subgraph of Γ^* on $\delta_1, \ldots, \delta_t$ with all edges coloured 0 or n deleted and all edges coloured 1 or $n-1$ replaced by a black edge; secondly, Γ_1^* is obtained from Γ^* by the following replacements:

(1) replace all vertices δ_i, ϕ_j by black vertices δ_i, ϕ_j;

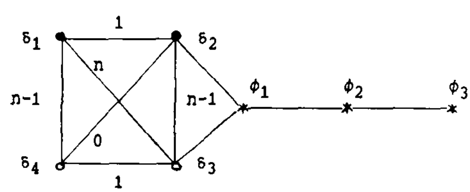
(2) replace any edge coloured 1 or n by a black edge;

(3) delete any edge coloured 0 or $n-1$.

Thus Γ_1^* can be regarded as an uncoloured graph.

We aim to obtain necessary and sufficient conditions for $\operatorname{Aut} \Gamma \cong S_n$ purely in terms of the smaller graphs Γ^*, Γ_0^* and Γ_1^*.

EXAMPLE. If Γ^* is



(* *represents a red vertex*)

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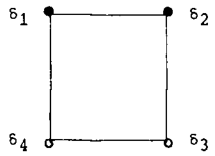
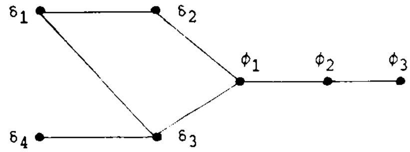
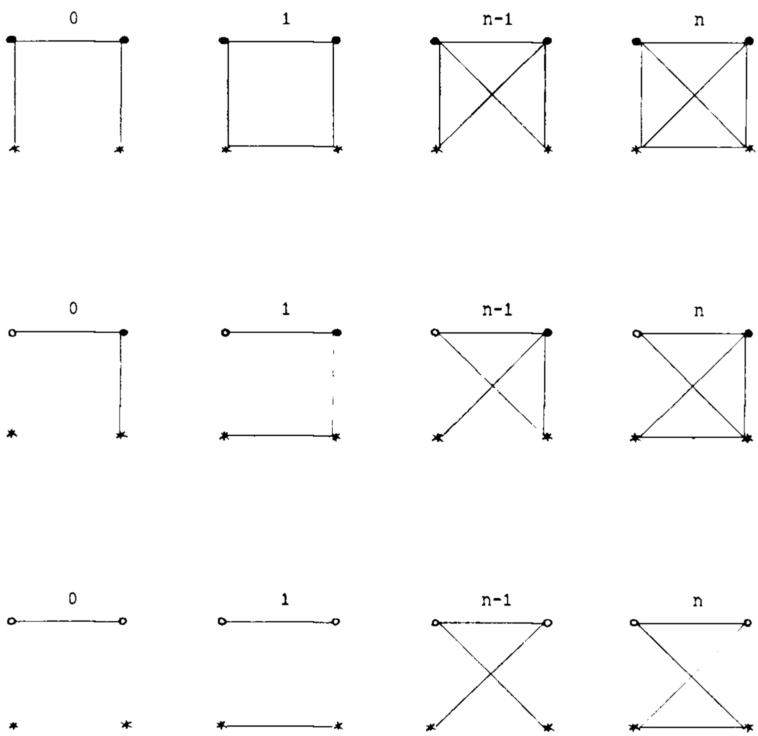
then Γ_0^*, Γ_1^* are respectively



and



Now consider a coloured graph Γ^* with $t = r = 2$, that is, with $V\Gamma^* = \{\delta_1, \delta_2, \phi_1, \phi_2\}$. Let Γ be the corresponding graph in $\mathscr{E}_n$ with $V\Gamma = \Delta_1 \cup \Delta_2 \cup \{\phi_1, \phi_2\}$. It is easy to see that $\operatorname{Aut}\Gamma$ contains a subgroup isomorphic to S_{n+1} having orbits $\Delta_1 \cup \{\phi_{i_1}\}$ and $\Delta_2 \cup \{\phi_{i_2}\}$ (where $\{i_1, i_2\} = \{1,2\}$) on $V\Gamma$ if and only if Γ^* is isomorphic to one of the following 12 graphs:



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(where ∘, •, ∗ represent white, black and red vertices respectively). Denote this set of 12 graphs by $\mathscr{C}_n$. Note that the graphs in $\mathscr{C}_n$ correspond to 6 graphs in $\mathscr{E}_n$ and their complements.

LEMMA 1.3. *Let $\Gamma \in \mathscr{E}_n$ have vertex set $\Delta_1 \cup \cdots \cup \Delta_t \cup \{\phi_1, \ldots, \phi_r\}$ as above and let Γ^* and Γ_1^* be the graphs corresponding to Γ as above. Suppose that* $\operatorname{Aut}\Gamma_1^*$ *contains an automorphism $x = (\delta_1\phi_{i_1})\cdots(\delta_t\phi_{i_t})$ $(i_1, \ldots, i_t$ all distinct$)$ such that*

(i) *ϕ_{i_j} is joined to δ_j in Γ^* if and only if δ_j is black $(j = 1, \ldots, t)$, and*

(ii) *for any distinct, k, l the subgraph $\{\delta_k, \delta_l, \phi_{i_k}, \phi_{i_l}\}$ of Γ^* lies in the set $\mathscr{C}_n$ of 12 graphs defined above.*

Then $\Gamma \in \mathscr{E}_{n+1}$.

PROOF. For $j = 1, \ldots, t$ put $\Delta'_j = \Delta_j \cup \{\phi_{i_j}\}$. By (i) each subgraph Δ'_j is either K_{n+1} or V_{n+1}. Write $\phi_{i_j} = \alpha_{j,n+1}$ $(j = 1, \ldots, t)$ and for any i, j, k define

$$\Gamma'_j(\alpha_{ik}) = \left\{\alpha_{jl} \in \Delta'_j \mid \alpha_{jl} \text{ is joined to } \alpha_{ik} \text{ in } \Gamma\right\}.$$

Then by (i) and (ii), $\Gamma'_j(\alpha_{ik})$ is one of the sets $\varnothing$, Δ'_j, $\{\alpha_{jk}\}$ and $\Delta'_j \setminus \{\alpha_{jk}\}$ and for any k, l, if $\Gamma'_j(\alpha_{ik})$ is $\varnothing$ $(\Delta'_j$, $\{\alpha_{jk}\}$, $\Delta'_j \setminus \{\alpha_{jk}\})$ then $\Gamma'_j(\alpha_{il})$ is $\varnothing$ $(\Delta'_j$, $\{\alpha_{jl}\}$, $\Delta'_j \setminus \{\alpha_{jl}\}$ respectively$)$. Also, since $x \in \operatorname{Aut}\Gamma_1^*$, for any $k \notin \{i_1, \ldots, i_t\}$ and any j, ϕ_k is joined to all or no vertices of Δ'_j. From these facts we see that $\operatorname{Aut}\Gamma$ contains a subgroup $H \cong S_{n+1}$ having orbits $\Delta'_1, \ldots, \Delta'_t$ and fixing ϕ_k for $k \notin \{i_1, \ldots, i_t\}$. Hence $\Gamma \in \mathscr{E}_{n+1}$.

THEOREM 1.4. *Let $\Gamma \in \mathscr{E}_n$ have vertex set $\Delta_1 \cup \cdots \cup \Delta_t \cup \{\phi_1, \ldots, \phi_r\}$ as above and let Γ^*, Γ_0^*, Γ_1^* be the graphs corresponding to Γ. Suppose that* $\operatorname{Aut}\Gamma \cong S_n$. *Then*

(a) $\operatorname{Aut}\Gamma^* = 1$;

(b) *Γ_0^* is connected (by the black edges);*

(c) $\operatorname{Aut}\Gamma_1^*$ *contains no automorphisms $(\delta_1\phi_{i_1})\cdots(\delta_t\phi_{i_t})$, with $i_1, \ldots, i_t$ distinct, such that*

  (i) *ϕ_{i_j} is joined to δ_j in Γ^* if and only if δ_j is black $(j = 1, \ldots, t)$,*

  (ii) *for any distinct k, l the subgraph $\{\delta_k, \delta_l, \phi_{i_k}, \phi_{i_l}\}$ of Γ^* lies in $\mathscr{C}_n$.*

PROOF. (a) Suppose that $h \in \operatorname{Aut}\Gamma^*$ with $h \neq 1$. Define a permutation g on $V\Gamma$ as follows

(1) if $\delta_i h = \delta_j$ put $\alpha_{ik} g = \alpha_{jk}$ $(k = 1, \ldots, n)$,

(2) for $i = 1, \ldots, r$ put $\phi_i g = \phi_i h$.

It is easy to check that $g \in \operatorname{Aut}\Gamma$, which contradicts the fact that since $\operatorname{Aut}\Gamma \cong S_n$, $\operatorname{Aut}\Gamma$ has orbits $\Delta_1, \ldots, \Delta_t$ and fixes each ϕ_j.

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(b) Suppose that Γ_0^* is disconnected and let $\{\delta_{i_1}, \ldots, \delta_{i_u}\}$ $(u < t)$ be a connected component of Γ_0^*; write $\Delta = \bigcup_{j=1}^{u} \Delta_{i_j}$. Then for any $\beta \in V\Gamma \setminus \Delta$ and any $j \in \{1, \ldots, u\}$, β is joined to all or no vertices in Δ_{i_j}. Hence $\operatorname{Aut}\Gamma$ contains a subgroup $H \cong S_n$ with orbits $\Delta_{i_1}, \ldots, \Delta_{i_u}$ and fixing every vertex in $V\Gamma \setminus \Delta$. Since $u < t$ it is clear that $H \neq \operatorname{Aut}\Gamma$, contradicting the fact that $\operatorname{Aut}\Gamma \cong S_n$. Thus Γ_0^* is connected.

(c) This follows directly from Lemma 1.3.

It seems likely that a general converse of Theorem 1.4 holds; since we have only been able to prove this when $t \leqslant 4$, we state the general case as a conjecture.

CONJECTURE 1.5. *Let n, t be positive integers with $n > t$. Let Γ^* be a graph on vertex set $\{\delta_1, \ldots, \delta_t, \phi_1, \ldots, \phi_r\}$ with 3 vertex-colours (white and black among the δ_i, red for the ϕ_i) and 5 edge-colours* $(0, 1, n-1, n$ *for edges between the δ_i, black for any other edges). Let Γ_0^* and Γ_1^* be the graphs defined from Γ^* as above and suppose that these satisfy conditions* (a), (b) *and* (c) *of Theorem* 1.4. *Then if Γ is the graph on $tn + r$ vertices corresponding as above to Γ^*, we have* $\operatorname{Aut}\Gamma \cong S_n$.

In the next sections we prove Conjecture 1.5 for $1 \leqslant t \leqslant 4$ and give some illustrations of its use in describing graphs Γ with $\operatorname{Aut}\Gamma \cong S_n$. It should be noted that we have introduced the condition $n > t$ in Conjecture 1.5 solely for convenience in the proofs in §2, and that it seems likely that the conjecture is true for any values of n and t with $n \geqslant 3$.

2. Proofs of Conjecture 1.5 for $1 \leqslant t \leqslant 4$

The case $t = 1$. We prove Conjecture 1.5 for $t = 1$. Let Γ^* be a graph on $\{\delta_1, \phi_1, \ldots, \phi_r\}$ coloured as in 1.5. We assume first that δ_1 is black. Writing $H = \operatorname{Aut}\Gamma_1^*$, condition (a) of Theorem 1.4 means that

(1) $H_{\delta_1} = 1$,

condition (b) is vacuously satisfied and condition (c) means that

(2) H contains no automorphism $(\delta_1 \phi_{i_1})$ with δ_1 joined to ϕ_{i_1}.

Suppose then that (1) and (2) hold and let Γ be the corresponding graph on $n + r$ vertices $\Delta_1 \cup \{\phi_1, \ldots, \phi_r\}$, where $\Delta_1 = \{\alpha_{11}, \ldots, \alpha_{1n}\}$ is K_n since δ_1 is black. Write $G = \operatorname{Aut}\Gamma$. We show that $G \cong S_n$.

Since $H_{\delta_1} = 1$ it is clear that $G_{\{\Delta_1\}}$ fixes $V\Gamma \setminus \Delta_1$ pointwise; thus $G_{\{\Delta_1\}} \cong S_n$. Suppose that there exists $g \in G \setminus G_{\{\Delta_1\}}$. Then $G_{\{\Delta_1 g\}}$ fixes $V\Gamma \setminus \Delta_1 g$ pointwise and $G_{\{\Delta_1 g\}} \cong S_n$, so if $\Delta_1 \cap \Delta_1 g = \varnothing$ then H_{δ_1} has a subgroup isomorphic to S_n, contradicting (1). Hence $\Delta_1 \cap \Delta_1 g \neq \varnothing$. Write $\Sigma = \Delta_1 \cup \Delta_1 g$. It is easy to see that $G_{\{\Sigma\}} = \langle G_{\{\Delta_1\}}, G_{\{\Delta_1 g\}} \rangle \cong \operatorname{Sym}(\Sigma)$ and that $G_{\{\Sigma\}}$ fixes $V\Gamma \setminus \Sigma$ pointwise.

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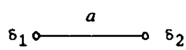
Consequently Σ is a complete subgraph of Γ and if we choose $\phi_{i_1} \in \Sigma \setminus \Delta_1$ then $(\alpha_{11}\phi_{i_1}) \in \operatorname{Aut}\Gamma$. This forces $(\delta_1\phi_{i_1}) \in \operatorname{Aut}\Gamma_1^*$, contradicting (2), as ϕ_{i_1} is joined to δ_1.

The case where δ_1 is white follows from the above argument by considering the complement of the corresponding graph Γ. Hence Conjecture 1.5 is proved for $t = 1$.

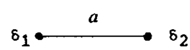
Descriptions of the graphs characterized by this result can be found in the Appendix.

The cases $2 \leqslant t \leqslant 4$. We prove Conjecture 1.5 just for $t = 4$, as the cases $t = 2$ and $t = 3$ are similar and easier. In the proof we shall need, for $2 \leqslant u \leqslant 4$ and $n > u$, a description of all coloured graphs Γ^* on $\{\delta_1, \ldots, \delta_u\}$ which give rise as in §1 to vertex-transitive graphs Γ on un vertices. We call Γ^* *vertex-monochrome* if all the vertices δ_i have the same colour.

$\underline{u = 2}$. The only graphs Γ^* on $\{\delta_1, \delta_2\}$ which give rise to a transitive graph Γ are

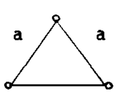


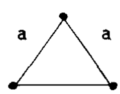
and



where a is 0, 1, $n - 1$ or n (that is, Γ^* is any vertex-monochrome graph on $\{\delta_1, \delta_2\}$).

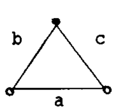
$\underline{u = 3}$. Suppose that Γ^* on $\{\delta_1, \delta_2, \delta_3\}$ gives rise to a transitive graph Γ. If Γ^* is vertex-monochrome then by the regularity of Γ it must be one of the following graphs:





$(a \in \{0, 1, n - 1, n\})$.

There are no further such graphs Γ^*. For suppose that Γ^* is not vertex-monochrome. Then we may take Γ^* to be



where $a, b, c \in \{0, 1, n - 1, n\}$. Since Γ is regular we have

$$a + b = a + c = b + c + n - 1$$

so that $b = c$ and $a = c + n - 1$. Thus c is 0 or 1, which forces Δ_3 to be the unique subgraph K_n of Γ, contradicting the transitivity of Γ.

$\underline{u = 4}$. Suppose that Γ^* on $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ gives rise to a transitive graph Γ. Let $a_{ij} \in \{0, 1, n - 1, n\}$ be the colour of the edge joining δ_i and δ_j. If Γ^* is vertex-monochrome then since Γ is regular of valency b, say, we have

$$a_{12} + a_{13} + a_{14} = a_{12} + a_{23} + a_{24} = a_{13} + a_{23} + a_{34} = a_{14} + a_{24} + a_{34} = b.$$

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This gives $a_{14} = a_{23}$, $a_{13} = a_{24}$, $a_{12} = a_{34}$, so $\operatorname{Aut}\Gamma^*$ contains the subgroup $V_4 = \langle(\delta_1\delta_2)(\delta_3\delta_4), (\delta_1\delta_3)(\delta_2\delta_4)\rangle$. If δ_1 is black and δ_2, δ_3, δ_4 are white then the regularity of Γ gives $2(a_{23} - a_{14}) = n - 1$ which is not possible since $n > u = 4$. And if δ_1, δ_2 are black and δ_3, δ_4 are white then it is easy to see that any vertex in $\Delta_1 \cup \Delta_2$ is contained in more subgraphs K_n of Γ than any vertex in $\Delta_3 \cup \Delta_4$, contradicting the transitivity of Γ.

We summarise these results in a lemma:

LEMMA 2.1. *Let* Γ^* *be a coloured graph on* $\{\delta_1, \ldots, \delta_u\}$ $(u \leqslant 4)$ *which gives rise as in* §1 *to a transitive graph* Γ *on un vertices* $(n > u)$. *Then* Γ^* *is vertex-monochrome and* $\operatorname{Aut}\Gamma^*$ *contains a subgroup* S, *where* $S = S_2$ *if* $u = 2$, $S = S_3$ *if* $u = 3$ *and* $S = V_4$ *if* $u = 4$.

PROOF OF CONJECTURE 1.5 FOR $t = 4$. Let Γ^* be a graph on $\{\delta_1, \delta_2, \delta_3, \delta_4, \phi_1, \ldots, \phi_r\}$ coloured as in 1.5 and suppose that (a), (b) and (c) of Theorem 1.4 hold. Let n be an integer with $n > t = 4$ and let Γ be the corresponding graph on $4n + r$ vertices $\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \cup \{\phi_1, \ldots, \phi_r\}$ (where $\Delta_i = \{\alpha_{i1}, \ldots, \alpha_{in}\}$ for $i = 1, \ldots, 4$). Write $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$ and $\Phi = \{\phi_1, \ldots, \phi_r\}$, and let Δ^* be the subgraph of Γ^* on $\{\delta_1, \delta_2, \delta_3, \delta_4\}$. Also write $\phi_i \sim \delta_j$ if ϕ_i is joined to δ_j in Γ^*. Put $G = \operatorname{Aut}\Gamma$. By the construction of Γ from Γ^* (explained in §1) it is clear that G has a unique subgroup $H \cong S_n$ having orbits Δ_1, Δ_2, Δ_3, Δ_4, fixing each ϕ_i, and such that $H_{\alpha_{ij}} = H_{\alpha_{ik}}$ for all i, j, k. We aim to show that $G_{\{\Delta\}} = H$, which we establish in the following two lemmas.

LEMMA 2.2. *If* $g \in G_{\{\Delta\}} \setminus H$ *then* $\Delta_i g \neq \Delta_i$ *for some* $i \in \{1, 2, 3, 4\}$.

PROOF. Suppose that $\Delta_i g = \Delta_i$ for all i. Then clearly $g^{\Phi} 1^{\Delta^*} \in \operatorname{Aut}\Gamma^*$, so $g^{\Phi} = 1$ by (a) of 1.4. Now $H^{\Delta_1} \cong S_n$, so $g^{\Delta_1} = h^{\Delta_1}$ for some $h \in H$. Then $g^{-1}h$ fixes $\Delta_1 \cup \Phi$ pointwise and $g^{-1}h \neq 1$ as $g \notin H$. Hence the sets

$$\Delta' = \bigcup\left\{\Delta_i \mid (g^{-1}h)^{\Delta_i} = 1\right\}, \qquad \Delta'' = \bigcup\left\{\Delta_i \mid (g^{-1}h)^{\Delta_1} \neq 1\right\}$$

are both nonempty. Let $K = \langle (g^{-1}h)^x \mid x \in H\rangle$. Then $K^{\Delta'} = 1$ and for $\Delta_i \subseteq \Delta''$ we have $K^{\Delta_i} \geqslant \operatorname{Alt}(\Delta_i)$ since $K^{\Delta_i} \triangleleft \operatorname{Sym}(\Delta_i)$. Hence for any $\Delta_i \subseteq \Delta''$ and any $\alpha_{jk} \in \Delta'$, α_{jk} is joined to all or no vertices of Δ_i. Thus in Γ^*, any edge between a vertex of $\{\delta_i \mid \Delta_i \subseteq \Delta'\}$ and a vertex of $\{\delta_i \mid \Delta_i \subseteq \Delta''\}$ must be coloured 0 or n. This forces $\{\delta_i \mid \Delta_i \subseteq \Delta'\}$ to be a union of connected components of the graph Γ_0^*, contradicting (b) of 1.4.

LEMMA 2.3. *We have* $G_{\{\Delta\}} = H$.

PROOF. Suppose false and pick $g \in G_{\{\Delta\}} \setminus H$. By Lemma 2.2 we have $\Delta_i g \neq \Delta_i$ for some i, so if $L = \langle H, g\rangle$ then L has at most 3 orbits on Δ. We prove the lemma by obtaining a contradiction to the fact that $\operatorname{Aut}\Gamma^* = 1$. There are several cases, depending on the number of orbits of L on Δ.

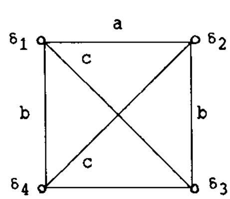
Case 1. *L is transitive on* Δ. For any $i, j \in \{1,2,3,4\}$ write $\Delta_i \to \Delta_j$ if there exist $\alpha_{ik} \in \Delta_i$, $\alpha_{jl} \in \Delta_j$ with $\alpha_{ik} g = \alpha_{jl}$. For distinct $i_1, \ldots, i_u \in \{1,2,3,4\}$ $(1 \leqslant u \leqslant 4)$ write $[\Delta_{i_1} \cdots \Delta_{i_u}]$ to mean that $\Delta_{i_1} \to \Delta_{i_2}, \Delta_{i_2} \to \Delta_{i_3}, \ldots, \Delta_{i_u} \to \Delta_{i_1}$.

Now H has orbits Δ_1, Δ_2, Δ_3, Δ_4 on Δ and $L = \langle H, g\rangle$ is transitive on Δ; it is not hard to see from this that we may assume that one of the following holds

(i) $[\Delta_1\Delta_2\Delta_3\Delta_4]$;
(ii) $[\Delta_1\Delta_2]$, $[\Delta_1\Delta_3]$ and $[\Delta_1\Delta_4]$;
(iii) $[\Delta_1\Delta_2]$, $[\Delta_1\Delta_3]$ and $[\Delta_2\Delta_4]$;
(iv) $[\Delta_1\Delta_2\Delta_3]$ and $[\Delta_1\Delta_4]$;
(v) $[\Delta_1\Delta_2\Delta_3]$ and $[\Delta_1\Delta_2\Delta_4]$.

Suppose that (i) holds. Then $\alpha_{1i_1} g = \alpha_{2i_2}$, $\alpha_{2j_2} g = \alpha_{3j_3}$, $\alpha_{3k_3} g = \alpha_{4k_4}$, $\alpha_{4l_4} g = \alpha_{1l_1}$ for some i_1, i_2, etc. Choose $\phi_a \in \Phi$. If ϕ_a is joined to α_{1i_1} then $\phi_a g$ is joined to α_{2i_2}, hence to every vertex in Δ_2, so $\phi_a g^2$ is joined to α_{3j_3}. Thus $\phi_a \sim \delta_1 \Rightarrow \phi_a g^2 \sim \delta_3$. In this way we see that $\phi_a \sim \delta_1 \Rightarrow \phi_a g^2 \sim \delta_3$, $\phi_a \sim \delta_2 \Rightarrow \phi_a g^2 \sim \delta_4$, $\phi_a \sim \delta_3 \Rightarrow \phi_a g^2 \sim \delta_1$ and $\phi_a \sim \delta_4 \Rightarrow \phi_a g^2 \sim \delta_2$. Also by Lemma 2.1 we have $(\delta_1\delta_3)(\delta_2\delta_4) \in \operatorname{Aut}\Delta^*$. It follows that $(g^2)^{\Phi}(\delta_1\delta_3)(\delta_2\delta_4) \in \operatorname{Aut}\Gamma^*$, contradicting the fact that $\operatorname{Aut}\Gamma^* = 1$.

If (ii) holds then for any $a \in \{1, \ldots, r\}$ we have $\phi_a \sim \delta_2 \Leftrightarrow \phi_a g \sim \delta_1 \Leftrightarrow \phi_a \sim \delta_3 \Leftrightarrow \phi_a \sim \delta_4$. Hence any permutation of $\{\delta_2, \delta_3, \delta_4\}$ fixing δ_1 and each ϕ_a will be an automorphism of Γ^* providing it is an automorphism of the subgraph Δ^*. By Lemma 2.1 we can take Δ^* to be



where $a, b, c \in \{0, 1, n-1, n\}$. If $a = b$ then $(\delta_2\delta_4) \in \operatorname{Aut}\Gamma^*$, if $b = c$ then $(\delta_3\delta_4) \in \operatorname{Aut}\Gamma^*$ and if $a = c$ then $(\delta_2\delta_3) \in \operatorname{Aut}\Gamma^*$, all of which are contradictions. Hence, a, b, c are distinct and we may assume that either $a = 0$, $b = 1$, $c \geqslant n-1$ or $a \leqslant 1$, $b = n-1$, $c = n$. Write $m(\alpha_{ij}, \alpha_{kl})$ for the number of mutual adjacencies of α_{ij} and α_{kl} in the subgraph Δ of Γ. Then for any i, j, k, l

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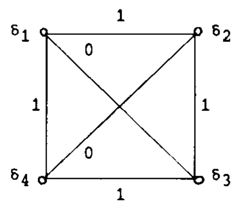
we have $m(\alpha_{1_i}, \alpha_{1_j}) \geqslant n - 2$ and $m(\alpha_{2_k}, \alpha_{3_l}) \leqslant 2$. However, by assumption (we are in case (ii)) there exist i, j, k, l such that $\alpha_{1_i}g = \alpha_{2_k}$, $\alpha_{1_j}g = \alpha_{3_l}$, which forces $m(\alpha_{1_i}, \alpha_{1_j}) = m(\alpha_{2_k}, \alpha_{3_l})$; hence $n - 2 \leqslant 2$ or $n \leqslant 4$, contradicting the fact that $n > t = 4$.

In case (iii) we have $\phi_a \sim \delta_1 \Leftrightarrow \phi_a g \sim \delta_2 \Leftrightarrow \phi_a \sim \delta_4$ and $\phi_a \sim \delta_2 \Leftrightarrow \phi_a g \sim \delta_1 \Leftrightarrow \phi_a \sim \delta_3$. Hence $(\delta_1\delta_4)(\delta_2\delta_3) \in \operatorname{Aut}\Gamma^*$ which is a contradiction.

In case (iv) we see similarly that $(\delta_1\delta_2)(\delta_3\delta_4) \in \operatorname{Aut}\Gamma^*$, again a contradiction.

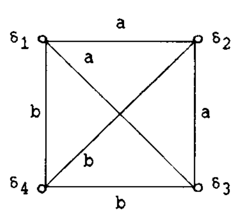
Finally, suppose that (v) holds. Then $\phi_a \sim \delta_3 \Leftrightarrow \phi_a g \sim \delta_1 \Leftrightarrow \phi_a \sim \delta_4$ so $(\delta_3\delta_4) \in \operatorname{Aut}\Gamma^*$ if $b = c$ in the subgraph Δ^*. Thus $b \neq c$. Suppose first that $c \leqslant 1$. Then $m(\alpha_{3_i}, \alpha_{4_j}) \leqslant 2$ for any i, j, so by application of g^{-1} we see that $m(\alpha_{1_k}, \alpha_{1_l}) \leqslant 2$ for any distinct k, l. This forces $a \leqslant 1$ and $b \leqslant 1$. Since $b \neq c$ we may take $b = 1$, $c = 0$; as Γ_0^* is connected we have $a = 1$ and Δ^* is



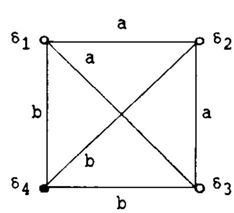
Thus the subgraph Δ of Γ consists of n disjoint squares. Now $\alpha_{1_i}g = \alpha_{2_j}$ for some i, j. Since α_{3_i} is the unique vertex of Δ opposite to α_{1_i} in the square containing α_{1_i} and α_{4_j} is similarly opposite to α_{2_j}, we must have $\alpha_{3_i}g = \alpha_{4_j}$. In this way we see that $[\Delta_3\Delta_4\Delta_1]$ and $[\Delta_3\Delta_4\Delta_2]$ also hold. The usual argument now shows that $\phi_a \sim \delta_1 \Leftrightarrow \phi_a \sim \delta_2 \Leftrightarrow \phi_a \sim \delta_3 \Leftrightarrow \phi_a \sim \delta_4$ so that $V_4 \leqslant \operatorname{Aut}\Gamma^*$, which is a contradiction. Similar arguments yield a contradiction if $c \geqslant n - 1$.

We have now dealt completely with Case 1.

Case 2. *L has orbits $\Delta_1 \cup \Delta_2 \cup \Delta_3$ and Δ_4 on Δ.* In this case we may assume that either (i) $[\Delta_1\Delta_2\Delta_3]$, or (ii) $[\Delta_1\Delta_2]$ and $[\Delta_1\Delta_3]$ holds. Using Lemma 2.2 for $u = 3$ and the fact that each vertex in $\Delta_1 \cup \Delta_2 \cup \Delta_3$ has the same valency, we see that the subgraph Δ^* can be taken to be one of



and



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for some $a, b \in \{0, 1, n-1, n\}$. If (i) holds then $g^{\Phi}(\delta_1\delta_2\delta_3) \in \operatorname{Aut}\Gamma^*$, while in case (ii) we have $\phi_a \sim \delta_2 \Leftrightarrow \phi_a g \sim \delta_1 \Leftrightarrow \phi_a \sim \delta_3$ so that $(\delta_2\delta_3) \in \operatorname{Aut}\Gamma^*$. These contradictions deal with Case 2.

Case 3. *L has orbits $\Delta_1 \cup \Delta_2$ and $\Delta_3 \cup \Delta_4$ or $\Delta_1 \cup \Delta_2$, Δ_3 and Δ_4 on Δ.* Then either (i) $[\Delta_1\Delta_2]$ and $[\Delta_3\Delta_4]$, or (ii) $[\Delta_1\Delta_2]$, $[\Delta_3]$ and $[\Delta_4]$, holds. In case (i) we have $g^{\Phi}(\delta_1\delta_2)(\delta_3\delta_4) \in \operatorname{Aut}\Gamma^*$ and in case (ii), $g^{\Phi}(\delta_1\delta_2) \in \operatorname{Aut}\Gamma^*$, neither of which can be so.

This completes the proof of Lemma 2.3.

To finish the proof of Conjecture 1.5 for $t = 4$ it remains to show that $G = G_{\{\Delta\}}$. Suppose then that there exists $g \in G \setminus G_{\{\Delta\}}$. Put $M = \langle G_{\{\Delta\}}, G^g_{\{\Delta\}} \rangle = \langle H, H^g \rangle$ and let $\Psi_1, \ldots, \Psi_s$ be the orbits of M on $\Delta \cup \Delta g$. For each i let $X_i = \{ j \mid \Delta_j \subseteq \Psi_i \}$.

LEMMA 2.4. *If $X_i \neq \varnothing$ then there is a block system $\mathscr{B}_i$ for M^{Ψ_i} (possibly with blocks of size 1), one of whose blocks B_i is contained in $\{ \alpha_{j1} \mid j \in X_i \}$, and such that $M^{\mathscr{B}_i} \geqslant \operatorname{Alt}(\mathscr{B}_i)$.*

PROOF. Pick $k \in X_i$, so that $\Delta_k \subseteq \Psi_i$. The lemma is certainly true if $\Delta_k = \Psi_i$ (for then we take $\mathscr{B}_i = \Psi_i$, that is, $\mathscr{B}_i$ to be the set of blocks of size 1); hence we may assume that $\Delta_k \subset \Psi_i$. Let $\mathscr{B}_i$ be a block system for M^{Ψ_i} such that $|\mathscr{B}_i| > 1$ and $\mathscr{B}_i$ contains blocks of maximum possible size. Then $M^{\mathscr{B}_i}$ is primitive. Let B_i be the block of $\mathscr{B}_i$ containing α_{k1}. Certainly either $\Delta_k \subseteq B_i$ or $\Delta_k \cap B_i = \{\alpha_{k1}\}$; we show that the latter must hold. Suppose then that $\Delta_k \subseteq B_i$. From the action of H we have

(1) $B_i \cap \Delta$ is a union of H-orbits Δ_j.

Next we show that

(2) $B_i \cap \Delta g$ is a union of H^g-orbits $\Delta_j g$.

We prove this as follows: if $|\Delta_k \cap \Delta_l g| \leqslant 1$ for all l then since $n > t$ there exists $\alpha_{km} \in \Delta_k \setminus \Delta g$, so that H^g fixes α_{km}. Now $\Delta_k \subset \Psi_i$ so we can find l such that $\Delta_k \cap \Delta_l g \neq \varnothing$. Also $\Delta_k \subseteq B_i$, so $B_i \cap \Delta_l g \neq \varnothing$. Since H^g fixes α_{km} this forces $\Delta_l g \subseteq B_i$. The action of H^g now gives (2). If $|\Delta_k \cap \Delta_l g| \geqslant 2$ for some l then $\Delta_l g \subseteq B_i$ again, from which (2) follows as before. Hence (2) is established. Now $M = \langle H, H^g \rangle$ and Ψ_i is a union of sets Δ_j and $\Delta_j g$ on which M is transitive. It follows from (1) and (2) that $B_i = \Psi_i$, contradicting the fact that $|\mathscr{B}_i| > 1$.

Thus we have shown that $\Delta_k \cap B_i = \{\alpha_{k1}\}$. Since $H_{\alpha_{k1}} = H_{\alpha_{j1}}$ for all j, it follows that $B_i \subseteq \{ \alpha_{j1} \mid j \in X_i \}$. Finally, $M^{\mathscr{B}_i}$ is primitive and contains the subgroup $H^{\mathscr{B}_i} \cong S_n$, so $M^{\mathscr{B}_i}$ contains an element of degree at most 8. From this it follows without much difficulty that $M^{\mathscr{B}_i} \geqslant \operatorname{Alt}(\mathscr{B}_i)$ (see for instance the papers of W. A. Manning referred to at the end of §15 of [4]).

If $X_i = \varnothing$ then $\Psi_i = \Delta_j g$ for some j and $M^{\Psi_i} = \mathrm{Sym}(\Psi_i)$. Put $\mathscr{B}_i = \Psi_i$ in this case (that is, let $\mathscr{B}_i$ be the set of blocks of size 1). Choose notation so that $X_i \neq \varnothing$ for $i = 1, \ldots, s_0$ and $X_i = \varnothing$ for $i = s_0 + 1, \ldots, s$. For $i \in \{1, \ldots, s_0\}$ let $\mathscr{B}_i'$ be the set of blocks of $\mathscr{B}_i$ contained in Δ. Then $|\mathscr{B}_i'| = r_i n$ for some positive integer r_i. Write $\mathscr{B} = \bigcup_{i=1}^{s} \mathscr{B}_i$.

LEMMA 2.5. *The following hold*:
(i) $s = s_0$;
(ii) $|\mathscr{B}_j| = |\mathscr{B}_k|$ *for all* $j, k \in \{1, \ldots, s\}$;
(iii) *if* $|\mathscr{B}_1| = b$ *then* $M \cong A_b$ *or* $M \cong S_b$ *and* M *acts similarly on each* $\mathscr{B}_j$ $(j = 1, \ldots, s)$.

PROOF. If K is the kernel of the action of M on $\mathscr{B}$ then $K \leqslant G_{\{\Delta\}}$, so $K = 1$ since $G_{\{\Delta\}} = H$. Hence M acts faithfully on $\mathscr{B}$. Write $N = M'$. Then by Lemma 2.4, N is a subdirect product of $\prod_{i=1}^{s} \mathrm{Alt}(\mathscr{B}_i)$ (that is, N projects surjectively onto each factor). Since each $\mathrm{Alt}(\mathscr{B}_i)$ is simple, N is isomorphic to a direct product of some of the groups $\mathrm{Alt}(\mathscr{B}_i)$ and if we choose i_0 such that $|\mathscr{B}_{i_0}| = \max\{|\mathscr{B}_i| \colon i = 1, \ldots, s\}$ then N has a minimal normal subgroup $N_0 \cong \mathrm{Alt}(\mathscr{B}_{i_0})$. Now $g \notin G_{\{\Delta\}}$, so $H^g \neq H$ and so $M \nleqslant G_{\{\Delta\}}$. Consequently $|\mathscr{B}_{i_0}| > n$. Hence if $X_i = \varnothing$ then $|\mathscr{B}_{i_0}| > |\mathscr{B}_i|$. Let

$$J = \left\{ j \mid N_0^{\mathscr{B}_j} = \mathrm{Alt}\left(\mathscr{B}_j\right) \right\} \quad \text{and} \quad \mathscr{B}_0 = \bigcup_{j \in J} \mathscr{B}_j.$$

Then $J \subseteq \{1, \ldots, s_0\}$, $|\mathscr{B}_j| = |\mathscr{B}_{i_0}|$ for all $j \in J$ and N_0 fixes $\mathscr{B} \setminus \mathscr{B}_0$ pointwise.

Write $H_0 = H'$; then $H_0 \cong A_n$ and $H_0^{\mathscr{B}_0} \leqslant N_0^{\mathscr{B}_0}$. It follows that N_0 acts similarly on all $\mathscr{B}_j$ $(j \in J)$ (whether $|\mathscr{B}_{i_0}| = 6$ or not), and hence that N_0 contains a nontrivial element x fixing each $\mathscr{B}_j'$ setwise ($J \in J$). Then x fixes $\mathscr{B} \setminus \mathscr{B}_0$ pointwise, so $x \in G_{\{\Delta\}}$ and so $x \in H$. This forces $J = \{1, \ldots, s_0\}$. If $s > s_0$ then N has a subgroup $L \cong A_n$ fixing $\bigcup_{i=1}^{s_0} \mathscr{B}_i$ pointwise; clearly $L \leqslant G_{\{\Delta\}}$, which is not possible as $G_{\{\Delta\}} = H$. Thus $s = s_0$, $J = \{1, \ldots, s\}$, $N = N_0$ and the lemma follows.

LEMMA 2.6. *We have* $|\mathscr{B}_i'| = n$, *that is*, $r_i = 1$ *for all* i.

PROOF. By Lemma 2.5, M has a subgroup N_1 fixing eacyh $\mathscr{B}_i'$ setwise and such that $N_1^{\mathscr{B}_i'} \geqslant \mathrm{Alt}(\mathscr{B}_i')$ $(i = 1, \ldots, s)$. Clearly $N_1 \leqslant G_{\{\Delta\}}$. Since $G_{\{\Delta\}} = H \cong S_n$ this forces $|\mathscr{B}_i'| = n$, that is, $r_i = 1$, for all i.

We can now complete the proof of Conjecture 1.5 for $t = 4$. First note that from the proof of Lemma 2.5, we have $M \nleqslant G_{\{\Delta\}}$. Hence there exists k such that

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$X_k \neq \varnothing$ and $\Psi_k \not\subseteq \Delta$ (equivalently $\mathscr{B}'_k \neq \mathscr{B}_k$). By Lemma 2.6 we have $|\mathscr{B}_k| = n + c$ where $c > 0$ is the number of blocks in $\mathscr{B}_k \setminus \mathscr{B}'_k$. Thus by Lemma 2.5, $X_i \neq \varnothing$, $|\mathscr{B}_i| = n + c$ and $M^{\mathscr{B}_i} \cong S_{n+c}$ $(i = 1, \ldots, s)$. Finally, choose $B'_1 \in \mathscr{B}_1 \setminus \mathscr{B}'_1$. There exists $m \in M$ with $m^{\mathscr{B}_i} = (B_1 B'_1)$. By Lemma 2.5, M acts similarly on all $\mathscr{B}_i$, so $m^{\mathscr{B}_i} = (B_i B'_i)$ for some $B'_i \in \mathscr{B}_i \setminus \mathscr{B}'_i$ $(i = 1, \ldots, s)$. Since the kernel of the action of M on $\mathscr{B}$ is trivial, we have $m^2 = 1$. Hence $m = (\alpha_{11}\phi_{i_1})(\alpha_{21}\phi_{i_2}) \cdots (\alpha_{t1}\phi_{i_t})$ for some $\phi_{i_j} \in \Phi$ $(j = 1, \ldots, t)$. From this it follows that $(\delta_1\phi_{i_1}) \cdots (\delta_t\phi_{i_t}) \in \operatorname{Aut}\Gamma_1^*$ and that for any distinct k, l the subgraph $\{\delta_k, \delta_l, \phi_{i_k}, \phi_{i_l}\}$ of Γ^* lies in the set $\mathscr{C}_n$ of 12 graphs defined in §1. This contradicts (c) of Theorem 1.4.

This completes the proof of Conjecture 1.5 for $t = 4$.

We summarise the results proved in this section:

THEOREM 2.7. *Let n, t be integers with $1 \leqslant t \leqslant 4$ and $n > t$, and let Γ^* be a graph on $\{\delta_1, \ldots, \delta_t, \phi_1, \ldots, \phi_r\}$ coloured as described in Conjecture* 1.5. *Suppose that* (a), (b) *and* (c) *of Theorem* 1.4 *are satisfied. Then if Γ is the corresponding graph on $tn + r$ vertices, we have* $\operatorname{Aut}\Gamma \cong S_n$.

The results 1.2, 1.4 and 2.7 give a description of all graphs Γ on v vertices with $\operatorname{Aut}\Gamma \cong S_n$ $(n > 6)$ and $v < \min\{5n, \frac{1}{2}n(n-1)\}$. This description is illustrated below in the Appendix. For values of n with $n \leqslant 6$ there are some extra possibilities which can easily be determined using the techniques of this paper.

REMARKS. 1. The restriction $n > t$ in Theorem 2.7 is in fact unnecessary—it is not hard to show that the result is true for any n, t with $1 \leqslant t \leqslant 4$, $n \geqslant 3$.

2. The obstacle to a general proof of Conjecture 1.5 seems to lie solely in proving Lemma 2.3 in the general case; the subsequent steps of the proof for $t = 4$ do not depend on the value of t and would remain largely unchanged in the general case.

3. The methods of this paper could be used to study graphs with automorphism group S_n having some orbit sizes greater than $\frac{1}{2}n(n-1)$. For example, suppose that we only restrict all orbits to have size less than $n(n-1)(n-2)/6$. Then for n large enough, the proofs of Propositions 1.1 and 1.2 show that all orbits have size 1, n or $\frac{1}{2}n(n-1)$ (with the action of S_n in the latter case being that on the set of pairs of points in an underlying set of size n). There are four possible subgraphs on an orbit of size $\frac{1}{2}n(n-1)$: these are the complete graph $K_{\frac{1}{2}n(n-1)}$, the triangular graph T_n and their complements. By introducing a suitable collection of colours to represent these subgraphs and the edges between them, we can proceed in similar fashion to §1.

Appendix

In this Appendix we give descriptions of some of the graphs characterized by Theorems 1.4 and 2.7. In particular we describe all graphs Γ with $\mathrm{Aut}\,\Gamma \cong S_n$ and $|V\Gamma| \leqslant 3n$ (with $n > 6$). The reader will have no difficulty in extending these descriptions to cover all graphs Γ with $\mathrm{Aut}\,\Gamma \cong S_n$ and

$$|V\Gamma| < \min\{5n, \tfrac{1}{2}n(n-1)\}.$$

It is unfortunately necessary to introduce some fairly complicated notation for these descriptions, so we include a number of small examples for illustration.

Throughout this Appendix, Γ denotes a graph on v vertices. For any n, let

$$\mathscr{F}_{v,n} = \{\Gamma \mid \mathrm{Aut}\,\Gamma \cong S_n\}.$$

For any t, r, n with $n > t$ define

$$\mathscr{G}_{t,r,n} = \{\Gamma \mid v = tn + r,\ \mathrm{Aut}\,\Gamma \cong S_n \text{ has } t \text{ orbits of size } n \text{ and } r \text{ fixed points on } V\Gamma\}.$$

Thus by Proposition 1.2, for $n > 6$ and $v < \frac{1}{2}n(n-1)$, we have

$$(1) \qquad \mathscr{F}_{v,n} = \bigcup_{1 \leqslant j \leqslant t} \mathscr{G}_{j,(t-j)n+r,n}$$

where $v = tn + r$ and $0 \leqslant r < n$. Thus to describe $\mathscr{F}_{v,n}$ we must describe the graphs in $\mathscr{G}_{t,r,n}$. This can be done for $t \leqslant 4$ using Theorems 1.4 and 2.7, and we now give such descriptions explicitly, starting with the simplest case $t = 1$. For convenience, if $\mathscr{S}$ is a set of graphs on v vertices, define

$$\mathscr{S}^* = \{\{\Gamma, \overline{\Gamma}\} \mid \Gamma \in \mathscr{S}\},$$

where $\overline{\Gamma}$ denotes the complement of Γ.

(A) *The case* $t = 1$. We describe $\mathscr{G}_{1,r,n}$. Let $\mathscr{H}_{1,r}$ be the set of (uncoloured) graphs Γ_1^* on $1 + r$ vertices $\{\delta_1, \phi_1, \ldots, \phi_r\}$ such that $H = \mathrm{Aut}\,\Gamma_1^*$ satisfies

(1) $H_{\delta_1} = 1$, and,

(2) H contains no element $(\delta_1 \phi_{i_1})$ with δ_1 joined to ϕ_{i_1}.

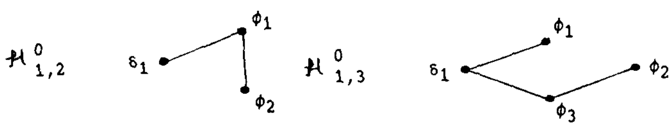
Each graph Γ_1^* in $\mathscr{H}_{1,r}$ corresponds as in Section 1 to a graph Γ on $n + r$ vertices as follows: Γ has vertex set $\Delta_1 \cup \{\phi_1, \ldots, \phi_r\}$, the subgraph Δ_1 is K_n, the subgraph $\{\phi_1, \ldots, \phi_r\}$ is as in Γ_1^*, and ϕ_i is joined to all or no vertices in Δ_1 according as ϕ_i is or is not joined to δ_1 in Γ_1^*. By Theorem 2.7 with $t = 1$, we have $\mathrm{Aut}\,\Gamma \cong S_n$, so that Γ is in $\mathscr{G}_{1,r,n}$.

Now the graphs in $\mathscr{G}_{1,r,n}$ are unlabelled, so we choose a subset $\mathscr{H}^0_{1,r}$ of $\mathscr{H}_{1,r}$ containing exactly one member of each orbit of $\mathrm{Sym}\{\phi_1, \ldots, \phi_r\}$ on $\mathscr{H}_{1,r}$. Then $\mathscr{G}^*_{1,r,n}$ is in 1-1 correspondence, as described above, with $\mathscr{H}^0_{1,r}$. We write this as

$$\mathscr{G}^*_{1,r,n} \leftrightarrow \mathscr{H}^0_{1,r}.$$

In particular, for $r < n$ we have by (1), $\mathcal{F}^*_{n+r,n} \leftrightarrow \mathcal{H}^0_{1,r}$. This gives the results of [3].

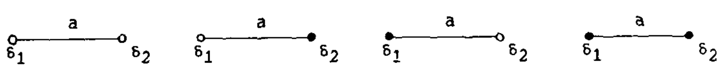
EXAMPLE. We illustrate this with $r = 2$ and $r = 3$. Each of $\mathcal{H}^0_{1,2}$ and $\mathcal{H}^0_{1,3}$ consists of just one graph:



Thus $\mathcal{G}_{1,2,n}$ and $\mathcal{G}_{1,3,n}$ consist, respectively, of the graphs

and their complements.

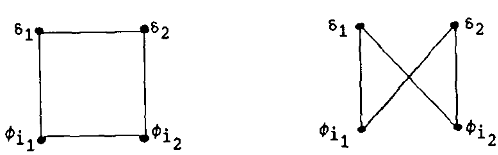
(B) *The case* $t = 2$. We now describe $\mathcal{G}_{2,r,n}$ $(n \geqslant 3)$. If $\Gamma \in \mathcal{G}_{2,r,n}$ and Γ^* is the corresponding coloured graph on $\{\delta_1, \delta_2, \phi_1, \ldots, \phi_r\}$ then by (b) of Theorem 1.4 the subgraph of Γ^* on δ_1, δ_2 is one of the following:



where a is 1 or $n - 1$. Let $\mathcal{H}_{2,r}$ be the set of (uncoloured) graphs Γ_1^* on $\{\delta_1, \delta_2, \phi_1, \ldots, \phi_r\}$ such that $H = \operatorname{Aut} \Gamma_1^*$ satisfies

(1) $H_{\{\delta_1\delta_2\}} = 1$, and

(2) H contains no element $(\delta_1\phi_{i_1})(\delta_2\phi_{i_2})$ such that the subgraph $\{\delta_1, \delta_2, \phi_{i_1}, \phi_{i_2}\}$ is one of



And let $\mathcal{H}'_{2,r}$ be the set of graphs Γ_1^* on $\{\delta_1, \delta_2, \phi_1, \ldots, \phi_r\}$ such that

(I) $H_{\delta_1\delta_2} = 1$, and

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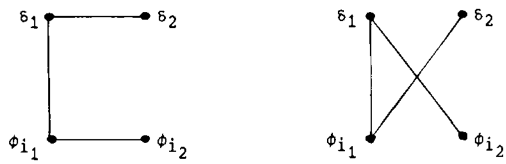
(II) H contains no element $(\delta_1\phi_{i_1})(\delta_2\phi_{i_2})$ such that the subgraph $\{\delta_1, \delta_2, \phi_{i_1}, \phi_{i_2}\}$ is one of



Choose a subset $\mathscr{H}_{2,r}^0$ of $\mathscr{H}_{2,r}$ containing exactly one member of each orbit of $\mathrm{Sym}\{\phi_1, \dots, \phi_r\} \times \mathrm{Sym}\{\delta_1, \delta_2\}$ on $\mathscr{H}_{2,r}$; and choose a subset $\mathscr{H}_{2,r}'^0$ containing exactly one member of each orbit of $\mathrm{Sym}\{\phi_1, \dots, \phi_r\}$ on $\mathscr{H}_{2,r}'$. Then each graph Γ_1^* in $\mathscr{H}_{2,r}^0$ corresponds to a unique graph Γ in $\mathscr{G}_{2,r,n}$ in which the subgraphs Δ_1 and Δ_2 are both K_n (and $a = 1$ if δ_1 and δ_2 are joined in Γ_1^*, $a = n - 1$ if not). And each Γ_1^* in $\mathscr{H}_{2,r}'^0$ corresponds to a unique graph Γ in $\mathscr{G}_{2,r,n}$ in which Δ_1 is K_n and Δ_2 is V_n; the complement $\bar{\Gamma}$ then corresponds to the graph $(\Gamma_1^*)^+$, which is the image of the complement $\bar{\Gamma}_1^*$ under the transposition $(\delta_1\delta_2)$. Hence if we write

$$\left(\mathscr{H}_{2,r}'^0\right)^+ = \left\{\left\{\Gamma_1^*, \left(\Gamma_1^*\right)^+\right\} \mid \Gamma_1^* \in \mathscr{H}_{2,r}'^0\right\}$$

then we have

$$\mathscr{G}_{2,r,n}^* \leftrightarrow \mathscr{H}_{2,r}^0 \cup \left(\mathscr{H}_{2,r}'^0\right)^+.$$

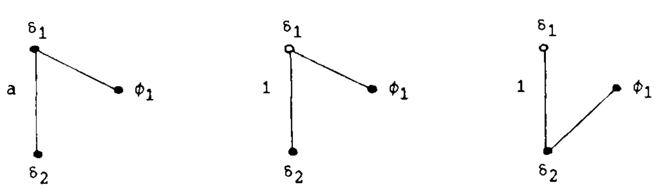
Note that if $v = 2n + r < 3n$ and $n > 6$, then by (1),

$$\mathscr{F}_{v,n} = \mathscr{G}_{1,n+r,n} \cup \mathscr{G}_{2,r,n}$$

so the description of $\mathscr{F}_{v,n}$ is given by (A) and the above.

EXAMPLE. We illustrate the above by producing the graphs in $\mathscr{G}_{2,0,n}$ and $\mathscr{G}_{2,1,n}$. Those in $\mathscr{G}_{2,0,n}$ correspond to the two coloured graphs

Thus $\mathscr{G}_{2,0,n}$ consists of the *corona* $K_n \circ K_1$ (which is K_n with each vertex joined to just one further vertex) and its complement. This answers a question raised in [3, §4]. The graphs in $\mathscr{G}_{2,1,n}$ are those corresponding to the coloured graphs



(where a is 1 or $n - 1$), together with their complements.

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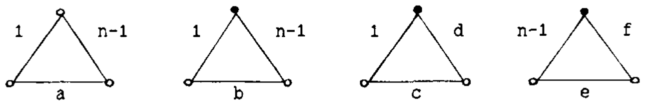
Descriptions similar to, but rather more complicated than those given in (A) and (B), exist for $t = 3$ and $t = 4$. We leave these to the reader, and offer just one further illustration.

(C) We describe $\mathscr{G}_{3,0,n}$ $(n \geqslant 4)$. By Theorems 1.4 and 2.7, $\mathscr{G}^*_{3,0,n}$ is in 1-1 correspondence with the following set $\mathscr{H}_{3,0}$ of coloured graphs:



(any $b \in \{0, 1, n-1, n\}$, $a, d, f \in \{0, n\}$, $c, e \in \{1, n-1\}$).

Hence for $n > 6$,

$$\begin{aligned}\mathscr{F}^*_{3n,n} &= \mathscr{G}^*_{1,2n,n} \cup \mathscr{G}^*_{2,n,n} \cup \mathscr{G}^*_{3,0,n} \\ &\leftrightarrow \mathscr{H}^0_{1,2n} \cup \mathscr{H}^0_{2,n} \cup \left(\mathscr{H}'^0_{2,n}\right)^+ \cup \mathscr{H}_{3,0}.\end{aligned}$$

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