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ON THE INPUT–OUTPUT STABILITY OF LINEAR CONTROLLABLE SYSTEMS

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1. **Introduction.** The extension of bounded input, bounded output criteria of Perron [1] for the case of a linear control system

$$(A, B) \qquad \dot{x}(t)=A(t)x(t)+B(t)f(t) \qquad \left(\dot{x}=\frac{dx}{dt}\right)$$

has been widely discussed (cf. e.g., references [2] to [10]).

As is well known ([2]) the equation

$$(A) \qquad \dot{x}(t)=A(t)x(t), \qquad t\geq 0$$

is *exponentially stable* iff there exist positive constant N and ν such that

$$\|U(t,s)\|\leq N\cdot e^{-\nu(t-s)}$$

for all $s\geq 0$ and for all $t\geq s$, where $U(t, s)$ is the evolution operator of the equation (A).

Throughout in this paper we shall assume that a constant $K>0$ exists such that

$$(1.1) \qquad \|A(t)\|\leq K, \qquad \|B(t)\|\leq K$$

for all $t\geq 0$.

A system satisfying (1.1) will be termed a *bounded system.*

Following [2], we say that the system (A, B) is (*zero-state*) *bounded-input, bounded-output* stable iff for every bounded input f there exists a constant N such that the output $x(t, t_0; 0; f)$ satisfies the condition

$$\|x(t, t_0; 0; f)\|\leq N, \quad \text{for all} \quad t\geq t_0\geq 0.$$

We denote that the bound N may depend on f.

For each system (A, B), we associate the self-adjoint positive operators

$$W(t,s)=\int_s^t U(s,\tau)B(\tau)B(\tau)^*U(s,\tau)^*\,d\tau$$

for $t\geq s\geq 0$.

Recall (see [5], [8]) that a bounded linear control system (A, B) is *uniformly*

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completely controllable (and we write u.c.c.) iff there exist $\delta>0$ and $m>0$ such that

$$W(s+\delta, s)\geq m\cdot I \quad \text{for all} \quad s\geq 0$$

(where I denotes the identity operator).

The B.I.B.O. stability of bounded finite dimensional systems is characterized by the following:

THEOREM 1.1. *The bounded system* (A, B) *is zero-state* B.I.B.O. *stable if and only if there exists a number* $M>0$ *such that*

$$\int_{t_0}^{t} \|U(t, s)B(s)\|\, ds\leq M \tag{1.2}$$

for every $t\geq t_0\geq 0$.

For the proof see [2] pp. 385–386 or [3] pp. 194–195. By Theorem 1.1 and a Theorem of Silverman and Anderson ([5] Theorem 3) we have

THEOREM 1.2. *If* (A, B) *is bounded and uniformly completely controllable then it is zero state* B.I.B.O. *stable if and only if the equation* (A) *is exponentially stable.*

This paper contains two major results. The first one is to prove that the Theorem 1.1 is not true in the case of infinite dimensional systems. This proves that the use of the finite dimensional method for the proof of the Theorem 1.2 (see [5]) is not applicable in infinite dimensional case. The second is to introduce a new concept of input–output stability ((L_a^p, L_b^q) stability) and to give a large class of linear systems (A, B) for which the (L_a^p, L_b^q) stability implies the exponential stability of (A).

2. **Definitions, notations, and preliminary results.** Let X be a separable Hilbert space and consider the time-varying linear system

$$\dot{x}(t)=A(t)x(t)+B(t)f(t), \qquad t\geq 0, \tag{A, B}$$

where $A\in L^1_{\text{loc}}(R_+, L(X))$, $B\in \tau(R_+, L(Y, X))$, $f\in F\subset L^1_{\text{loc}}(R_+, Y)$ in which $R_+=[0,\infty)$, $L(Y, X)$—the Banach space of the bounded linear operators from the Banach space Y to X (particularly $L(X, X)=L(X)$), $L^1_{\text{loc}}(R_+, Z)$—the space of Z—valued functions f defined almost everywhere on R_+, such that f is strongly measurable and locally Bochner integrable and, finally $\tau(R_+, L(Y, X))$—the space of continuous operator valued functions from R_+ to $L(Y, X)$.

By the above hypotheses, the solution of (A, B) with initial data $x(t_0)=x_0$ is ([10])

$$x(t, t_0; x_0; f)=U(t, t_0)x_0+\int_{t_0}^{t} U(t, s)B(s)f(s)\, ds$$

where $U(t, t_0)$ is the solution of Cauchy problem

$$\begin{cases} \dot{X}(t) = A(t)X(t) \\ X(t_0) = I. \end{cases}$$

As a preliminary, we prove the following lemma which gives a characterization of uniform complete controllability.

LEMMA 2.1. *The system* (A, B) *is* u.c.c. *if and only if there exist* $\delta > 0$ *and* $k > 0$ *such that for every* $x \in X$ *and for any time* $s \geq 0$ *there exists an input* f *such that*

(i) $\|f(t)\| \leq k \cdot \|x\|$ for $t \in [s, s+\delta]$ and

(ii) $x = \displaystyle\int_s^{s+\delta} U(s, \tau)B(\tau)f(\tau)\, d\tau.$

Proof. If (A, B) is u.c.c. then $W(s+\delta, s)$ are invertible operators and

$$\|W(s+\delta, s)^{-1}\| \leq 1/m \quad \text{for every} \quad s \geq 0.$$

Let $x \in X$ and $s \geq 0$. Define

$$f(t) = \begin{cases} -B(t)^* U(s, t)^* W(s+\delta, s)^{-1} x, & \text{if} \quad t \in [s, s+\delta] \\ 0, & \text{if} \quad t \notin [s, s+\delta] \end{cases}$$

It is clear that

$$\|f(t)\| \leq k \cdot \|x\|, \quad \text{where} \quad k = Ke^{K\delta}/m$$

and

$$x(s+\delta, s; x; f) = U(s+\delta, s)(x - W(s+\delta, s)W(s+\delta, s)^{-1}x) = 0$$

which implies that

$$x = \int_s^{s+\delta} U(s, \tau)B(\tau)f(\tau)\, d\tau.$$

The converse implication will be established by contradiction. If (A, B) is not u.c.c. then for each $\delta > 0$ and for any $\varepsilon > 0$ there exists a vector $x_0 \in X$ with $\|x_0\| = 1$ such that

$$\langle W(s_0+\delta, s_0)x_0, x_0 \rangle = \int_{s_0}^{s_0+\delta} \|B(\tau)^* U(s_0, \tau)^* x_0\|^2\, d\tau < \varepsilon$$

for some $s_0 \geq 0$.

Assume that (i) and (ii) hold. Then for x_0 there exists an input f_0 such that $\|f_0(t)\| \leq k_0$ for $t \in [s_0, s_0+\delta]$ and

$$x_0 = \int_{s_0}^{s_0+\delta} U(s_0, \tau)B(\tau)f_0(\tau)\, d\tau.$$

By the Schwarz inequality it follows

$$1=\|x_0\|^2=\int_{s_0}^{s_0+\delta}\langle f_0(t), B(t)^*U(s_0,t)^*x_0\rangle\,dt$$

$$\leq\left(\int_{s_0}^{s_0+\delta} f_0(t)\|^2\,dt\right)^{1/2}\cdot\left(\int_{s_0}^{s_0+\delta}\|B(t)^*U(s_0,t)^*x_0\|^2\,dt\right)^{1/2}$$

$$\leq\sqrt{(\delta\cdot\varepsilon)}\cdot k_0,$$

which is a contradiction since ε can be found arbitrarily small.

The following result can be found in Ref. [10].

Lemma 2.2. *The equation* (A) *is exponentially stable if and only if there exists a constant* $N>0$ *such that*

$$\int_{t_0}^{\infty}\|U(t,t_0)x\|^q\,dt\leq N, \qquad (1\leq q<\infty)$$

for all $t_0\geq 0$ *and for all* $x\in X$ *with* $\|x\|=1$.

Let $a\geq 0$ *and* $1\leq p\leq\infty$. $L_a^p(X)$ *shall denote the Banach space of all* X*-valued, strongly measurable functions defined* a.e. *on* R_+ *such that*

$$\|f\|_{(a,p)}=\int_{R_+}\|f(s)\|^p\cdot e^{pas}\,ds<\infty \quad \text{if} \quad p<\infty$$

$$\|f\|_{(a,\infty)}=\operatorname*{ess\,sup}_{s\geq 0}\|f(s)\|e^{as}<\infty \quad \text{if} \quad p=\infty.$$

(*Particularly* $L_0^p(X)=L^p(X)$).

3. Example. The following example shows that, in contrast to the case where X is finite dimensional, zero state B.I.B.O. stability does not imply the inequality (1.2).

Let l^2 be the Hilbert space of all real sequences $x=(x_0, x_1, x_2, \ldots, x_n, \ldots)$ such that $\sum_{n=0}^{\infty}|x_n^2|<\infty$.

On l^2 we define the bounded linear operator $B(t)$ defined by

$$B(t)x=\frac{1}{[t]+1}(0,0,\ldots,0,x_{[t]},0,\ldots)$$

where $[t]$ denotes the integral part of t.

It is not difficult to verify that

$$\|B(t)\|=\frac{1}{[t]+1}\leq 1$$

for every $t\geq 0$.

On l^2 we will consider the control system $(0, B)$. Since

$$\int_0^{\infty}\|U(t,s)B(s)\|\,ds=\int_0^{\infty}\|B(s)\|\,ds=\int_0^{\infty}\frac{1}{1+[s]}\,ds=\infty$$

it follows that the inequality (1.2) is not true for the system $(0, B)$.

Let $t \geq t_0 \geq 0$ and let f be an input such that $\|f(s)\| \leq M$ for every $s \geq t_0$.

Then there exists two positive integer numbers m and n such that $m < t_0 \leq m+1$ and $n < t \leq n+1$.

Since $\|f(s)\| \leq M$ it follows that

$$\left| \int_{k-1}^{k} f_{k-1}(s)\, ds \right| \leq M,$$

for every $k \geq m+1$, where $f(s) = (f_0(s), f_1(s), \ldots, f_k(s), \ldots)$.

By virtue of definition of norm on l^2 and by the Cauchy–Schwartz inequality we obtain:

$$\begin{aligned}
\|x(t, t_0; 0; f)\| &= \left\| \int_{t_0}^{t} U(t, s)B(s)f(s)\, ds \right\| = \left\| \frac{1}{m+1} \int_{t_0}^{m+1} (0, 0, \ldots, f_m(s), \ldots)\, ds \right.\\
&\quad + \frac{1}{m+2} \int_{m+1}^{m+2} (0, 0, \ldots, f_{m+1}(s), \ldots)\, ds + \cdots \\
&\quad \left. + \frac{1}{n+1} \int_{n+1}^{t} (0, 0, \ldots, f_n(s), \ldots)\, ds \right\| \\
&= \left\| \left(0, \ldots, 0, \frac{1}{m+1} \int_{t_0}^{m+1} f_m(s)\, ds, \frac{1}{m+2} \right.\right.\\
&\quad \times \left.\left. \int_{m+1}^{m+2} f_{m+1}(s)\, ds, \ldots, \frac{1}{n+1} \int_{n}^{t} f_n(s)\, ds, 0, \ldots \right) \right\| \\
&= \left[\sum_{j=m+2}^{n} \frac{1}{j^2} \left(\int_{j-1}^{j} f_{j-1}(s)\, ds \right)^2 + \frac{1}{(m+1)^2} \left(\int_{t}^{m+1} f_m(s)\, ds \right)^2 \right.\\
&\quad \left. + \frac{1}{(n+1)^2} \left(\int_{n}^{t} f_n(s)\, ds \right)^2 \right]^{1/2} \\
&\leq M \left(\sum_{j=m+1}^{n+1} \frac{1}{j^2} \right)^{1/2} \leq M \cdot \left(\sum_{j=1}^{\infty} \frac{1}{j^2} \right)^{1/2} = M \cdot \frac{\pi}{\sqrt{6}} < \infty,
\end{aligned}$$

and hence the system $(0, B)$ is zero state B.I.B.O. stable. Hence the Theorem 1.1 is false in the infinite dimensional case.

4. (L_a^p, L_b^q) **stability.** Let $a, b \geq 0$ and $1 \leq p, q \leq \infty$.

Definition 4.1. The linear control system is said to be (L_a^p, L_b^q) *stable* iff for every $f \in L_a^p(Y)$ the output

$$x(t, 0; 0; f) = \int_0^t U(t, s)B(s)f(s)\, ds \quad \text{is in} \quad L_b^q(X).$$

Lemma 4.1. *If (A, B) is (L_a^p, L_b^q) stable then the operator $\Lambda : L_a^p(X) \to L_b^q(X)$*

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defined by

$$(\Lambda f)(t)=\int_0^t U(t, s)B(s)f(s)\,ds$$

is a bounded linear operator.

Proof. Let $f_n \to f$ in $L^p_a(X)$ and $\Lambda f_n \to g$ in $L^q_b(X)$.

Hence $h_n \to h$ in $L^p(X)$, where $h_n(s)=f_n(s)e^{as}$ and $h(s)=f(s)e^{as}$.

Since we are dealing with L^p spaces, we can find a subsequence h_{n_k} such that $h_{n_k}(t)\to h(t)$ a.e. on R_+ (and hence $f_{n_k}(t)\to f(t)$ a.e. on R_+) and

$$\int_0^t U(t, s)B(s)f_{n_k}(s)\,ds \to g(t) \quad \text{a.e. on} \quad R_+.$$

Because $U(t, s)B(s)$ is strongly continuous on X, this means that

$$(\Lambda f_{n_k})(t)=\int_0^t U(t, s)B(s)f_{n_k}(s)\,ds \to \int_0^t U(t, s)B(s)f(s)\,ds$$

for all $t\in R_+$. Hence, $g(t)=\int_0^t U(t, s)B(s)f(s)\,ds$ a.e. on R_+.

The closed graph theorem ([12]) shows that Λ is continuous.

Corollary 4.2. *If (A, B) is (L^p_a, L^∞_b) stable then there exists $M>0$ such that*

$$e^{bt}\cdot\left\|\int_0^t U(t, s)B(s)f(s)\,ds\right\|\le M\cdot\|f\|_{(p,a)}.$$

Theorem 4.3. *If the system (A, B) is* u.c.c. *and (L^p_a, L^∞_b) stable then there exists $N>0$ such that*

$$\|U(t, t_0)\|\le N\cdot e^{at_0}e^{-bt}$$

for all $t\ge t_0\ge 0$.

Proof. By u.c.c. of (A, B) we have that there exist $\delta>0$ and $m>0$ such that $W(s+\delta, s)$ are invertible operators and

$$\|W(s+\delta, s)^{-1}\|\le 1/m, \quad \text{for all} \quad s\ge 0.$$

Let $t\ge\delta$ and let n be a positive integer such that $t\ge(n+1)\delta$. Let f_n be the control function given by

$$f_n(s)=\begin{cases} K_1B(s)^*U(n\delta, s)^*x, & \text{if} \quad s\in[n\delta, (n+1)\delta] \\ 0, & \text{if} \quad s\notin[n\delta, (n+1)\delta], \end{cases}$$

where $x\in X$ with $\|x\|=1$ and $K_1=1/(K\cdot e^{K\delta}\cdot \sqrt[p]{\delta})$.

Clearly $f_n\in L^p_a(Y)$ and by

$$\|U(n\delta, s)\|\le e\int_{n\delta}^s \|A(s)\|\,ds\le e^{K\delta}, \quad \text{for} \quad s\le(n+1)\delta \quad ([10])$$

and by (2.1) we then obtain $\|f_n\|_{(p,a)} \le e^{(n+1)a\delta}$. Using the Corollary 4.2. it follows:

$$Me^{(n+1)a}e^{-bt} \ge \left\| \int_0^t U(t, s)B(s)f_n(s)\, ds \right\|$$
$$= K_1 \|U(t, n\delta)W((n+1)\delta, n\delta)x\|$$

and hence

$$\|U(t, n\delta)W((n+1)\delta, n\delta)\| \le \frac{M}{K_1} \cdot e^{(n+1)a\delta} \cdot e^{-bt}$$

for every $t \ge (n+1)\delta$. By u.c.c. of (A, B) we obtain

$$\|U(t, n\delta)\| \le \|U(t, n\delta)W((n+1)\delta, n\delta)\| \cdot \|W((n+1)\delta, n\delta)^{-1}\|$$
$$\le M_1 e^{(n+1)a\delta} \cdot e^{-bt},$$

for $t \ge (n+1)\delta$, where $M_1 = M/m \cdot K_1$.

(i) Let $t \ge \delta$, $t_0 \in [\delta, t]$ and let n be the positive integer such that $(n+1)\delta \le t_0 < (n+2)\delta$. Then

$$\|U(t, t_0)\| \le \|U(t, n\delta)\| \cdot \|U(n\delta, t_0)\| \le M_1 e^{(n+1)a\delta} \cdot e^{-bt} \cdot e^{K(t_0 - n\delta)}$$
$$\le M_1 e^{2K\delta} e^{at_0} e^{-bt} = M_2 e^{at_0} e^{-bt}.$$

(ii) If $t \ge \delta \ge t_0 \ge 0$ then

$$\|U(t, t_0)\| \le \|U(t, \delta)\| \cdot \|U(\delta, t_0)\| \le M_2 e^{a\delta} \cdot e^{-bt} \cdot e^{K(\delta - t_0)}$$
$$= M_3 e^{at_0} e^{-bt}, \quad \text{where} \quad M_3 = e^{\delta(K+a)}.$$

(iii) Let $t_0 \ge 0$ and $t \ge t_0$.

If $m_4 = \sup\{\|U(t, t_0)\| e^{-at_0} \cdot e^{bt},\ 0 \le t_0 \le t \le \delta\}$ and $N = \max\{M_2, M_3, M_4\}$ then we have

$$\|U(t, t_0)\| \le Ne^{-at_0} \cdot e^{bt}$$

for all $t_0 \ge 0$ and $t \ge t_0$.

Corollary 4.4. *If the system* (A, B) *is* u.c.c. *and* (L_a^p, L_b^∞) *stable* (*where* $1 \le p \le \infty$ *and* $0 \le a < b$ *or* $0 < a \le b$) *then the equation* (A) *is exponentially stable.*

Theorem 4.5. *If the system* (A, B) *is* u.c.c. *and* (L_0^p, L_b^q) *stable* (*where* $b \ge 0$, $1 \le p \le \infty$, *and* $1 \le q < \infty$) *then the equation* (A) *is exponentially stable.*

Proof. Let $x \in X$ with $\|x\| = 1$. From Lemma 2.1. it follows that there exist $\delta > 0$ and $k > 0$ such that for all $t_0 \ge 0$, there exists an input f which satisfies $\|f(s)\| \le k$ for $s \in [t_0, t_0 + \delta]$ and

$$x = \int_{t_0}^{t_0+\delta} U(t_0, s)B(s)f(s)\, ds.$$

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Let

$$g(s)=\begin{cases} f(s), & \text{if } s\in[t_0, t_0+\delta] \\ 0, & \text{if } s\notin[t_0, t_0+\delta]\end{cases}$$

Clearly $g\in L_a^p(Y)$ and

$$(\Lambda g)(t)=x(t, 0; 0; g)=\begin{cases} 0, & \text{if } t\le t_0 \\ \int_{t_0}^{t} U(t, s)B(s)f(s)\, ds, & \text{if } t_0\le t\le t_0+\delta \\ \int_{t_0}^{t_0+\delta} U(t, s)B(s)f(s)\, ds = U(t, t_0)x, & \text{if } t>t_0+\delta. \end{cases}$$

Using the (L_0^p, L_b^q) stability we have

$$\int_{t_0}^{\infty} \|U(t, t_0)x\|^q e^{qbt}\, ds \le \int_0^{\infty} \|(\Lambda g)(t)\|^q e^{qbt} = \|\Lambda_g\|^q_{(q,b)}$$
$$\le M\cdot\|g\|^q_{(p,0)} \le (M\cdot k\cdot\delta)^q$$

and hence

$$\int_{t_0}^{\infty} \|U(t, t_0)x\|^q\, dt \le \int_{t_0}^{t_0+\delta} \|U(t, t_0)x\|^q\, dt + \int_{t_0+\delta}^{\infty} \|U(t, t_0)x\|^q\, e^{qbt}\, dt$$
$$\le N = e^{K\delta q} + (Mk\delta)^q,$$

for every $t_0\ge 0$ and for all $x\in X$ with $\|x\|=1$. By Lemma 2.2 it follows that (A) is exponentially stable.

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